Research Article

Green's Function for a Slice of the Korányi Ball in the Heisenberg Group $\mathbb{H}_n$

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We give a representation formula for solution of the inhomogeneous Dirichlet problem on the upper half Korányi ball and for the slice of the Korányi ball in the Heisenberg group $\mathbb{H}_n$ by obtaining explicit expressions of Green-like kernel when the given data has certain radial symmetry.

1. Introduction

The Heisenberg groups, in discrete and continuous versions, appear in several streams of mathematics, including Fourier analysis, several complex variables, geometry, and topology. The well known concepts of Green, Neumann, and Robin functions are important for representing solutions to certain boundary value problems for elliptic equations and to realize their smoothness properties. While the existence of these particular fundamental solutions for admissible domains are presented in all textbooks (see, e.g., [1–3]), explicit expressions for these functions are rare. In case of the Laplace operator mostly just the unit ball serves as an example.

The objective of this paper is to continue the search for explicit Green's functions for domains other than the Korányi ball in the Heisenberg group. There is little hope to get explicit kernels that work for arbitrary continuous boundary data, but it is possible to find some if one is restricted to boundary data having certain symmetry properties. This line of investigation was started by Gaveau et al. in [4] where they dealt with the case of the unit ball in the 3-dimensional Heisenberg group $\mathbb{H}_1$ and functions invariant under a circle action. This result was extended in [5] to the general Heisenberg group $\mathbb{H}_n$ with its natural metric, for functions invariant under the unitary group $U(n)$. Further, in the case of $\mathbb{H}_n$, it has been shown that the method of [5] works for the much larger class of circular functions, that is, functions invariant under a circle

action [6]. The Dirichlet problem on the Heisenberg group and the existence of unique solution was discussed in [7]. Green's function for circular data in the Heisenberg group has been studied for various domains, for example, for half space in [6], for quarter space in [8], and for annulus in [9].

In following sections, we obtain the circular Green's function for the upper half Korányi ball and a slice of the Korányi ball by two parallel planes. We apply the method of infinitely many reflections along the boundaries of the domain, which was introduced by Courant and Hilbert in [2], generalized for the annulus in the Heisenberg group [10].

2. Analysis on the Heisenberg Group

We begin by recalling some notions from [11], which have laid the foundation for harmonic analysis on the Heisenberg group.

Definition 1. The Heisenberg group $\mathbb{H}_n$ (of degree n) is the Lie group structure on $\mathbb{C}^n \times \mathbb{R}$ whose group law is given by

$$[z, t] \cdot [z', t'] = [z + z', t + t' + 2\operatorname{Im}(z \cdot \bar{z}')],$$

$$z \cdot \bar{z}' = \sum_{j=1}^n z_j \cdot \bar{z}'_j. \quad (1)$$

We write $z_j = x_j + iy_j$, $j = 1, \dots, n$, and we define on $\mathbb{H}_n$ the vector fields

$$\begin{aligned} X_j &= \partial_{x_j} + 2y_j \partial_t, \\ Y_j &= \partial_{y_j} - 2x_j \partial_t, \\ T &= \partial_t, \\ Z_j &= \frac{1}{2} (X_j - iY_j) = \partial_{z_j} + i\bar{z}_j \partial_t, \\ \bar{Z}_j &= \frac{1}{2} (X_j + iY_j) = \partial_{\bar{z}_j} - iz_j \partial_t. \end{aligned} \quad (2)$$

One may note the following:

$$\begin{aligned} [X_j, Y_k] &= -4\delta_{jk}T, \\ [X_j, X_k] &= [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0, \\ \forall 1 \leq j, k \leq n. \end{aligned} \quad (3)$$

Equivalently,

$$\begin{aligned} [Z_j, \bar{Z}_k] &= -2i\delta_{jk}T, \\ [Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] &= [Z_j, T] = [\bar{Z}_j, T] = 0, \\ \forall 1 \leq j, k \leq n. \end{aligned} \quad (4)$$

Here $[\cdot, \cdot]$ is the Lie bracket.

Observe that Z_j, T are left invariant vector fields generated by the tangent vectors $\partial_{z_j}, \partial_t$ at e .

If $\xi = [z, t] \in \mathbb{H}_n$, the natural gauge on $\mathbb{H}_n$ is given by

$$N(z, t) = (|z|^4 + t^2)^{1/4}. \quad (5)$$

On the Heisenberg group we have an analogue of the Laplacian which was first studied by Folland and Stein [11]. The sub-Laplacian L on $\mathbb{H}_n$ is explicitly given by

$$L = -\sum_{j=1}^n (X_j^2 + Y_j^2). \quad (6)$$

Let L_0 denote the slightly modified subelliptic operator $(-1/4)L$.

The fundamental solution for the sub-Laplacian L_0 on the Heisenberg group was first given by Folland in [12] with pole at identity by

$$g_e(\xi) = g_e([z, t]) = a_0 (|z|^4 + t^2)^{-n/2}, \quad (7)$$

where

$$a_0 = 2^{n-2} \frac{(\Gamma(n/2))^2}{\pi^{n+1}}, \quad (8)$$

is the normalization constant.

The fundamental solution of L_0 with pole at η is given by

$$g_\eta(\xi) = g_e(\eta^{-1}\xi). \quad (9)$$

From [6], for $\eta = [z', t']$ and $\xi = [z, t]$,

$$g_\eta(\xi) = a_0 |C_{[z', t']} (z, t) - P_{[z', t']} (z, t)|^{-n}, \quad (10)$$

where

$$\begin{aligned} C_{[z', t']} (z, t) &= |z|^2 + |z'|^2 + i(t - t'), \\ P_{[z', t']} (z, t) &= 2z \cdot \bar{z}'. \end{aligned} \quad (11)$$

For an integrable function f on $\mathbb{H}_n$, we denote the average of f by

$$\bar{f}([z, t]) = \frac{1}{2\pi} \int_0^{2\pi} f([e^{i\theta} z, t]) d\theta. \quad (12)$$

A function f is said to be circular, that is, invariant under circle action if $f([z, t]) = \bar{f}([z, t])$, for $[z, t] \in \mathbb{H}_n$.

The average of the fundamental solution with pole at η is given in [6] as

$$\begin{aligned} \bar{g}_\eta(\xi) &= a_0 |C_{[z', t']} (z, t)|^{-n} F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', t']} (z, t)|^2}\right), \end{aligned} \quad (13)$$

where F is the Gaussian hypergeometric function [13]. We can easily see that the average of the fundamental solution is also L_0 -harmonic away from η .

3. Green's Function for the Upper Half Korányi Ball

In this section, the domain D is defined as $D = B \cap H$, where B denotes the Korányi ball in $\mathbb{H}_n$; that is, $B = \{\xi = [z, t] \in \mathbb{H}_n : N(\xi) < 1\}$ and $H = \{\xi = [z, t] \in \mathbb{H}_n : t > 0\}$ denotes the half space in $\mathbb{H}_n$.

W. Thomson (Lord Kelvin) proved in 1847 the following fact: if U is harmonic on $\mathbb{R}^3$ then the function V defined by

$$V(x, y, z) := ar^{-1}U(a^2r^{-2}x, a^2r^{-2}y, a^2r^{-2}z), \quad (14)$$

($r = \sqrt{x^2 + y^2 + z^2}$, $a > 0$) is harmonic on $\mathbb{R}^3 \setminus \{0\}$ [14, p. 232]. For this reason the transformation $U \mapsto V$ is called the Kelvin transform. An analogue of the classical Kelvin transform is defined for functions on the Heisenberg group and it is shown that this transform preserves the class of functions annihilated by the left invariant sub-Laplacian [15]. For a function f on $\mathbb{H}_n$, the Kelvin transform is defined by

$$Kf = N^{-2n} foh, \quad (15)$$

where h is the inversion defined as

$$h([z, t]) = \left[\frac{-z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2} \right], \quad (16)$$

for $[z, t] \in \mathbb{H}_n \setminus \{e\}$. The Kelvin transform sends a L_0 -harmonic function on $\mathbb{H}_n \setminus \{e\}$ to a L_0 -harmonic function. It was shown in [5] that, for a circular function f on $\mathbb{H}_n \setminus \{e\}$, we have

$$K(f)(\xi^{-1}) = f(\xi), \quad (17)$$

for all ξ with $N(\xi) = 1$.

From [5, (3.3)] we have, for $\eta \neq e \in \mathbb{H}_n$

$$K(g_\eta) = N(\eta)^{-2n} g_{\eta^\dagger}, \quad (18)$$

where we wrote $\eta^\dagger$ for $h(\eta)$.

For D , let

$$\begin{aligned} G_D(\eta, \xi) &= \bar{g}_\eta(\xi) - K(\bar{g}_\eta)(\xi^{-1}) - \bar{g}_{\eta^*}(\xi) \\ &\quad + K(\bar{g}_{\eta^*})(\xi^{-1}), \end{aligned} \quad (19)$$

η^* being the reflection of the point η with respect to the boundary $t = 0$; that is, $\eta^* = [z', -t']$.

In the next result, we take the differential operator L_0 with respect to the variable ξ .

Theorem 2. For each $\eta \in D$, the function $G_D(\eta, \xi)$ as defined in (19) is a smooth function on $D \setminus \{\eta\}$ which extends continuously up to ∂D and satisfies the following:

- (i) $L_0 G_D(\eta, \xi) = \delta_\eta$.
- (ii) Limits of the function $G_D(\eta, \xi)$ vanish at the boundaries of the upper half Korányi ball, that is, at $N(\xi) = 1$ and $t = 0$.

Proof. (i) For $\eta = [z', t'] \in D$, we have

$$K(\bar{g}_\eta)(\xi^{-1}) = (N(\eta))^{-2n} \bar{g}_{\eta^\dagger}(\xi^{-1}), \quad (20)$$

since $\eta^\dagger = h(\eta)$ lies outside D and we know that $\bar{g}_\eta(\xi)$ is L_0 -harmonic away from η therefore $K(\bar{g}_\eta)(\xi^{-1})$ is L_0 -harmonic on D .

Similarly $\bar{g}_{\eta^*}(\xi)$ and $K(\bar{g}_{\eta^*})(\xi^{-1})$ are L_0 -harmonic on D because $\eta^* = [z', -t']$ and $h(\eta^*) = h([z', -t'])$ lie outside D .

Hence, $L_0(G_D(\eta, \xi)) = L_0(\bar{g}_\eta(\xi)) = \delta_\eta$.

(ii) We have

$$\begin{aligned} \bar{g}_{\eta^*}(\xi) &= a_0 |C_{[z', -t']}(z, t)|^{-n} F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P_{[z', t']}(z, t)|^2}{|C_{[z', -t']}(z, t)|^2}\right), \\ \bar{g}_\eta(\xi) &= a_0 |C_{[z', t']}(z, t)|^{-n} F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P_{[z', t']}(z, t)|^2}{|C_{[z', t']}(z, t)|^2}\right), \end{aligned} \quad (21)$$

where $C_{[z', \pm t']}(z, t) = |z|^2 + |z'|^2 + i(t \mp t')$ and $P_{[z', t']}(z, t) = 2z \cdot \bar{z}'$.

At $t = 0$, $C_{[z', t']}(z, t) = C_{[z', -t']}(z, t)$, therefore,

$$\begin{aligned} \bar{g}_{\eta^*}(\xi) &= \bar{g}_\eta(\xi), \\ K(\bar{g}_{\eta^*})(\xi^{-1}) &= K(\bar{g}_\eta)(\xi^{-1}). \end{aligned} \quad (22)$$

Hence $G_D(\eta, \xi) = 0$ at $t = 0$.

Since $\bar{g}_\eta(\xi)$ is circular, so, by using (17) we get

$$\begin{aligned} \bar{g}_{\eta^*}(\xi) &= K(\bar{g}_{\eta^*})(\xi^{-1}), \\ \bar{g}_\eta(\xi) &= K(\bar{g}_\eta)(\xi^{-1}) \end{aligned} \quad (23)$$

for $N(\xi) = 1$.

Hence $G_D(\eta, \xi) = 0$ at $N(\xi) = 1$. $\square$

By using the circular Green's function, we can easily solve the inhomogeneous Dirichlet problem

$$\begin{aligned} L_0 u &= f \quad \text{in } D, \\ u &= h \quad \text{on } \partial D, \end{aligned} \quad (24)$$

where f and h are continuous circular functions and the solution is given by the representation formula

$$\begin{aligned} u(\eta) &= \int_D G_D(\eta, \xi) f(\xi) dv(\xi) \\ &\quad + \int_{\partial D} P_D(\eta, \xi) h(\xi) d\sigma(\xi), \end{aligned} \quad (25)$$

where

$$P_D(\eta, \xi) = -\frac{1}{4} \frac{\partial}{\partial n_0} G_D(\eta, \xi), \quad \xi \in \partial D \quad (26)$$

is the Poisson kernel and $d\sigma$ denote the Riemannian surface element, dv being the corresponding Riemannian volume element on $\mathbb{H}_n$; for detailed study refer [5].

4. Green's Function for a Slice of the Korányi Ball

In this section S denotes a slice of the Korányi ball in $\mathbb{H}_n$ which is defined as follows:

$$\begin{aligned} S &= \{\xi = [z, t] \in \mathbb{H}_n : N(\xi) < 1, c_1 < t < c_2\}, \\ &\quad -1 < c_1 < c_2 < 1. \end{aligned} \quad (27)$$

For $\xi = [z, t]$ and $\eta = [z', t'] \in \mathbb{H}_n$ we define

$$\begin{aligned} H_\xi(z', t') &= a_0 |C_{[z', t']}(z, t)|^{-n} F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P_{[z', t']}(z, t)|^2}{|C_{[z', t']}(z, t)|^2}\right), \end{aligned} \quad (28)$$

where $C_{[z', t']}(z, t)$, $P_{[z', t']}(z, t)$ are as in (11). A differential operator, whenever applied to function $H_\xi(z', t')$, will be

with respect to the variable ξ . Now, we assert that the Green function for the domain S is given by

$$\begin{aligned}
 G_S(\eta, \xi) = & \sum_{m=0}^{\infty} \left[H_{\xi}(z', 2m(c_2 - c_1) + t') \right. \\
 & - K \left(H_{\xi^{-1}}(z', 2m(c_2 - c_1) + t') \right) \\
 & + \sum_{m=1}^{\infty} \left[H_{\xi}(z', 2m(c_1 - c_2) + t') \right. \\
 & - K \left(H_{\xi^{-1}}(z', 2m(c_1 - c_2) + t') \right) \\
 & - \sum_{m=1}^{\infty} \left[H_{\xi}(z', (2mc_2 - (2m-2)c_1) - t') \right. \\
 & - K \left(H_{\xi^{-1}}(z', (2mc_2 - (2m-2)c_1) - t') \right) \\
 & - \sum_{m=1}^{\infty} \left[H_{\xi}(z', (2mc_1 - (2m-2)c_2) - t') \right. \\
 & \left. \left. - K \left(H_{\xi^{-1}}(z', (2mc_1 - (2m-2)c_2) - t') \right) \right] \right], \quad (29)
 \end{aligned}$$

where K denotes the Kelvin transform in $\mathbb{H}_n$. In the following lemma, we prove that $G_S(\eta, \xi)$ is well defined for all η, ξ with $|z| \neq |z'|$ or $t \neq t'$.

Lemma 3. For a fixed $\eta = [z', t']$ and $\xi = [z, t]$ such that $|z| \neq |z'|$ or $t \neq t'$, the series on RHS of (29) are absolutely and uniformly convergent on a compact neighbourhood of ξ .

Proof.

Case 1. Suppose that $t = t'$, we choose a relatively compact neighbourhood U of ξ such that $||z| - |z'|| > \delta, \delta > 0$. Then,

$$\begin{aligned}
 & \frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} \\
 &= \frac{4|z|^2 |z'|^2}{|z|^4 + |z'|^4 + 2|z|^2 |z'|^2 + (t - (2m(c_2 - c_1) + t'))^2} \quad (30) \\
 &\leq \frac{M}{4m^2 (c_2 - c_1)^2}, \quad \text{where } M = \sup_U |z|^2 \text{ for } m > 0,
 \end{aligned}$$

and therefore, for sufficiently large m ,

$$\frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} < 1 - \epsilon \quad \text{for some } \epsilon > 0. \quad (31)$$

On the set $[0, 1 - \epsilon]$, the function $F(n/2, n/2; n; |P_{[z', t']} (z, t)|^2 / |C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2)$ is bounded and so,

for sufficiently large m , $H_{\xi}(z', 2m(c_2 - c_1) + t')$ are uniformly bounded. Also, for smaller positive values of m

$$\begin{aligned}
 & \frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} \leq \frac{2|z|^2 |z'|^2}{2|z|^2 |z'|^2 + 4m^2 (c_2 - c_1)^2} \\
 &\leq \frac{1}{1 + 2m^2 (c_2 - c_1)^2 / |z|^2 |z'|^2} \quad (32) \\
 &\leq \frac{1}{1 + 2m^2 (c_2 - c_1)^2 / M^2 |z'|^2} < 1.
 \end{aligned}$$

Hence $H_{\xi}(z', 2m(c_2 - c_1) + t')$ are uniformly bounded for all $m > 0$. For $m = 0$, the term is

$$\frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} = \frac{4|z|^2 |z'|^2}{|z|^4 + |z'|^4 + 2|z|^2 |z'|^2}. \quad (33)$$

In case $|z| = 0$ or $|z'| = 0$, above expression is equal to zero. For $|z| \neq 0$ and $|z'| \neq 0$,

$$\begin{aligned}
 & 1 - \frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} \\
 &= \frac{(|z|^2 - |z'|^2)^2}{|z|^4 + |z'|^4 + 2|z|^2 |z'|^2} \quad (34) \\
 &\geq \frac{\delta^2}{|z|^4 + |z'|^4 + 2|z|^2 |z'|^2},
 \end{aligned}$$

on a neighbourhood U of ξ

and therefore, $F(n/2, n/2; n; |P_{[z', t']} (z, t)|^2 / |C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2)$ is uniformly bounded on U . Now,

$$\begin{aligned}
 & |C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^{-n} \\
 &= \frac{1}{| |z|^2 + |z'|^2 + i(t - 2m(c_2 - c_1) - t') |^n} \\
 &= \frac{1}{| |z|^2 + |z'|^2 - 2m(c_2 - c_1) i |^n} \quad (35) \\
 &\leq \frac{1}{(2m(c_2 - c_1))^n}.
 \end{aligned}$$

Thus,

$$\sum_{m=0}^{\infty} H_{\xi}(z', 2m(c_2 - c_1) + t') < C \sum_{m=0}^{\infty} \frac{1}{(2m(c_2 - c_1))^n}. \quad (36)$$

Case 2. Suppose that $|z| = |z'|$, we choose a compact neighbourhood O of ξ such that $|t - t'| > \delta_1$, $\delta_1 > 0$, for all $t' \in O$.

$$\frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2}$$

$$= \frac{4|z|^4}{4|z|^4 + (t - t' - 2m(c_2 - c_1))^2}$$

$$= \frac{1}{1 + ((t - t' - 2m(c_2 - c_1))/2|z|^2)^2} < 1 - \epsilon_1$$

for some $\epsilon_1 > 0$.

(37)

So, the functions $F(n/2, n/2; n; |P_{[z', t']} (z, t)|^2 / |C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2)$ are uniformly bounded on O . Now

$$\begin{aligned} |C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^{-n} &= \frac{1}{|z|^2 + |z'|^2 + i(t - 2m(c_2 - c_1) - t')}^n = \frac{1}{(4|z|^4 + (t - 2m(c_2 - c_1) - t')^2)^{n/2}} \\ &= \frac{1}{(4m^2(c_2 - c_1)^2)^{n/2} (|z|^4/m^2(c_2 - c_1)^2 + ((t - t')/2m(c_2 - c_1))^2)^{n/2}} < \frac{1}{(2m(c_2 - c_1))^n}. \end{aligned} \quad (38)$$

Case 3. Now, suppose that $|z| \neq |z'|$, $t \neq t'$, and consider

$$\begin{aligned} &\frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} \\ &= \frac{4|z|^2|z'|^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2 + (t - (2m(c_2 - c_1) + t'))^2} \quad (39) \\ &\leq \frac{4|z|^2|z'|^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 - \frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} & \\ &\geq \frac{(|z|^2 - |z'|^2)^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2}. \end{aligned} \quad (40)$$

We can choose a suitable compact neighbourhood of ξ such that $|z|^2 - |z'|^2 > \epsilon$ for some $\epsilon > 0$. We have

$$1 - \frac{|P_{[z', t']} (z, t)|^2}{|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2} > \frac{\epsilon^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2} \quad (41)$$

$> \epsilon_1$.

This implies that $|P_{[z', t']} (z, t)|^2 / |C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^2 < 1 - \epsilon_1$. Clearly,

$$|C_{[z', 2m(c_2 - c_1) + t']} (z, t)|^{-n} \leq (2m(c_2 - c_1))^{-n} \quad (42)$$

for sufficiently large m .

Hence, in all the three cases

$$\sum_{m=0}^{\infty} H_{\xi}(z', 2m(c_2 - c_1) + t') < C \sum_{m=0}^{\infty} \frac{1}{(2m(c_2 - c_1))^n}, \quad (43)$$

for some positive constant C . Thus, $\sum_{m=0}^{\infty} H_{\xi}(z', 2m(c_2 - c_1) + t')$ is absolutely and uniformly convergent for $\eta \neq \xi$. Similarly, one can easily show that the other series in the RHS of (29) are uniformly and absolutely convergent for $\eta \neq \xi$.

Therefore, $G_S(\eta, \xi)$ is well defined for all $\eta \neq \xi$. $\square$

In the next lemma, we show that $G_S(\eta, \xi) = 0$ on the boundaries of the domain S .

Lemma 4. For $m \geq 0$,

$$\begin{aligned} \lim_{t \rightarrow c_1} H_{\xi}(z', 2m(c_2 - c_1) + t') &= \lim_{t \rightarrow c_1} H_{\xi}(z', \\ &\quad (2(m+1)c_1 - (2(m+1)-2)c_2) - t'), \\ \lim_{t \rightarrow c_1} K[H_{\xi^{-1}}(z', 2m(c_2 - c_1) + t')] & \\ &= \lim_{t \rightarrow c_1} K[H_{\xi^{-1}}(z', \\ &\quad (2(m+1)c_1 - (2(m+1)-2)c_2) - t')], \end{aligned} \quad (44)$$

and for $m \geq 1$

$$\begin{aligned} \lim_{t \rightarrow c_1} H_{\xi}(z', (2mc_2 - (2m-2)c_1) - t') & \\ &= \lim_{t \rightarrow c_1} H_{\xi}(z', 2m(c_1 - c_2) + t'), \\ \lim_{t \rightarrow c_1} K[H_{\xi^{-1}}(z', (2mc_2 - (2m-2)c_1) - t')] & \\ &= \lim_{t \rightarrow c_1} K[H_{\xi^{-1}}(z', 2m(c_1 - c_2) + t')]. \end{aligned} \quad (45)$$

Similarly, for $m \geq 0$,

$$\begin{aligned} \lim_{t \rightarrow c_2} H_{\xi}(z', 2m(c_1 - c_2) + t') &= \lim_{t \rightarrow c_2} H_{\xi}(z', \\ &\quad (2(m+1)c_2 - (2(m+1)-2)c_1) - t'), \end{aligned}$$

$$\begin{aligned}
& \lim_{t \rightarrow c_2} K \left[H_{\xi^{-1}} \left(z', 2m(c_1 - c_2) + t' \right) \right] \\
&= \lim_{t \rightarrow c_2} K \left[H_{\xi^{-1}} \left(z', \right. \right. \\
&\quad \left. \left. (2(m+1)c_2 - (2(m+1)-2)c_1) - t' \right) \right],
\end{aligned} \tag{46}$$

and for $m \geq 1$

$$\begin{aligned}
& \lim_{t \rightarrow c_2} H_{\xi} \left(z', (2mc_1 - (2m-2)c_2) - t' \right) \\
&= \lim_{t \rightarrow c_2} H_{\xi} \left(z', 2m(c_2 - c_1) + t' \right), \\
& \lim_{t \rightarrow c_2} K \left[H_{\xi^{-1}} \left(z', (2mc_1 - (2m-2)c_2) - t' \right) \right] \\
&= \lim_{t \rightarrow c_2} K \left[H_{\xi^{-1}} \left(z', 2m(c_2 - c_1) + t' \right) \right].
\end{aligned} \tag{47}$$

Finally, for $m \geq 0$

$$\begin{aligned}
& \lim_{N(\xi) \rightarrow 1} H_{\xi} \left(z', 2m(c_2 - c_1) + t' \right) \\
&= \lim_{N(\xi) \rightarrow 1} K \left[H_{\xi^{-1}} \left(z', 2m(c_2 - c_1) + t' \right) \right],
\end{aligned} \tag{48}$$

and for $m \geq 1$

$$\begin{aligned}
& \lim_{N(\xi) \rightarrow 1} H_{\xi} \left(z', 2m(c_1 - c_2) + t' \right) \\
&= \lim_{N(\xi) \rightarrow 1} K \left[H_{\xi^{-1}} \left(z', 2m(c_1 - c_2) + t' \right) \right], \\
& \lim_{N(\xi) \rightarrow 1} H_{\xi} \left(z', (2mc_2 - (2m-2)c_1) - t' \right) \\
&= \lim_{N(\xi) \rightarrow 1} K \left[H_{\xi^{-1}} \left(z', (2mc_2 - (2m-2)c_1) - t' \right) \right], \\
& \lim_{N(\xi) \rightarrow 1} H_{\xi} \left(z', (2mc_1 - (2m-2)c_2) - t' \right) \\
&= \lim_{N(\xi) \rightarrow 1} K \left[H_{\xi^{-1}} \left(z', (2mc_1 - (2m-2)c_2) - t' \right) \right].
\end{aligned} \tag{49}$$

Proof. Consider

$$\begin{aligned}
& \left| C_{[z', 2m(c_2 - c_1) + t']} (z, t) \right| = \left| |z|^2 + |z'|^2 \right. \\
& \quad \left. + i \left(t - (2m(c_2 - c_1) + t') \right) \right| = \left| |z|^2 + |z'|^2 \right. \\
& \quad \left. + i \left(c_1 - 2mc_2 + 2mc_1 - t' \right) \right| \\
& \quad \quad \quad \text{at } t = c_1 \\
&= \left| |z|^2 + |z'|^2 \right. \\
& \quad \left. + i \left((2(m+1)c_1 - (2(m+1)-2)c_2) - t' \right) \right| \\
&= \left| C_{[z', (2(m+1)c_1 - (2(m+1)-2)c_2) - t']} (z, t) \right|
\end{aligned} \tag{50}$$

for $m \geq 0$.

Therefore,

$$\begin{aligned}
& \lim_{t \rightarrow c_1} H_{\xi} \left(z', 2m(c_2 - c_1) + t' \right) \\
&= \lim_{t \rightarrow c_1} H_{\xi} \left(z', (2(m+1)c_1 - (2(m+1)-2)c_2) \right. \\
&\quad \left. - t' \right).
\end{aligned} \tag{51}$$

On the same lines, one can easily prove the limits agree over the other boundaries also. $\square$

Now, we show that $G_S(\eta, \xi)$ works as Green's function for domain S when applied to circular data.

Theorem 5. For each $\eta \in S$, the function $G_S(\eta, \xi)$ as defined in (29) is a smooth function on $S \setminus \{\eta\}$ which extends continuously up to ∂S and satisfies the following:

- (i) $L_0 G_S(\eta, \xi) = \delta_{\eta}(\xi)$.
- (ii) Limits of the function $G_S(\eta, \xi)$ vanish at the boundaries of Korányi ball slice, that is, at $t = c_1$, $t = c_2$, and $N(\xi) = 1$.

Proof. (i) For $\eta = [z', t'] \in S$, $m \geq 1$, the functions $H_{\xi}(z', 2m(c_2 - c_1) + t')$, $H_{\xi}(z', 2m(c_1 - c_2) + t')$, $H_{\xi}(z', (2mc_2 - (2m-2)c_1) - t')$, $H_{\xi}(z', (2mc_1 - (2m-2)c_2) - t')$, $K(H_{\xi^{-1}}(z', 2m(c_2 - c_1) + t'))$, $K(H_{\xi^{-1}}(z', 2m(c_1 - c_2) + t'))$, $K(H_{\xi^{-1}}(z', (2mc_2 - (2m-2)c_1) - t'))$, and $K(H_{\xi^{-1}}(z', (2mc_1 - (2m-2)c_2) - t'))$ are all harmonic at $\xi = [z, t]$ because poles of these functions lie outside the domain S . Also we have $K(H_{\xi^{-1}}(z', t'))$ is harmonic and from the definition of $H_{\xi}(z', t')$ it follows that

$$L_0 H_{\xi}(z', t') = \delta_{\eta}(\xi). \tag{52}$$

Therefore, by using Lemma 3

$$\begin{aligned}
L_0 G_S(\eta, \xi) &= \sum_{m=0}^{\infty} L_0 \left[H_{\xi}(z', 2m(c_2 - c_1) + t') \right. \\
&\quad \left. - K(H_{\xi^{-1}}(z', 2m(c_2 - c_1) + t')) \right] \\
&\quad + \sum_{m=1}^{\infty} L_0 \left[H_{\xi}(z', 2m(c_1 - c_2) + t') \right. \\
&\quad \left. - K(H_{\xi^{-1}}(z', 2m(c_1 - c_2) + t')) \right] \\
&\quad - \sum_{m=1}^{\infty} L_0 \left[H_{\xi}(z', (2mc_2 - (2m-2)c_1) - t') \right. \\
&\quad \left. - K(H_{\xi^{-1}}(z', (2mc_2 - (2m-2)c_1) - t')) \right] \\
&\quad - \sum_{m=1}^{\infty} L_0 \left[H_{\xi}(z', (2mc_1 - (2m-2)c_2) - t') \right. \\
&\quad \left. - K(H_{\xi^{-1}}(z', (2mc_1 - (2m-2)c_2) - t')) \right] \\
&= \delta_{\eta}(\xi).
\end{aligned} \tag{53}$$

(ii) This part easily follows from Lemma 4.

Hence the theorem is established. $\square$

Now, we have circular Green's function for a slice of the Korányi ball so we can obtain explicit expression of the Poisson kernel for this domain to solve inhomogeneous Dirichlet boundary value problem. The Poisson kernel is the normal derivative of Green's function and, from [5, 6], is given by

$$P(\eta, \xi) = -\frac{1}{4} \frac{\partial}{\partial n_0} G(\eta, \xi), \quad \xi \in \partial S, \quad (54)$$

where

$$\frac{\partial}{\partial n_0} = \begin{cases} \frac{1}{|z|} (\overline{AE} + A\overline{E}) & \text{at } \{\xi \in \mathbb{H}_n : N(\xi) = 1\} \\ \frac{i}{|z|} (E - \overline{E}) & \text{at } \{\xi \in \mathbb{H}_n : t = c_1, t = c_2\}. \end{cases} \quad (55)$$

Theorem 6. *Green's function and the Poisson kernel which we have obtained solve the inhomogeneous Dirichlet boundary value problem for S and the solution for BVP*

$$\begin{aligned} L_0 u &= f \quad \text{in } S, \\ u &= h \quad \text{on } \partial S \end{aligned} \quad (56)$$

is given by

$$\begin{aligned} u(\eta) &= \int_S G(\eta, \xi) f(\xi) dv(\xi) \\ &+ \int_{t=c_1} P(\eta, \xi) h(\xi) d\sigma(\xi) \\ &+ \int_{t=c_2} P(\eta, \xi) h(\xi) d\sigma(\xi) \\ &+ \int_{N(\xi)=1} P(\eta, \xi) h(\xi) d\sigma(\xi), \end{aligned} \quad (57)$$

where f and h are continuous circular functions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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