Research Article

Solving the Linear 1D Thermoelasticity Equations with Pure Delay

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We propose a system of partial differential equations with a single constant delay $\tau > 0$ describing the behavior of a one-dimensional thermoelastic solid occupying a bounded interval of $\mathbb{R}^1$. For an initial-boundary value problem associated with this system, we prove a well-posedness result in a certain topology under appropriate regularity conditions on the data. Further, we show the solution of our delayed model to converge to the solution of the classical equations of thermoelasticity as $\tau \to 0$. Finally, we deduce an explicit solution representation for the delay problem.

1. Introduction

Over the past half-century, the equations of thermoelasticity have drawn a lot of attention from the side of both mathematical and physical communities. Starting with the late 50s and early 60s of the last century, the necessity of a rational physical description for elastic deformations of solid bodies accompanied by thermal stresses motivated the more prominent mathematicians, physicists, and engineers to focus on this problem (see, e.g., [1, 2]). As a consequence, many theories emerged, mainly in the cross-section of (nonlinear) field theory and thermodynamics, making it possible for the equations of thermoelasticity to be interpreted as an anelastic modification of the equations of elasticity (cf. [3] and the references therein). Both linear and nonlinear models and solution theories were proposed.

An initial-boundary value problem for the general linear equations of classical thermoelasticity in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ was studied by Dafermos in [4]. Here, $[u_i]$ and $T$ denote the (unknown) displacement vector field and the absolute temperature, respectively. Further, $\rho > 0$ is the material density, $\theta_0$ is a reference temperature rendering the body free of thermal stresses, $c_D$ is the specific heat capacity, $[C_{ijkl}]$ stands for Hooke’s tensor, $[m_{ij}]$ is the stress-temperature tensor, $[K_{ij}]$ is the heat conductivity tensor, $[f_i]$ represents the specific external body force, and $r$ is the external heat supply. Under usual initial conditions, appropriate normalization conditions to rule out the rigid motion as a trivial solution, and general boundary conditions

$$u_i = 0 \text{ in } \Gamma_1 \times (0, \infty),$$

$$(C_{ijkl}u_{ki} - m_{ij}T) n_j + A_{ij} u_j = 0 \text{ in } \operatorname{int}(\Gamma_1^r) \times (0, \infty),$$

$$T = 0 \text{ in } \Gamma_2 \times (0, \infty),$$

$$(K_{ij} T_{ij}) n_i + BT = 0 \text{ in } \operatorname{int}(\Gamma_2^r) \times (0, \infty),$$

(3)

where $\Gamma_1, \Gamma_2 \subset \partial \Omega$ are relatively open, $[A_{ij}]$ denotes the “elasticity” modulus, and $B$ is heat transfer coefficient, Dafermos proved the global existence and uniqueness of finite energy solutions and studied their regularity as well as asymptotics.
as $t \to \infty$. In 1D, even an exponential stability result for (1)-(2) under all "reasonable" boundary conditions was shown by Hansen in [5].

In his work [6], Slemrod studied the nonlinear equations of 1D thermoelasticity in the Lagrangian coordinates

$$\partial_t u = \psi_{TF} (\partial_x u + 1, \theta + T_0) \partial_{xx} u + \psi_{TT} (\partial_x u + 1, \theta + T_0) \partial_{x} \theta \quad \text{in } (0, 1) \times (0, \infty),$$

$$\rho (\theta + T_0) (\psi_{TT} (\partial_x u + 1, \theta + T_0) \partial_{x} \theta + \psi_{FF} (\partial_x u + 1, \theta + T_0) \partial_{xx} u)$$

$$= q' (\partial_x \theta) \partial_{xx} \theta \quad \text{in } (0, 1) \times (0, \infty)$$

for the unknown functions $u$ denoting the displacement of the rod and $\theta$ being a temperature difference to a reference temperature $T_0$ rendering the body free of thermal stresses. The functions $\psi$ and $q'$ denote the Helmholtz free energy and the heat flux, respectively, and are assumed to be given. Finally, $\rho > 0$ is the material density in the references configuration. Under appropriate boundary conditions (when the boundary is free of tractions and is held at a constant temperature or when the body is rigidly clamped and thermally insulated) as well as usual initial conditions for both unknown functions, a local existence theorem for (4) was proved by additionally imposing a regularity and compatibility condition. For sufficiently small initial data, the local classical solution could be globally continued. At the same time, when studying (4) in the whole space, large data can be considered. Together with (1)-(2), (5), this leads to the so-called dual phase lag constitutive equation

$$q_i (x, t + \tau_i) + K_{ij} T_j (x, t) = 0 \quad \text{for } (x, t) \in \Omega \times (-\infty, 0).$$

Racke and Shibata studied in [8] the equations of thermoelasticity with pure delay derived in Section 2, we propose a new approach in this paper. Though their approach can essentially be carried over to the equations of hyperbolic thermoelasticity (1), (5)-(6) have been studied in the literature. See, for example, [11] by Messaoudi and Said-Houari for a proof of global well-posedness of the 1D system in the whole space or Irmscher's work [12] for the global well-posedness of nonlinear problem for rotationally symmetric data in a bounded rotationally symmetric domain of $\mathbb{R}^3$. In a bounded 1D domain, a quantitative stability comparison between the classical and the hyperbolic system was presented by Irmscher and Racke in [13]. For a detailed overview on hyperbolic thermoelasticity, we refer the reader to [14] by Chandrasekhar and Racke.

A unified approach establishing a connection between the classical and hyperbolic thermoelasticity was developed by Tzou in [16, 17]. Namely, he proposed to view (6) with $\tau_{ij} \equiv \tau$ as a first-order Taylor approximation of the equation

$$q_i (x, t + \tau_i) + K_{ij} T_j (x, t - \tau_j) = 0 \quad \text{for } (x, t) \in \Omega \times (0, \infty).$$

More generally, higher-order Taylor expansion to the dual phase lag constitutive equation

$$q_i (x, t + \tau_i) + K_{ij} T_j (x, t + \tau_j)$$

$$= 0 \quad \text{for } (x, t) \in \Omega \times (-\infty, 0).$$

can be considered. Together with (1)-(2), (5), this leads to the equations of hyperbolic thermoelasticity originated from delayed material laws. One of the first attempts to obtain a well-posedness result for a partial differential equation with pure delay is due to Rodrigues et al. In their paper [20], Rodrigues et al. studied a heat equation with pure delay in an appropriate Fréchet space and showed the delayed Laplacian to generate a $C_0$-semigroup on this space. Further, they investigated the spectrum of the infinitesimal generator. Though their approach can essentially be carried over to the equations of thermoelasticity with pure delay derived in Section 2, we propose a new approach in this paper preserving the Hilbert space structure of the space and thus the connection to the classical equations of thermoelasticity. To the authors' best knowledge, no results on thermoelasticity with delay in the highest order terms have been previously published in the literature. At the same time, we refer the
reader to the works by Khusainov et al. [21–24], in which
the authors studied the well-posedness and controllability
for the heat and/or the wave equation on a finite time
horizon. In their recent paper [25], Khusainov et al. exploited
the $L^2$-maximum regularity theory to prove a global well-
posedness and asymptotic stability results for a regularized
heat equation with delay.

The present paper has the following outline. In Section 2,
we give a physical model for linear thermoelasticity based
on delayed material laws. For the sake of simplicity, we
present a 1D model though our approach can easily be
carried over to the general multidimensional case. Next, in
Section 3, we prove the well-posedness of this model in an
appropriate Hilbert space framework and discuss the small
parameter asymptotics, that is, the behavior of solutions
as $\tau \to 0$. Further, in Section 4, we deduce an explicit
solution representation formula. Finally, in the Appendix, we
summarize some seminal results on the delayed exponential
function and Cauchy problems with pure delay.

2. Model Description

We consider a solid body occupying an axis-aligned rectan-
gular domain of $\mathbb{R}^3$. Assuming that the body motion is purely
longitudinal with respect to the first space variable $x$ (cf. [6,
page 100]), deformation gradient, stress and strain tensors,
and so forth are diagonal matrices and a complete rational
description of the original 3D body motion can be reduced to
studying the 1D projection $\Omega = (0, l)$, $l > 0$, of the body onto
the $x$-axis as displayed in Figure 1. Hence, in the following, we
restrict ourselves to considering the relevant physical values
only in $x$-direction.

Let the functions $u : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ and $\theta : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ denote the body displacement and its
relative temperature measured with respect to a reference
temperature $\theta_0 > 0$ rendering the body free of thermal
stresses, respectively. We restrict ourselves to the Lagrangian
coordinates and write $\sigma, e, S, q : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$
for the stress field, strain field, entropy field, or the heat flux,
respectively. With $\rho > 0$ denoting the material density, the
momentum conservation law as well as the linearized entropy
balance law reads as

\[ \rho \partial_t u + \partial_x \sigma = \rho r \quad \text{for } x \in \Omega, \ t > 0, \]
\[ \partial_t \theta + \partial_x S = h \quad \text{for } x \in \Omega, \ t > 0, \]

(10)

where $r : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ and $h : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ are a
known volume force acting on the body and an internal heat
source.

Assuming physical linearity for the strain field, the strain
can be decomposed into elastic strain $\varepsilon^e$ and thermal stress $\varepsilon^\theta$.
Further, assuming $|\theta(t, x)/\theta_0| \ll 1$ uniformly with respect to
$x \in \bar{\Omega}, t \geq 0$, we can postulate

\[ \varepsilon^e(x, t) = \alpha \theta(x, t) \quad \text{for } x \in \Omega, \ t > 0, \]

(11)

\[ \varepsilon^\theta(x, t) = \frac{\rho c_p}{\theta_0} \theta(x, t) \quad \text{for } x \in \Omega, \ t > 0, \]

(12)

with $\varepsilon^e, \varepsilon^\theta > 0$ standing for the specific heat capacity and $B \in \mathbb{R}$
denoting the bulk modulus.

In our further considerations, we depart from the classical
material laws and use their delay counterparts. Let $\tau > 0$ be a
positive time delay. In the sequel, all functions are supposed
to be defined on $\bar{\Omega} \times [-\tau, \infty)$. Assuming a delay feedback
between the stress and the strain as well as the heat flux and
the temperature gradient, Hooke's law with pure delay reads as
(cf. [1])

\[ \sigma(x, t) = \left( B + \frac{4}{3} G \right) \varepsilon^e(x, t-\tau) + B \varepsilon^\theta(x, t-\tau) \]

(13)

for $x \in \Omega, \ t > 0$, with $G > 0$ denoting the shear modulus. Similarly, we
consider a delay version of Fourier's law given as

\[ q(x, t) = -\kappa \partial_x \theta(x, t-\tau) \quad \text{for } x \in \Omega, \ t > 0, \]

(14)

where $\kappa > 0$ stands for the thermal conductivity. Assuming
the elastic strain tensor to be equal to the displacement
gradient, we have

\[ \varepsilon^e(x, t) = \partial_x u(x, t) \quad \text{for } x \in \Omega, \ t > 0, \]

(15)

We further postulate the following relation:

\[ \partial_t \partial_x u(x, t) = \partial_x \partial_t u(x, t-\tau) \quad \text{for } x \in \Omega, \ t > 0 \]

(16)


which is a delayed counterpart of Schwarz's theorem. Finally,
we also modify (12) to introduce a delay feedback between the
entropy, the elastic strain tensor, and the temperature

\[ S(x, t) = \alpha B e^e(x, t-\tau) + \frac{\rho c_p}{\theta_0} \theta(x, t-\tau) \]

(17)

for $x \in \Omega, \ t > 0$. where $\alpha > 0$ denotes the thermal expansion coefficient.

Exploiting now (10), (13)–(17), we obtain

\[
\rho \partial_{tt} u(x, t) - \left( B + \frac{4}{3} G \right) \partial_{xx} u(x, t - \tau) + aB \partial_{x} \theta(x, t - \tau) = f(x, t) \quad \text{for } x \in \Omega, \ t > 0,
\]

\[
\rho c \partial_{t} \theta(x, t) - k \partial_{xx} \theta(x, t - \tau) + a \theta \partial_{x} B \partial_{xx} u(x, t - \tau) = h(x, t) \quad \text{for } x \in \Omega, \ t > 0,
\]

\[
\partial_{x} \partial_{tt} u(x, t) - \partial_{t} \partial_{xx} u(x, t - \tau) = 0 \quad \text{for } x \in \Omega, \ t > 0.
\]

(18)

To close (18), appropriate boundary and initial conditions for \( u \) and \( \theta \) are required. In the following, we prescribe homogeneous Dirichlet boundary conditions for \( u \) and homogeneous Neumann boundary conditions for \( \theta \) given as

\[
u(0, t) = u(l, t) = 0,
\]

\[
\partial_{x} \theta(0, t) = \partial_{x} \theta(l, t) = 0
\]

(19)

for \( t > 0 \).

This particular choice of boundary conditions not only turns out to be convenient for our further mathematical considerations but also is a physically relevant one. Similar to the thermoelasticity with second sound, it is one of the combinations typically arising when studying micro- and nanoscopic strings or plates (cf. [13]).

The initial conditions are given over the whole history period \( (\tau, 0) \) and read as

\[
u(x, 0) = u^0(x), \quad u(x, t) = u^1(x, t)
\]

for \( x \in \Omega, \ t \in (\tau, 0), \)

\[
\partial_{t} u(x, 0) = u^1_1(x), \quad \partial_{t} u(x, t) = u^1_2(x, t)
\]

(20)

for \( x \in \Omega, \ t \in (\tau, 0), \)

\[
\theta(x, 0) = \theta^0(x), \quad \theta(x, t) = \theta^0_1(x, t)
\]

for \( x \in \Omega, \ t \in (\tau, 0), \)

with known \( u^0, u^1, \theta^0 : \Omega \to \mathbb{R} \) and \( u^0_1, u^1_2, \theta^0_1 : \Omega \times (\tau, 0) \to \mathbb{R} \).

3. Well-Posedness and Limit \( \tau \to 0 \)

Letting \( a := (B + (4/3) G)/\rho, b := \alpha B / \rho, c := \kappa / \rho c_p, d := \alpha \theta_B / \rho c_p \) and \( f(x, t) := r(x, t), \) and \( g(x, t) := (1 / \rho c_p) h(x, t) \) for \( x \in \Omega, \ t \geq 0, \) (18) can be rewritten as

\[
\partial_{tt} u(x, t) - a \partial_{xx} u(x, t - \tau) + b \partial_{x} \theta(x, t - \tau) = f(x, t) \quad \text{for } x \in \Omega, \ t > 0,
\]

\[
\partial_{t} \theta(x, t) - c \partial_{xx} \theta(x, t - \tau) + d \partial_{x} u(x, t - \tau) = g(x, t) \quad \text{for } x \in \Omega, \ t > 0,
\]

\[
\partial_{x} \partial_{tt} u(x, t) - \partial_{t} \partial_{xx} u(x, t - \tau) = 0 \quad \text{for } x \in \Omega, \ t > 0.
\]

(21)

subject to the boundary conditions from (19) and initial conditions from (20). Introducing a new vector of unknown functions,

\[
V(x, t) = \begin{pmatrix} V^1(x, t) \\ V^2(x, t) \\ V^3(x, t) \end{pmatrix} := \begin{pmatrix} \partial_{tt} u(x, t) \\ \partial_{t} \theta(x, t) \end{pmatrix}
\]

(22)

for \( x \in \Omega, \ t \in [-\tau, T]. \)

Equations (21) can be transformed to

\[
\partial_{t} V(x, t) + BV(x, t - \tau) = F(x, t) \quad \text{for } x \in \Omega, \ t \in (0, T)
\]

(23)

with the differential matrix operator and the right-hand side

\[
B := \begin{pmatrix} 0 & -a \partial_{xx} & b \partial_{x} \\ -\partial_{x} & 0 & 0 \\ d \partial_{xx} & 0 & -c \partial_{xx} \end{pmatrix}, \quad F(x, t) := \begin{pmatrix} f(x, t) \\ g(x, t) \end{pmatrix}
\]

(24)

respectively.

Exploiting (19) and the definition of \( V \), the boundary conditions for \( V \) read as

\[
V^1(0, t) = V^1(l, t) = 0, \quad \partial_{x} V^1(0, t) = \partial_{x} V^1(l, t) = 0
\]

(25)

for \( t > 0, \)

whereas the initial conditions are given by

\[
V(x, 0) = V^0(x), \quad V(x, t) = V^0_{x}(x, t)
\]

(26)

for \( x \in \Omega, \ t \in (-\tau, 0) \)

with

\[
V^0(x) = \begin{pmatrix} u^0 \\ u^1 \\ \theta^0 \end{pmatrix}, \quad V^0_{x}(x, t) = \begin{pmatrix} u^1_1(x, t) \\ u^1_2(x, t) \\ \theta^0_1(x, t) \end{pmatrix}
\]

(27)

for \( x \in \Omega, \ t \in [-\tau, 0]. \)

Note that (18)–(20), (23), (25), and (26) are equivalent for; if the vector \( V \) is known, \( u \) and \( \theta \) are uniquely determined by

\[
u(x, t) = \begin{cases} u^0(x) + \int_{0}^{t} V^1(x, s) ds, & \text{for } t \geq 0, \\ u^0_{x}(x, t), & \text{for } t \in [-\tau, 0), \end{cases}
\]

(28)

\[
\theta(t, x) = \begin{cases} V^3(x, t), & \text{for } t \geq 0, \\ \theta^0_{x}(x, t), & \text{for } t \in [-\tau, 0). \end{cases}
\]
Therefore, in the sequel, we consider the following equivalent first-order-in-time problem:

\[
\begin{align*}
\partial_t V(x,t) + BV(x,t-\tau) &= F(x,t) \quad \text{for } x \in \Omega, \ t > 0, \\
V^1(0,t) &= V^1(t) = 0, \\
\partial_t V^3(0,t) &= \partial_x V^3(t) = 0 \\
& \quad \text{for } t > 0, \\
V(x,0) &= V^0(x), \\
V(x,t) &= V^0_t(x,t) \\
& \quad \text{for } x \in \Omega, \ t \in (\tau,0).
\end{align*}
\]

(29)

For our well-posedness investigations, we need a solution notion for (29). To this end, appropriate functional spaces have to be introduced. We start with the “naive” approach by using the case \( \tau = 0 \) as a reference situation. We introduce the Hilbert space \( X := L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \) equipped with the dot product

\[
\langle V, W \rangle_X := \langle V^1, W^1 \rangle_{L^2(\Omega)} + a \langle V^2, W^2 \rangle_{L^2(\Omega)}
\]

\[ + \frac{b}{d} \langle V^3, W^3 \rangle_{L^2(\Omega)} \quad \text{for } V, W \in X \]

and define the operator

\[
\mathcal{B} : D(\mathcal{B}) \subset X \rightarrow X, \quad V \mapsto BV \quad \text{(30)}
\]

with the domain

\[
D(\mathcal{B}) := \left\{ V \in H^1_0(\Omega) \times H^1(\Omega) \times H^1(\Omega) \left| \partial_x V^3|_{\partial \Omega} = 0 \right. \right\}. \quad \text{(31)}
\]

See [26, Section 3] for the definition of Sobolev spaces. With this notation, (29) can be written in the equivalent form

\[
\begin{align*}
\partial_t V(x,t) + \mathcal{B} V(x,t-\tau) &= F(x,t) \quad \text{for } x \in \Omega, \ t > 0, \\
V(x,0) &= V^0(x), \\
V(x,t) &= V^0_t(x,t) \\
& \quad \text{for } x \in \Omega, \ t \in (\tau,0).
\end{align*}
\]

(33)

Under a classical solution to (33) on \([-\tau, T]\) for any \(T > 0\), one would naturally understand a function \(V \in C^0([-\tau, T], D(\mathcal{B})) \cap C^1([0, T], X)\) satisfying the equations pointwise.

We know from [5] that the linear operator \(\mathcal{B}\) is accretive and satisfies \(D(\mathcal{B}) = D(\mathcal{B}^\ast)\). Its spectrum \(\sigma(\mathcal{B})\) only consists of isolated eigenvalues \(\lambda_n \in \mathbb{C}, n \in \mathbb{N}_0\), of finite multiplicity with \(\text{Re } \lambda_n \geq 0, n \in \mathbb{N}_0\), and \(\lambda_n \to \infty\) as \(n \to \infty\). The corresponding eigenfunctions \(\{\Psi_n\}_{n \in \mathbb{N}_0} \subset D(\mathcal{B})\) build an orthonormal basis of \(X\). Unfortunately, from [18, Theorem 1.1] we know that (33) is ill-posed in \(X\). Hence, a different solution notion should be adopted. As we already mentioned in Section I, we want to preserve the Hilbert space structure of the problem and thus cannot follow the approach developed by Rodrigues et al. in [20].

For \(T > 0\), we define the space \(X_T := \{ V \in \bigcap_{i=0}^\infty D(\mathcal{B}^i) \mid \| V \|_{X_T} < \infty \}\) equipped with the scalar product induced by the norm

\[
\| V \|_{X_T} := \left\| e^{T|\mathcal{B}|} V \right\|_{D(\mathcal{B}^2)}
\]

\[ = \left( \sum_{n=0}^\infty \left( 1 + |\lambda_n|^2 + |\lambda_n|^4 \right) \right)^{1/2} \cdot \exp \left( 2T|\lambda_n| \right) \left\| \langle V, \Psi_n \rangle_X \right\|^{1/2}
\]

(34)

for \(V \in X_T\).

Hence, \(X_T\) is closed subspace of \(X\) and thus a Hilbert space. Moreover, \(X_T\) is dense in \(X\) since \((\Phi_n)_{n \in \mathbb{N}} \subset X_T\). Indeed, for \(n \in \mathbb{N}\), we have \(\Psi_n \in \bigcap_{i=0}^\infty D(\mathcal{B}^i)\) and \(\| \Psi_n \|_{X_T} = (1 + |\lambda_n|^2 + |\lambda_n|^4) \exp(2T|\lambda_n|) < \infty\).

Restricting \(\mathcal{B}\) to the closed subspace \(X_T\) of \(X\), we trivially obtain a bounded linear operator \(\mathcal{B}_T : X_T \rightarrow D(\mathcal{B}^2)\) since for \(V \in X_T\)

\[
\| \mathcal{B}_T V \|_{D(\mathcal{B}^2)}^2 = \| \mathcal{B}_T V \|_{X_T}^2 + \| \mathcal{B}_T V \|_{X_T}^2 + \| \mathcal{B}_T V \|_{X_T}^2
\]

\[ = \sum_{n=1}^\infty \left( 1 + |\lambda_n|^2 + |\lambda_n|^4 \right) |\lambda_n|^2 \left\| \langle V, \Psi_n \rangle_X \right\|^{1/2}
\]

\[ \leq \frac{1}{2T^2} \sum_{n=1}^\infty \left( 1 + |\lambda_n|^2 + |\lambda_n|^4 \right) \left\| \langle V, \Psi_n \rangle_X \right\|^{1/2}
\]

\[ \leq \frac{1}{2T^2} \| V \|_{X_T}^2. \quad \text{(35)}
\]

Now, restricting (33) to \(X_T\), we obtain

\[
\begin{align*}
\partial_t V(x,t) + \mathcal{B}_T V(x,t-\tau) &= F(x,t) \quad \text{for } x \in \Omega, \ t \in (0,T), \\
V(x,0) &= V^0(x), \\
V(x,t) &= V^0_t(x,t) \\
& \quad \text{for } x \in \Omega, \ t \in (\tau,0).
\end{align*}
\]

(36)

Following the approach in [21], we introduce for \(t \in \mathbb{R}\) the delayed exponential function

\[
\exp_T (\mathcal{B}_T t) := \begin{cases}
0_{L(X)}, & t < -\tau, \\
\text{id}_X + \sum_{k=1}^{\left\lfloor \frac{t}{\tau} \right\rfloor} \frac{(t-(k-1)\tau)^k}{k!} \mathcal{B}_T^k, & t \geq -\tau.
\end{cases}
\]

(37)
is a classical solution to
\[ \partial_t Z(x, t) + B_T Z(x, t - \tau) = 0 \quad \text{for} \quad x \in \Omega, \quad t \in (0, T), \]
\[ Z(x, 0) = 0, \quad Z(x, t) = 0 \quad \text{for} \quad x \in \Omega, \quad t \in (0, -\tau). \]  
(41)

Figure 2 displays the delayed exponential function for the case where $\mathcal{B}_T$ is a real number.

Obviously, for any $t \in \mathbb{R}$, $\exp_t (- \mathcal{B}_T, t) \in \bigcap_{k=0}^{\infty} D(\mathcal{B}^k)$. Moreover, we have
\[ \|\exp_t (- \mathcal{B}_T, t)\|_{L(X, Y)} \leq 1 \quad \text{for} \quad t \in (-\infty, T] \]  
(38)
uniformly in $\tau > 0$ since
\[
\|\exp_t (- \mathcal{B}_T, t)\|_{L(X, Y)}^2 = \|\exp_t (- \mathcal{B}_T, t)\|^2_X + \|\mathcal{B}\exp_t (- \mathcal{B}_T, t)\|^2_X \\
= \sum_{n=0}^{\infty} \left(1 + |\lambda_n|^2 + |\lambda_n|^4 \right) |\exp_t (-\lambda_n, t)|^2 |\langle \Psi, \Psi_n \rangle| X^2 \\
\leq \sum_{n=0}^{\infty} \left(1 + |\lambda_n|^2 + |\lambda_n|^4 \right) \\
\cdot \left(1 + \sum_{k=1}^{\infty} \frac{(t - (k-1))\tau}{k!} |\lambda_n|^k \right)^2 \\
\cdot |\langle \Psi, \Psi_n \rangle| X^2 \\
\leq \sum_{n=0}^{\infty} \left(1 + |\lambda_n|^2 + |\lambda_n|^4 \right) \exp (2T |\lambda_n|) \\
\cdot |\langle \Psi, \Psi_n \rangle| X^2 = \|\Psi\|_{X_T}^2 \quad \text{for} \quad \Psi \in X_T. 
\]  
(39)

Here, $L(X, Y)$ denotes the space of bounded, linear operators from $X$ to $Y$ equipped with the standard operator topology.

Now, we can prove the following well-posedness result.

**Theorem 1.** For $T > 0$, let $V^0 \in X_T$, $V^0_t \in C^\infty([-\tau, 0], X_T)$ with $V^0_t(0, 0) = V^0$ and let $F \in C^\infty([0, T], X_T)$. Then (36) possess a unique classical solution $V \in C^\infty([-\tau, T], D(\mathcal{B})) \cap C^1([0, T], X)$ explicitly given by

\[ V(t, t), \quad t \in [-\tau, 0), \]
\[ V^0, \quad t = 0, \]
\[ \exp_t (- \mathcal{B}_T, t - \tau) V^0 \]
\[ = - \mathcal{B}_T \int_{-\tau}^t \exp_s (- \mathcal{B}_T, t - 2\tau - s) \cdot \langle V, F(s) \rangle ds, \quad t \in (0, T]. 
\]  
(40)

**Proof.** First, we prove uniqueness of classical solutions. Assuming $V$ and $W$ to be classical solutions of (36), we conclude that their difference $Z := V - W \in C^\infty([-\tau, T], D(\mathcal{B})) \cap C^1([0, T], X)$ is a classical solution to

\[ \partial_t Z(x, t) + \mathcal{B}_T Z(x, t - \tau) = 0 \quad \text{for} \quad x \in \Omega, \quad t \in (0, T), \]
\[ Z(x, 0) = 0, \quad Z(x, t) = 0 \quad \text{for} \quad x \in \Omega, \quad t \in (-\tau, 0). 
\]  
(41)
Multiplying these equations with $\Psi_n$ in the inner product of $X$, we further deduce that $Z_n(t, \cdot) := \langle Z(t, \cdot), \Psi_n \rangle_X$, $t \in [-\tau, T]$, is a classical solution to the scalar delay differential equation

$$\dot{Z}_n(t) + \lambda_n Z_n(t - \tau) = 0 \quad \text{for } t \in (0, T),$$

$$Z_n(0) = 0, \quad Z_n(t) = 0 \quad \text{for } x \in \Omega, \ t \in (-\tau, 0).$$

(42)

From Theorem A.4 in the Appendix, the later equations are known to be uniquely solvable by $Z_n \equiv 0$. Hence, $Z_n \equiv 0$ for all $n \in \mathbb{N}$. With $(\Psi_n)_{n \in \mathbb{N}}$ being a basis of $X$, this implies $Z \equiv 0$, and, therefore, $V \equiv W$.

For the existence proof, we show that the function $V$ in (40) is a classical solution to (36). Performing the diagonalization, we obtain for $t \in [0, T]$:

$$V(t, t) = \sum_{n=1}^{\infty} \exp(-\lambda_n t - \tau) \left( \langle V^0, \Psi_n \rangle_X \right) \Psi_n \left( \int_0^t \exp(-\lambda_n t - 2\tau - s) \langle F(s), \Psi_n \rangle_X \right) ds \right) \Psi_n$$

$$\equiv \sum_{n=1}^{\infty} V_n(t) \Psi_n,$$

(43)

From [25], we know $V_n \in C^0([-\tau, T], X)$ and $C^1((0, T], X)$ for all $n \in \mathbb{N}$. Further, by the virtue of (38), the series converges uniformly in $D(\mathcal{B})$ and its derivative converges uniformly in $X$. Hence, the limiting function lies in $C([-\tau, T], D(\mathcal{B})) \cap C^1((0, T], X)$ and thus possesses the regularity of a classical solution. Finally, using the properties of scalar delay exponential (cf. [25]), we easily verify that $V$ solves (36).

Taking into account inequality (38) and applying Hölder’s inequality to (40), we obtain the following estimate.

**Corollary 2.** The solution $V$ continuously depends on the data in sense of the estimate

$$\|V\|_{C^0([0,T],X)} \leq \|V^0\|_{X_T} + \|V^0\|_{C^0([0,T],X_T)}$$

$$+ \sqrt{T} \|F\|_{L^2(0,T;X_T)} \quad \text{for } T > 0$$

(44)

For the rest of this section, we want to study the behavior of system (33) as $\tau \to 0$. Let $T_0 > 0$ and $T > 0$ be fixed. Similar to $X_T$, we consider the Hilbert space $Y_T := \{V \in \bigcap_{k=0}^{\infty} D(\mathcal{B}^k) \ | \ \|V\|_{Y_T} < \infty \}$ equipped with the scalar product induced by the norm

$$\|V\|_{Y_T} = \left( \sum_{n=0}^{\infty} (1 + |\lambda_n|^2) \right)^{1/2} \left( \langle V, \Psi_n \rangle_X \right)^{1/2} \quad \text{for } V \in Y_T.$$

(45)

Obviously, $Y_T \hookrightarrow X_T$. For simplicity, despite a slight abuse of notation, we let $\mathcal{B}_T$ now denote the part of $\mathcal{B}$ in $Y_T$ (namely, [27, page 139]). Formally, the limiting system of (33) as $\tau \to 0$ is given by

$$\partial_\tau V(x, t) + \mathcal{B}_T V(x, t) = F(x, t) \quad \text{for } x \in \Omega, \ t \in (0, T),$$

$$V(x, 0) = V^0(x) \quad \text{for } x \in \Omega.$$

(46)

From [27, Corollary 3.3.13], we know that $-\mathcal{B}_T$ generates a uniformly bounded $C_0$-semigroup $(\exp(-\mathcal{B}_T t))_{t \geq 0}$ of bounded linear operators on $Y_T$. The unique classical solution to (46) can then be written using Duhamel’s formula as

$$\widetilde{V}(t, t) = \exp(-\mathcal{B}_T t) V^0$$

$$+ \int_0^t \exp(-\mathcal{B}_T (t - s)) F(s) ds \quad \text{for } t \in [0, T].$$

(47)

**Theorem A.3** from the Appendix implies the following.

**Lemma 3.** There holds

$$\|\exp(-\mathcal{B}_T t - \tau) - \exp(-\mathcal{B}_T t)\|_{L^2(Y_T;D(\mathcal{B}))} \leq \tau$$

for $t \in [0, T]$.

Now, we can prove the following.

**Theorem 4.** Let $V^0 \in Y_T$, $F \in C^0([0,\infty), Y_T)$ be fixed. For $\tau > 0$, let $V^0 \in C^0([-\tau, 0], Y_T)$ with $V^0(0) = V^0$ and $\limsup_{\tau \to 0} \|V^0\|_{L^1(0;Y_T)} < \infty$. Denoting with $V(\tau)$ the classical solution of (36) corresponding to the initial data $V^0$, $V^0$ and the right-hand side $F$, one has

$$\|V(\tau) - V(\tau)\|_{C^0([0,T],X)} = O(\tau) \quad \text{as } \tau \to 0.$$

(49)

**Proof.** Using the representation formulas for $V$ and $\widetilde{V}$, we can estimate for any $t \in [0, T]$

$$\|V(t) - \widetilde{V}(\tau; t)\|_{X_T}$$

$$\leq \|\exp(-\mathcal{B}_T t - \tau) - \exp(-\mathcal{B}_T t)\|_{L^2(Y_T;X_T)} \|V^0\|_Y$$

$$+ \int_0^t \|\mathcal{B}_T \exp(-\mathcal{B}_T t - 2\tau - s)\|_{L^2(Y_T;X_T)} \|V^0\|_Y ds$$

$$+ \int_0^t \|\exp(-\mathcal{B}_T t - \tau - s) - \exp(-\mathcal{B}_T t - s)\|_{L^2(Y_T,X_T)} ds$$

$$\leq \tau \|V^0\|_Y + \tau (1 + \tau) \limsup_{\tau \to 0} \|V^0\|_{L^1(\tau;0,Y_T)}$$

$$+ \tau T \|F\|_{L^1(0;Y_T)} = O(\tau) \quad \text{as } \tau \to 0.$$

(50)

This finishes the proof.
4. Explicit Solution Representation

In this section, we want to deduce an explicit representation of solutions to (36) in the form of a Fourier series with respect to an orthogonal basis \( (\Phi_n)_{n \in \mathbb{N}_0} \) of \( X \) (and thus of \( X_T \)) given by

\[
\Phi_n(x) = \begin{cases} 
\sqrt{\frac{T}{2l}}(0,1) \quad &\text{if } n = 0, \\
\sqrt{\frac{T}{3l}}(\sin(v_n x), \cos(v_n x)) \quad &\text{otherwise}, 
\end{cases}
\]

for \( x \in \overline{\Omega}, \quad n \in \mathbb{N}_0 \)  

(51)

with 

\[ v_n := \frac{m}{l} \quad \text{for } n \in \mathbb{N}_0. \]  

(52)

Note that the sequence \( (\Phi_n)_{n \in \mathbb{N}_0} \) does not coincide, in general, with the eigenfunctions \( (\Psi_n)_{n \in \mathbb{N}_0} \) but, at the same time, \( (\Phi_n)_{n \in \mathbb{N}_0} \subset D(\partial_T) \) constitutes a basis of \( D(\partial_T) \). To this end, we assume that the conditions of Theorem 1 are satisfied which yields a unique classical solution \( V \in C^0((-\tau, \infty), D(\partial_T)) \cap C^1([0, \infty), X) \).

Denoting \( \Phi_n = (\Phi_n^1, \Phi_n^2, \Phi_n^3)^T \) and computing the componentwise Fourier coefficients

\[
V^{nk} = \langle V^{nk}, \Phi_n^k \rangle_{L^2(\Omega)},
\]

\[
V^{nk}(t) = \langle V^{nk}(t), \Phi_n^k \rangle \quad \text{for } t \in [-\tau, 0],
\]

\[
F^{nk}(t) = \langle F^{nk}(t), \Phi_n^k \rangle \quad \text{for } t \geq 0
\]

for \( n \in \mathbb{N}_0 \) and \( k = 1, 2, 3 \), we get the following Fourier expansions:

\[
V^0 = \sum_{n=0}^{\infty} \sum_{k=0}^{3} V^{nk} \Phi_n^k, 
\]

\[
V_n(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{3} (V_{1,n}^1 \Phi_n^1 + V_{2,n}^2 \Phi_n^2 + V_{3,n}^3 \Phi_n^3) \quad \text{for } t \in [-\tau, 0],
\]

\[
F(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{3} (F_{1,n}^1 \Phi_n^1 + F_{2,n}^2 \Phi_n^2 + F_{3,n}^3 \Phi_n^3) \quad \text{for } t \geq 0
\]

(54)

uniformly in \( \bar{\Omega} \) (due to the Sobolev embedding theorem). Similarly, the solution \( V \) can be expanded into Fourier series

\[
V(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{3} (V_{1,n}^1(t) \Phi_n^1 + V_{2,n}^2(t) \Phi_n^2 + V_{3,n}^3(t) \Phi_n^3)
\]

(55)

for some \( V_{nk} \in C^0((-\tau, \infty), C) \cap C^1([0, \infty), C), n \in \mathbb{N}_0, k = 1, 2, 3 \), to be determined later. Using this ansatz and letting

\[
B_n := \begin{pmatrix} 0 & a_{\nu_n} & -b_{\nu_n} \\
-a_{\nu_n} & 0 & 0 \\
-b_{\nu_n} & 0 & c_{\nu_n}^2 \\
\end{pmatrix},
\]

(56)

we observe that (36) decompose into a sequence of ordinary delay differential equations

\[
\begin{align*}
V_n(t) &= -B_n V_n(t - \tau) + F_n(t) & \text{for } t > 0, \\
V_n(0) &= V_n^0, \quad V_n(t) = V_{n,\tau}^0(t) & \text{for } t \in (-\tau, 0).
\end{align*}
\]

(57)

By the virtue of Theorem A.4, for any \( n \in \mathbb{N}_0 \), the unique solution to (57) is given by

\[
\begin{align*}
V_n(t) &= \begin{cases} 
V_{n,\tau}^0(t), & t \in [-\tau, 0), \\
V_n^0, & t = 0, \\
\exp_t(-B_n, t - \tau) V_n^0, & t \in (0, T] \\
\int_0^t \exp_s(-B_n, t - \tau - s) V_n^0(s) ds + \int_0^t \exp_s(-B_n, t - \tau) F_n(s) ds, & t \in (0, T].
\end{cases}
\end{align*}
\]

(58)

To explicitly compute the function given in (58), we need to diagonalize the matrix \( B_n \).

Lemma 5. Let

\[
\Delta_0 = \frac{c_n^2 \nu_n^2}{3} - 3(a + bd) \nu_n^2,
\]

\[
\Delta_1 = -2c_n^4 \nu_n^6 + 9c(a + bd) \nu_n^4 - 27ac \nu_n^2,
\]

\[
C = \sqrt{\frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3} \right)}.
\]

(59)

where \( \sqrt{\cdot} \) and \( \sqrt[3]{\cdot} \) stand for the main branch of complex square and cubic roots. The spectrum of \( B_n \) consists of three eigenvalues

\[
\mu_{n,k} = \begin{cases} 
0, & n = 0, \\
\frac{1}{3} \left( c_{\nu_n} - C e^{2ik\tau/3} - e^{-2ik\tau/3} \frac{\Delta_0}{C} \right), & \text{otherwise}
\end{cases}
\]

(60)

for \( k = 0, 1, 2 \) with \( i \) denoting the imaginary unit.

Proof. For \( n = 0 \), we have \( \nu_n = 0 \) and therefore \( B_n = 0_{3 \times 3} \). Hence, 0 is the only eigenvalue of \( B_n \) with an algebraic multiplicity of 3.

Now, let us assume \( n > 1 \). To compute the eigenvalues of \( B_n \), we consider the characteristic polynomial

\[
P_n(\mu) := \det(B_n - \mu I_{3 \times 3}) = \mu^3 - c_{\nu_n}^2 \mu^2
\]

\[
+ (a + bd) \nu_n^2 \mu - a c_{\nu_n}^4.
\]

(61)

Since the matrix

\[
B_n := \begin{pmatrix} 0 & a_{\nu_n} & -b_{\nu_n} \\
-a_{\nu_n} & 0 & 0 \\
-b_{\nu_n} & 0 & c_{\nu_n}^2 \\
\end{pmatrix}
\]

(62)
has real components and is skew-symmetrizable, it has to possess one real and two complex-conjugate eigenvalues. Thus, introducing the expressions

\[
\Delta_0 = c^2 v_n^4 - 3 (a + bd) v_n^2, \\
\Delta_1 = -2c^2 v_n^6 + 9c (a + bd) v_n^4 - 27acv_n^2, \\
C = \sqrt{\frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3} \right)},
\]

we obtain the three roots \( \mu_{n,1}, \mu_{n,2}, \mu_{n,3} \) of the characteristic equation \( P_n \) (cf. [28, page 179])

\[
\mu_{n,k} = \frac{1}{3} \left( \sqrt[3]{C}, \sqrt[3]{C}, \sqrt[3]{C} \right), \quad k = 0, 1, 2
\]

where \( \sqrt[3]{C} \) and \( \sqrt[3]{C} \) stand for the main branch of complex square and cubic roots.

Lemma 6. Eigenvectors \( v_{nk} \), \( k = 0, 1, 2 \), of \( B_n \) corresponding to the eigenvalues \( \mu_{n,k} \) of \( B_n \) from Lemma 5 are given by

\[
v_{nk} = \begin{cases} \begin{pmatrix} e_1 & \text{if } n = 0, \\ -bv_n \mu_{n,k} & \frac{by_n}{\mu_{n,k}} \end{pmatrix}, & \text{otherwise} \\ e_2 & -c v_n \mu_{n,k} + \frac{b}{\mu_{n,k}} \end{cases}
\]

with \( e_1 = (1, 0, 0)^T \), \( e_2 = (0, 1, 0)^T \), and \( e_3 = (0, 0, 1)^T \).

Proof. Since the first case for \( n = 0 \) is obvious, we only consider the case \( n > 1 \). For \( k \in \{0, 1, 2\} \), we consider the matrix

\[
\mu_{n,k} I_{3n} - B_n = \begin{pmatrix} \mu_{n,k} - av_n & bv_n \\ v_n & \mu_{n,k} - av_n \\ -dv_n & \mu_{n,k} - av_n \end{pmatrix}.
\]

The latter is singular since \( \alpha_{n,k} \) is an eigenvalue of \( B_n \). Further, due to the fact that

\[
det(B_n) = acv_n^4 > 0,
\]

\( B_n \) is invertible and, therefore, \( \mu_{n,k} \neq 0 \). We want to find a nontrivial vector \( v_{nk} \in \mathbb{R}^3 \) satisfying

\[
(\alpha_{n,k} I_{3n} - B_n) v_{nk} = 0_{3x1}.
\]

Thus, we can apply a Gauss-Jordan iteration to the former matrix and find

\[
\mu_{n,k} I_{3n} - B_n = \begin{pmatrix} \mu_{n,k} - av_n & bv_n \\ 0 & \mu_{n,k} - av_n \\ -dv_n & \mu_{n,k} - av_n \end{pmatrix}.
\]

Since the latter matrix must be singular, the third row must be proportional to the second one. Thus, (68) is equivalent to

\[
\begin{pmatrix} \mu_{n,k} - av_n & bv_n \\ 0 & \mu_{n,k} + av_n & -bv_n \end{pmatrix} v_{nk} = 0_{3x1}.
\]

Since the rank of this matrix is 2, the equation above yields only one eigenvector

\[
v_{nk} = \begin{pmatrix} -bv_n \mu_{n,k} \\ by_n \mu_{n,k} \\ \mu_{n,k}^3 + \mu_{n,k}^2 \end{pmatrix}
\]

being determined up to a multiplicative constant. \( \square \)

Note that \( v_{n,1}, v_{n,2}, \) and \( v_{n,3} \) are linearly independent, but, in general, not orthonormal.

Letting now

\[
D_n := \text{diag}(\mu_{n,1}, \mu_{n,2}, \mu_{n,3}),
\]

we obtain a singular value decomposition for \( B_n \)

\[
B_n = S_n D_n S_n^{-1}
\]

with an invertible matrix

\[
S_n = (v_{n,1} v_{n,2} v_{n,3})^T.
\]

Exploiting now Corollary A.2 from Appendix, (58) can finally be written as

\[
V_n(t) = \begin{cases} V^0, & t \in [-\tau, 0), \\ V^0, & t = 0, \\ S_n \exp \left(-D_n (t - \tau)\right) S_n^{-1} V^0 \int_{-\tau}^{t} \exp \left(-D_n (t - 2\tau - s)\right) \\
\cdot S_n^{-1} F_n(s) ds, & t \in (0, T], 
\end{cases}
\]

where the inverse of \( S_n \) is given by the Laplace formula

\[
S_n^{-1} = \begin{pmatrix} S_n^{22} S_n^{33} - S_n^{32} S_n^{33} & -S_n^{12} S_n^{33} + S_n^{13} S_n^{32} & S_n^{12} S_n^{23} - S_n^{13} S_n^{22} \\ -S_n^{21} S_n^{33} + S_n^{23} S_n^{31} & S_n^{11} S_n^{33} - S_n^{13} S_n^{31} -S_n^{12} S_n^{23} + S_n^{13} S_n^{21} \\ -S_n^{21} S_n^{32} - S_n^{23} S_n^{31} & S_n^{11} S_n^{32} - S_n^{12} S_n^{31} +S_n^{13} S_n^{22} - S_n^{13} S_n^{21} \end{pmatrix}
\]

\[
\cdot \left(S_n^{11} S_n^{22} S_n^{33} + S_n^{12} S_n^{23} S_n^{31} +S_n^{13} S_n^{21} S_n^{32} - S_n^{11} S_n^{22} S_n^{33} \right)^{-1}.
\]
Appendix

Ordinary Delay Differential Equations

Let $X$ be a (real or complex) Hilbert space and let $\tau > 0$ be arbitrary. For a bounded linear operator $\mathcal{B}$ on $X$, we consider the following ordinary delay differential equation:

$$\begin{align*}
\frac{\partial}{\partial t} u(t) &= \mathcal{B} u(t - \tau) + f(t) \quad \text{for } t > 0, \\
u(0) &= u^0, \\
u(t) &= u^0 \quad \text{for } t \in (-\tau, 0)
\end{align*}$$

(A.1)

for some $u^0 \in X$, $u^0 \in L^2(-\tau, 0; X)$, and $f \in L^2_{\text{loc}}(0, \infty; X)$.

Since the delay exponential function

$$\exp_{\tau}(\mathcal{B}, t) = \begin{cases} 0_{L(X)}, & t < -\tau, \\
\text{id}_X + \sum_{k=1}^{\lfloor (1/\tau) \rfloor} \frac{(t - (k - 1)\tau)^k}{k!} \mathcal{B}^k, & t \geq -\tau
\end{cases}$$

(A.2)

is an operator polynomial in $\mathcal{B}$ piecewise with respect to $t$, we obviously have the following representation.

**Theorem A.1.** Let $\mathcal{S}: X \rightarrow X$ be an isomorphism. Then

$$\exp_{\tau}(\mathcal{B}, t) = \mathcal{S} \exp_{\tau} \left( \mathcal{S}^{-1} \mathcal{B} \mathcal{S}, t \right) \mathcal{S}^{-1} \quad \text{for } t \in \mathbb{R}.$$  \hspace{1cm} (A.3)

**Corollary A.2.** If $X \subseteq \{\mathbb{R}^d, \mathbb{C}^d\}$, $d \in \mathbb{N}$, and $\mathcal{B}$ is diagonalizable over $\mathbb{C}$, that is, if there exists a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_d)$, $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$, and an invertible $S \in \mathbb{C}^{d \times d}$ such that $A = S D S^{-1}$, then

$$\exp_{\tau}(\mathcal{B}, t) = S \exp_{\tau}(D,t) S^{-1} = S \text{diag} (\exp_{\tau}(\lambda_1,t), \ldots, \exp_{\tau}(\lambda_d,t)) S^{-1} \quad \text{for } t \in \mathbb{R}.$$  \hspace{1cm} (A.4)

**Theorem A.3.** Let $b \in C$, $T > 0$, and $\tau > 0$ and let $\alpha := 1 + |b| \exp(\tau_0 |b|)$. Then, for any $\tau \in (0, \tau_0]$,

$$\begin{align*}
\left| \exp_{\tau}(b,t-\tau) - \exp(b t) \right| &\leq \tau \exp(\alpha T |b|) \quad \text{for } t \in [0,T], \\
\end{align*}$$

(A.5)

**Proof.** Let $\tau \in (0, \tau_0]$. For $t \in [0, \tau]$, the claim immediately follows from the mean value theorem for integration. We use mathematical induction to prove for any $k \in \mathbb{N}$

$$\begin{align*}
\left| \exp_{\tau}(b,t-\tau) - \exp(b t) \right| &\leq \tau \exp(\alpha \tau |b|) \\
&\text{for } t \in ((k - 1)\tau, k\tau].
\end{align*}$$

(A.6)

Indeed, assuming that the claim is true for some $k \in \mathbb{N}$, the fundamental theorem of calculus yields for $t \in (k\tau, (k + 1)\tau]$

$$\begin{align*}
\left| \exp_{\tau}(b,t-\tau) - \exp(b t) \right| &\leq \tau \exp(\alpha \tau |b|) \\
&\quad + \frac{d}{ds} \left| \int_{k\tau}^{(k+1)\tau} \exp_{\tau}(b,s-t) - \exp(b s) \, ds \right| \, ds \\
&\leq \tau \exp(\alpha \tau |b|) \\
&\quad + |b| \left| \int_{k\tau}^{(k+1)\tau} \exp_{\tau}(b,s-2\tau) - \exp(b(s-\tau)) \right| \, ds \\
&\leq \tau \exp(\alpha \tau |b|) \\
&\quad + |b| \int_{k\tau}^{(k+1)\tau} \left| \exp_{\tau}(b,s-2\tau) - \exp(b(s-\tau)) \right| \, ds \\
&\leq \tau \exp(\alpha \tau |b|) + \tau^2 |b| \exp(\alpha \tau |b|) \\
&\quad + \tau^2 |b|^2 \exp(\alpha \tau |b|) \\
&\leq \tau \exp(\alpha \tau |b|) \left( 1 + \tau \exp(\alpha \tau |b|) \right) \\
&\leq \tau \exp(\alpha \tau |b|) \exp(\alpha \tau |b|) \exp(\alpha \tau |b|) \\
&\leq \tau \exp(\alpha \tau |b|) \exp(\alpha \tau |b|) \exp(\alpha \tau |b|) \\
&\leq \tau \exp(\alpha \tau |b|) \exp(\alpha |b|) \\
&\leq \tau \exp(\alpha |b|).
\end{align*}$$

(A.7)

as $\alpha \geq 1$. Finally, the claim follows by induction. \hfill \Box

According to [25, Theorem 3.12], we have the following well-posedness result for (A.1).

**Theorem A.4.** The delay differential equation (A.1) possesses a unique strong solution $u \in L^2_{\text{loc}}(\mathbb{R}, C^1 C^0 X) \cap H^1_{\text{loc}}(0, \infty; X)$ given by

$$\begin{align*}
u(t) &= \begin{cases} \varphi(t), & t \in [-\tau, 0), \\
u^0, & t = 0, \\
\exp_{\tau}(\mathcal{B}, t-\tau) u^0 \\
+ \int_{-\tau}^{t} \exp_{\tau}(\mathcal{B}, t-2\tau-s) u^0(s) \, ds \\
+ \int_{0}^{t} \exp_{\tau}(\mathcal{B}, t-\tau-s) f(s) \, ds, & t \geq 0.
\end{cases}
\end{align*}$$

(A.8)
If $u_t^\tau$ lies in $C^0([-\tau, 0], \mathbb{X})$ and satisfies the compatibility condition $u_t^\tau(0) = u^\tau$, then the strong solution is even a classical solution; that is, $u \in C^0([-\tau, \infty), \mathbb{X}) \cap C^1([0, \infty), \mathbb{X})$.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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