Research Article

Certain Properties of q -Hypergeometric Functions

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The quotients of certain q -hypergeometric functions are presented as g -fractions which converge uniformly in the unit disc. These results lead to the existence of certain q -hypergeometric functions in the class of either q -convex functions, $\mathcal{P}\mathcal{C}_q$, or q -starlike functions $\mathcal{P}\mathcal{S}_q^*$.

1. Introduction and Preliminaries

Let $\mathcal{A}_k$ be the class of analytic functions f in the unit disc $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathcal{U}). \quad (1)$$

In this paper, we extend some results obtained in the theory of functions to the q -theory and to achieve this we write out some standard notations and basic definitions used in this paper.

Definition 1. The q -shift factorial, the multiple q -shift factorial, and the q -binomial coefficients are defined by

$$(a_1, a_2, \dots, a_n; q)_k = \begin{cases} 1, & (k = 0, j = 1, a_1 = a), \\ \prod_{n=0}^{k-1} (1 - aq^n), & (j = 1, k \neq 0, a_1 = a, n \in \mathbb{N}), \\ \prod_{j=1}^n (a_j; q)_n, & (j = 1, 2, \dots, n; n \in \mathbb{N}, k \in \mathbb{Z}), \end{cases}$$

$$\begin{aligned} \begin{bmatrix} a \\ k \end{bmatrix}_q &= \begin{cases} 1, & (k = 0), \\ \frac{(1 - q^a)(1 - q^{a-1}) \cdots (1 - q^{a-k+1})}{(q; q)_k}, & (k \in \mathbb{N}), \end{cases} \end{aligned} \quad (2)$$

where $a, q \in \mathbb{C}$.

In [1, 2] Jackson defined the q -derivative operator $\mathcal{D}_{z,q}$ as follows.

Definition 2. Consider the following:

$$\mathcal{D}_{z,q} f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (z \in \mathbb{C} - \{0\}; 0 < q < 1), \quad (3)$$

$$\mathcal{D}_{z,q} f(z) \Big|_{z=0} = f'(0).$$

From (3), one has

$$\mathcal{D}_{z,q} f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad (z \neq 0), \quad (4)$$

where $[k]_q = (1 - q^k)/(1 - q)$ and as $q \rightarrow 1$, $[k]_q \rightarrow k$.

Definition 2 leads to the n th order q -derivative derived by Sofonea [3] as follows:

$$\begin{aligned} \mathcal{D}_{z,q} (\mathcal{D}_{z,q}^{n-1} f)(z) &= (\mathcal{D}_{z,q}^n f)(z) = (-1)^n (1 - q)^{-n} \\ &\cdot z^{-n} q^{-n(n-1)/2} \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k(k-1)/2} f(q^{n-k} z), \end{aligned} \quad (5)$$

where $n \in \mathbb{N}$, $(\mathcal{D}_{z,q}^0 f)(z)$ denotes the identity operator, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q -binomial coefficients defined in Definition 1.

Definition 3. For complex parameters a_i , b_i , and q with $b_i \in \mathbb{C} \setminus \mathbb{Z}^-$, $i \in \mathbb{N}$, and $|q| < 1$. ${}_r\Phi_s$ denotes the generalized basic (or q -) hypergeometric functions

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q, z \right) \quad (6)$$

$$\begin{aligned} &= {}_r\Phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ &= \begin{bmatrix} a \\ 0 \end{bmatrix}_q + \sum_{k=1}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} z^k (-q^{(k-1)/2})^{k(s+1-r)}. \end{aligned} \quad (7)$$

Let $r = 2$, $s = 1$, $r = s + 1$, $|q| < 1$ with complex parameters $a_1 = a$, $a_2 = b$, and $b_1 = c \in \mathbb{C} \setminus \mathbb{Z}^-$. Then (6) becomes the basic (q -) hypergeometric functions ${}_2\Phi_1$ written as

$${}_2\Phi_1(a, b; c, q, z) = \sum_{k=0}^{\infty} \frac{(a, q)_k (b, q)_k}{(q, q)_k (c, q)_k} z^k, \quad z \in \mathcal{U}, \quad (8)$$

where $(a, q)_k$ is the q -shifted factorial defined in Definition 1.

In [4], Srivastava and Owa summarized some properties of functions that belong to the class $\mathcal{PS}_q^*$ of q -starlike functions in $\mathcal{U}$, introduced and investigated by Ismail et al. in [5]. Srivastava and Owa [4] further proposed the study of properties of functions that belong to the class $\mathcal{PS}_q^*(\alpha)$ of q -starlike functions of order α , $0 \leq \alpha \leq 1$, and also of the functions that belong to the class $\mathcal{PC}_q$ of q -convex functions in $\mathcal{U}$. The authors [4] also defined the class of functions $\mathcal{PC}_q$ on the function $f \in \mathcal{A}_k$, $f(0) = f'(0) - 1 = 0$ as follows:

$$\left| \frac{z \mathcal{D}_{z,q}^2 f(z)}{\mathcal{D}_{z,q} f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (0 < q < 1; z \in \mathcal{U}), \quad (9)$$

while [5] defined the class of functions $\mathcal{PS}_q^*$, on the function $f \in \mathcal{A}_k$, $f(0) = f'(0) - 1 = 0$ as

$$\left| \frac{z \mathcal{D}_{z,q} f(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (0 < q < 1; z \in \mathcal{U}). \quad (10)$$

Meanwhile, Agrawal and Sahoo in [6] defined and studied some properties of functions that belong to the class $\mathcal{PS}_q^*(\alpha)$ and also Sahoo and Sharma in [7] (see also [8]) defined and studied the class $\mathcal{PK}_q$ of q -close-to-convex functions. Kanas and Răducanu in [9] also used the Ruscheweyh q -differential operator to introduce and study some properties of (q, k) uniformly starlike functions of order α . Other works related to q -hypergeometric can also be traced in [10–13].

In addition, in [14] Baricz and Swaminathan used the Alexander duality between starlike and convex function to define the following.

Definition 4. If $\mathcal{PC}_q$ is the class of q -convex functions, then $f \in \mathcal{PC}_q$ if and only if $z \mathcal{D}_{z,q}(f)(z) \in \mathcal{PS}_q^*$. Hence, as $q \rightarrow 1^-$, $z \mathcal{D}_{z,q}(f)(z) \rightarrow z f'(z)$, and $\mathcal{PS}_q^*$ reduces to $\mathcal{S}^*$, the class of starlike functions.

They used Definition 4 to establish Theorem 5 as follows.

Theorem 5. If $0 < a, b, c, q < 1$ satisfies the following two more conditions

- (i) $abq \leq c \leq ac \leq b$,
- (ii) $[(1 - bq)(c - 2a + acq)q - (1 - cq)(1 - q)(abq + c)]^2 \leq (b(1 - q) + (1 - bq))(1 - q)(1 - cq)^2 4acq$,

then

$$\frac{(1 - c)(1 - q)}{(1 - a)(1 - b)} {}_2\Phi_1[a, b; c; q, z] \in \mathcal{PC}_q. \quad (11)$$

Apart from Theorem 5, to the best of our knowledge, other properties of functions in the class of functions $\mathcal{PC}_q$ are yet to be studied. For more results and further studies on q -calculus see [15–17].

Motivated by the numerous studies, of the abovementioned authors, we aimed, in this paper, at using the parameters a, b, c as real parameters and placing some constraints on quotients of two or more hypergeometric functions to establish the following:

- (♦i) $z {}_2\Phi_1[a, b, c; q, z] \in \mathcal{PC}_q$,
- (♦ii) $z {}_2\Phi_1[a, b, c; q, qz] \in \mathcal{PS}_q^*$.

We describe the procedures to achieve (♦i) as follows:

- (i) Calculate necessary and sufficient conditions for a function $f(z) = {}_2\Phi_1[a, b, c; q, z]$ to be in the class $\mathcal{PC}_q$.
- (ii) Calculate suitable, contiguous relations for hypergeometric functions.

- (iii) Use the contiguous relation to derive equation with a quotient of one or more hypergeometric functions. Let the absolute value of the derived quotient, on the left-hand side of the equation, constrain the function ${}_2\Phi_1[a, b, c; q, z]$ to be in $\mathcal{P}\mathcal{C}_q$, while the other side of the equation has continued fraction expansion.
- (iv) Calculate the continued fraction of the hypergeometric quotient on the right-hand side of the equation and convert the continued fractions to a g -fraction which converges uniformly.
- (v) Hence, the g -fractions lead to the geometric properties of Pick functions.
- (vi) Then substituting last result in the third outline with simple calculation on the outcome gives the required results.

The description (i) to (vi) can also be used to establish (♦ii).

First we write out the known results needed to establish ours. Ismail and Libis noted in [18] that the hypergeometric function

$$\psi(z) := {}_2\Phi_1(a, b, c; q, z) \quad (12)$$

satisfies the q -difference equation

$$\begin{aligned} (abz - cq^{-1})\psi(q^2z) - [(a+b)z - 1 - cq^{-1}]\psi(qz) \\ = (1-z)\psi(z). \end{aligned} \quad (13)$$

They rewrite (13) in the form

$$\begin{aligned} \frac{\psi(qz)}{(1-z)\psi(z)} \\ = \frac{1}{[1 - (a+b)z + cq^{-1} + (abz - cq^{-1})(\psi(q^2z)/\psi(qz))]}, \end{aligned} \quad (14)$$

iterated the functional relationship (14), and got

$$\begin{aligned} \frac{\psi(qz)}{(1-z)\psi(z)} = \frac{1}{b_0} - \frac{a_1}{b_1} - \frac{a_2}{b_2} - \dots - \frac{a_k}{b_k} \\ - \frac{(cq^{-1} - abzq^k)\psi(q^{k+2}z)}{\psi(q^{k+1}z)}. \end{aligned} \quad (15)$$

They also noted that if these iterations converge, this will give rise to the continued fraction

$$\frac{\psi(qz)}{(1-z)\psi(z)} = \frac{1}{b_0} - \frac{a_1}{b_1} - \frac{a_2}{b_2} - \dots, \quad (16)$$

where

$$\begin{aligned} a_{k+1} &= (cq^{-1} - abzq^k)(1 - q^{k+1}z), \\ b_k &= 1 + cq^{-1} - (a+b)zq^k, \\ k &\in \mathbb{N} \cup \{0\}. \end{aligned} \quad (17)$$

They also modified a result in [19] to Theorem 6. Later on, we will modify Theorem 6 to suit our results.

Theorem 6 (see [18]). Let $|q| \neq 1$ with exception of the zeros of ${}_2\Phi_1[aq, bq, cq; q, z]$; the continued fraction of a meromorphic function of z , which is equal to the function ${}_2\Phi_1(a, b; c; q, z)/{}_2\Phi_1(aq, bq, cq; q, z)$, is represented by

$$\frac{{}_2\Phi_1(a, b; c; q, z)}{{}_2\Phi_1(aq, bq, cq; q, z)} = e_0 + \frac{e_1}{f_1} + \frac{e_2}{f_2} + \frac{e_3}{f_3} + \dots, \quad (18)$$

throughout the z -plane with

$$\begin{aligned} e_{k+1} &= \frac{(1 - aq^{k+1})(1 - bq^{k+1})(c - abzq^{k+1})zq^k}{(1 - cq^k)(1 - cq^{k+1})}, \\ f_k &= 1 - \frac{[a + b - abq^k(1 + q)]zq^k}{(1 - cq^k)}, \quad k \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (19)$$

It is equal to this ratio in a neighborhood z at origin and furnishes the analytic continuation of it throughout the finite z -plane.

We review that the sequence of positive numbers, $\{e_k\}$, $k \geq 0$, with $e_k = 1$, is called a Hausdorff moment sequence if there exists a positive (Borel) measure ϑ on the close interval $[0, 1]$ such that

$$e_k = \int_0^1 \zeta^k d\vartheta(\zeta), \quad k \in \mathbb{N} \cup \{0\}. \quad (20)$$

Equation (20) can also be represented by

$$\Theta(z) = \sum_{k=0}^{\infty} e_k z^k = \int_0^1 \frac{1}{(1 - \zeta z)} d\vartheta(\zeta), \quad (21)$$

with Θ analytic in the slit complex domain $\mathbb{C} \setminus [1, \infty]$, and also Θ belongs to the set of Pick functions.

Lemma 7 bridges the gaps between total monotone sequence, Hausdorff moment sequence, the set of Pick functions, and g -fractions.

Lemma 7 (see [14, 20]). Let $\{e_k\}$, $k \geq 0$ be a real sequence; then the following are equivalent:

- (i) $\{e_k\}$, $k \geq 0$ is totally monotone sequence; that is, for $C_{i,k} = \binom{i}{k}$, $k \in \mathbb{N} \cup \{0\}$,

$$\Delta^i e_k = e_0 - C_{i,1}e_{1+k} + C_{i,2}e_{2+k} - \dots + (-1)^i e_{i+k} \geq 0, \quad i, k \in \mathbb{N} \cup \{0\}. \quad (22)$$

- (ii) $\{e_k\}$, $k \geq 0$ is a Hausdorff moment sequence; that is, there exists a positive (Borel) measure ϑ on the interval $[0, 1]$ whose sequence of moment is

$$e_k = \int_0^1 \zeta^k d\vartheta(\zeta), \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (23)$$

- (iii) The power series $\sum_{k=0}^{\infty} e_k z^k$ is analytic in $\mathcal{U}$, has the analytic continuation in the complex plane slit $\mathbb{C} \setminus [1, \infty)$, and also has a corresponding g -fraction

$$\frac{g_1}{1} + \frac{(1-g_1)g_2 z_2}{1} + \frac{(1-g_2)g_3 z_3}{1} + \dots$$

$$= \int_0^1 \frac{1}{1-\zeta z} d\vartheta(\zeta), \quad (24)$$

with $0 < g_k, g_{k+1} < 1, k \in \mathbb{N} \cup \{0\}$.

For more studies on Hausdorff moment of sequence and hypergeometric mappings see [20–26] and their references. Theorem 8 displays the constraints on the constant g that made the g -fraction uniformly convergent.

Theorem 8 (see [20]). Letting $g_1, g_2, g_3, \dots$ be constant satisfying one of the following conditions:

- (i) $0 < g_k \leq 1, (k \in \mathbb{N})$,
- (ii) $0 \leq g_k < 1, (k \in \mathbb{N})$,
- (iii) $0 < g_k < 1, (k \in \mathbb{N})$,

then the continued fraction

$$\frac{g_1}{1} + \frac{(1-g_1)g_2 z_2}{1} + \frac{(1-g_2)g_3 z_3}{1} + \dots \quad (25)$$

converges uniformly for $|z_k| \leq 1, (k \in \mathbb{N} \setminus \{1\})$.

We write Remark 9, which is some comments made by Wall in [20]. This remark enables us to convert our originally derived continued fraction into a g -fraction which converges uniformly.

Remark 9. By means of equivalence transformation, the continued fraction of the form

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots \quad (26)$$

can be transformed into continued fraction of the form (25), by first reducing the partial denominator to unity; hence we obtain

$$\frac{a_1/b_1}{1} + \frac{a_2/b_1 b_2}{1} + \frac{a_3/b_2 b_3}{1} + \dots \quad (27)$$

Also (10) converges uniformly if there exist constants g_k, g_{k+1} such that

$$\left| \frac{a_{k+1}}{b_k b_{k+1}} \right| \leq (1-g_k) g_{k+1}, \quad (28)$$

$$0 < g_k, g_{k+1} < 1, k \in \mathbb{N} \cup \{0\}.$$

We state necessary and sufficient condition for a function f to be in the class $\mathcal{PS}_q^*$ established in [5].

Lemma 10 (see [5]). A function $f(z)$ is in the class $\mathcal{PS}_q^*$ if and only if

$$\left| \frac{f(qz)}{f(z)} \right| \leq 1, \quad (f(z) \neq 0, z \in \mathcal{U}). \quad (29)$$

We calculate the necessary and sufficient condition for a function f to be in the class $\mathcal{PE}_q$.

Lemma 11. A function $f(z)$ is in the class $\mathcal{PE}_q$ if and only if

$$\left| \frac{f(q^2 z) - f(qz)}{f(qz) - f(z)} \right| \leq q, \quad (f(qz) \neq f(z), z \in \mathcal{U}). \quad (30)$$

Proof. The proof of Lemma 11 follows from substituting

$$\frac{z(\mathcal{D}_{z,q}^2 f)(z)}{(\mathcal{D}_{z,q} f)(z)} = \frac{1}{q(1-q)} \left\{ q - \frac{f(q^2 z) - f(qz)}{f(qz) - f(z)} \right\}, \quad (31)$$

in (9). $\square$

2. Contiguous Relations and Main Results

In this section we write out some contiguous relations of hypergeometric functions, which can be used to derive a continued fraction expansion, for some quotients of two or more hypergeometric functions. We will also state our results. Suppose

$$f(z) = \psi(z) := {}_2\Phi_1(a, b; c; q, z),$$

$$f(qz) = \psi(qz) := {}_2\Phi_1(a, b; c; q, qz), \quad (32)$$

$$f(q^2 z) = \psi(q^2 z) := {}_2\Phi_1(a, b; c; q, q^2 z).$$

Rearranging (13), we have

$$[cq^{-1} - abz] \frac{[\psi(q^2 z) - \psi(qz)]}{[\psi(qz) - \psi(z)]}$$

$$= 1 - z + \frac{(a-1)(1-b)z\psi(qz)}{\psi(z) - \psi(qz)}. \quad (33)$$

Substituting the contiguous relation,

$$\psi(z) - \psi(qz)$$

$$- \frac{(1-a)(1-b)}{(1-c)} z {}_2\Phi_1[aq, bq; cq; q, z] = 0, \quad (34)$$

calculated in [27] (see also [18]) in (32) gives

$$[cq^{-1} - abz] \frac{[\psi(q^2 z) - \psi(qz)]}{[\psi(qz) - \psi(z)]}$$

$$= 1 - (a+b-ab)z - \frac{(1-c)\psi(z)}{{}_2\Phi_1[aq, bq; cq; q, z]}. \quad (35)$$

Theorem 12. Let a, b, c, q be nonnegative real numbers with $a, b, c, q \in (0, 1)$ and $0 < cq^k < 1$ for all $k \in \mathbb{N} \cup \{0\}$. Then hypergeometric quotient ${}_2\Phi_1(a, b; c; q, z) / {}_2\Phi_1(aq, bq; cq; q, z)$ with exception of the zeros of ${}_2\Phi_1(aq, bq; cq; q, z)$ has the continued fraction

$$\frac{{}_2\Phi_1(a, b; c; q, z)}{{}_2\Phi_1(aq, bq; cq; q, z)} = \rho_0 + \frac{\delta_1}{\rho_1} + \frac{\delta_2}{\rho_2} + \frac{\delta_3}{\rho_3} + \dots + \frac{\delta_{k+1}}{\rho_{k+1}} + \dots, \quad (36)$$

where

$$\delta_{k+1} = \frac{zq^k(1 - aq^{k+1})(1 - bq^{k+1})(c - abzq^{k+1})}{(1 - cq^k)(1 - cq^{k+1})}, \quad (37)$$

$$\rho_k = \frac{[1 - cq^k - zq^k\{a + b - abq^k(1 + q)\}]}{1 - cq^k}. \quad (38)$$

Theorem 13. If the hypothesis of Theorem 12 holds and

- (i) $0 < bq^{k+1} \leq aq^{k+1} < 1$,
- (ii) $[q^k\{1 + q^{k+1}[abq^{k+1} - (a + b)]\}]^2 \leq (1 - cq^k)\{(1 - cq^k) - 2\operatorname{Re}\{z\}q^k[a + b - abq^k(q + 1)] + z\bar{z}[a + b - abq^k(q + 1)]^2\}$,
- (iii) $0 < c[c - 2ab\operatorname{Re}\{z\}q^{k+1}] + z\bar{z}a^2b^2q^{2(k+1)} \leq (1 - cq^{k+1})[(1 - cq^{k+1}) - 2\operatorname{Re}\{z\}q^{k+1}[a + b - abq^{k+1}(1 + q)] + z\bar{z}q^{2(k+1)}[a + b - abq^{k+1}(1 + q)]^2]$,

then the continued fraction (36) can be represented as

$$\frac{{}_2\Phi_1(a, b; c; q, z)}{{}_2\Phi_1(aq, bq; cq; q, z)} = \rho_0 + \frac{g_1}{1} + \frac{(1 - g_1)g_2z_2}{1} + \frac{(1 - g_2)g_3z_3}{1} + \dots, \quad (39)$$

$$(|z_k| \leq 1, \quad k \in \mathbb{N} \setminus \{1\}),$$

where

$$\left| \frac{\rho_{k+1}}{\delta_k \delta_{k+1}} \right| \leq (1 - g_k)g_{k+1}, \quad (40)$$

$$0 < g_k, \quad g_{k+1} < 1, \quad k \in \mathbb{N} \cup \{0\},$$

where δ_k is (37) and ρ_{k+1} is (38). Hence the hypergeometric quotient, ${}_2\Phi_1(a, b; c; q, z) / {}_2\Phi_1(aq, bq; cq; q, z)$, converges uniformly.

Theorem 14. If the hypothesis of Theorem 13 holds and there exists an increasing function, ϑ_k , mapping $[0, 1]$ into itself with $\int_0^1 d\vartheta(\zeta) = 1$, then

$$\frac{{}_2\Phi_1[a, b; c; q, z]}{{}_2\Phi_1[aq, bq; cq; q, z]} = \rho_0 + \int_0^1 \frac{1}{1 - \zeta z} d\vartheta(\zeta) \quad (41)$$

for all $z \in \mathbb{C} \setminus [1, \infty)$. Further,

$$\frac{{}_2\Phi_1[a, b; c; q, z]}{{}_2\Phi_1[aq, bq; cq; q, z]} = z\rho_0 + \int_0^1 \frac{z}{1 - \zeta z} d\vartheta(\zeta), \quad (42)$$

for all $z \in \mathbb{C} \setminus [1, \infty)$.

Theorem 15. If the hypothesis of Theorem 14 and inequality (43) holds,

$$\begin{aligned} & [c(1 + q^2) + q^2(abq - 1) + q(ab - q)]^2 \\ & \leq 4abq[1 + q^2][c + q^2(2c - 3)], \end{aligned} \quad (43)$$

then $z {}_2\Phi_1[a, b; c; q, z] \in \mathcal{PE}_q$.

Theorem 16 (see [18]). Let $a, b, c, q \in (0, 1)$, $z \neq 1$, with nonzeros of ${}_2\Phi_1[a, b; c; q, z]$. Then continued fraction of a meromorphic function of qz , ${}_2\Phi_1(a, b; c; q, qz) / (1 - z) {}_2\Phi_1(a, b; c; q, z)$, is represented by

$$\frac{{}_2\Phi_1(a, b; c; q, qz)}{(1 - z) {}_2\Phi_1(a, b; c; q, z)} = \frac{1}{b_0} - \frac{a_1}{b_1} - \frac{a_2}{b_2} - \dots, \quad (44)$$

where

$$\begin{aligned} a_{k+1} &= (cq^{-1} - abzq^k)(1 - q^{k+1}z), \\ b_k &= 1 + cq^{-1} - (a + b)zq^k, \end{aligned} \quad (45)$$

$$k \in \mathbb{N} \cup \{0\}.$$

Theorem 17. If the hypothesis of Theorem 16 holds and

- (i) $0 < q^2[z\bar{z}q^{2(k+1)} + 1 - 2\operatorname{Re}\{z\}q^{k+1}] < (q + c)\{q + c - \operatorname{Re}\{z\}(a + b)q^{k+1} + z\bar{z}(a + b)^2q^{2(k+1)}\}$,
- (ii) $0 < z\bar{z}a^2b^2q^{2(k+1)} + c\{c - 2ab\operatorname{Re}\{z\}q^{k+1}\} < (q + c)\{q + c - \operatorname{Re}\{z\}(a + b)q^{k+2} + z\bar{z}(a + b)^2q^{2(k+2)}\}$,

then

$$\begin{aligned} & \frac{{}_2\Phi_1(a, b; c; q, qz)}{(1 - z) {}_2\Phi_1(a, b; c; q, z)} \\ &= \frac{g_{2,1}}{1} + \frac{(1 - g_{2,1})g_{2,2}z_2}{1} + \frac{(1 - g_{2,2})g_{2,3}z_3}{1} \\ &+ \dots, \quad (|z_k| \leq 1, \quad k \in \mathbb{N} \setminus \{1\}) \end{aligned} \quad (46)$$

with

$$\begin{aligned} & \left| \frac{a_{k+1}}{b_k b_{k+1}} \right| \leq (1 - g_{2,k})g_{2,k+1}, \\ & 0 < g_{2,k}, \quad g_{2,k+1} < 1, \quad k \in \mathbb{N} \cup \{0\}, \quad g_{2,0} = 0, \end{aligned} \quad (47)$$

where a_{k+1}, b_k are as in (45). Hence ${}_2\Phi_1(a, b; c; q, qz) / (1 - z) {}_2\Phi_1(a, b; c; q, z)$ converges uniformly.

Theorem 18. If the hypothesis of Theorem 17 holds and there exists an increasing function, ϑ_k , mapping $[0, 1]$ into itself, with $\int_0^1 d\vartheta(\zeta) = 1$, then

$$\frac{{}_2\Phi_1(a, b; c; q, qz)}{(1-z){}_2\Phi_1(a, b; c; q, z)} = \int_0^1 \frac{1}{1-\zeta z} d\vartheta(\zeta) \quad (48)$$

for all $z \in \mathbb{C} \setminus [1, \infty)$. Also,

$$\frac{z{}_2\Phi_1(a, b; c; q, qz)}{(1-z){}_2\Phi_1(a, b; c; q, z)} = \int_0^z \frac{z}{1-\zeta z} d\vartheta(\zeta). \quad (49)$$

Theorem 19. If the hypothesis of Theorem 18 holds, then

$$z{}_2\Phi_1[a, b; c; q, qz] \in \mathcal{PS}_q^*. \quad (50)$$

3. Proof of the Main Results

Proof of Theorem 12. To calculate the continued fraction of ${}_2\Phi_1[a, b; c; q, z]/{}_2\Phi_1[aq, bq; cq; q, z]$, we let

$$\begin{aligned} \varphi_0 &= {}_2\Phi_1[a, b; c; q, z], \\ \varphi_1 &= {}_2\Phi_1[aq, bq; cq; q, z], \\ \varphi_2 &= {}_2\Phi_1[aq^2, bq^2; cq^2; q, z], \\ &\vdots \\ \varphi_k &= {}_2\Phi_1[aq^k, bq^k; cq^k; q, z], \\ &\quad (k \in \mathbb{N} \cup \{0\}). \end{aligned} \quad (51)$$

We use the following

$$\begin{aligned} &(1-c)(1-cq)\varphi_0 \\ &\quad - (1-cq)[1-c-z\{a+b-ab(1+q)\}]\varphi_1 \\ &= z(1-aq)(1-bq)(c-abzq)\varphi_2, \end{aligned} \quad (52)$$

stated in [18], to calculate the continued fraction of φ_0/φ_1 . Solving (52) gives

$$\frac{\varphi_0}{\varphi_1} = \delta_1 \frac{\varphi_2}{\varphi_1} + \rho_0 = \rho_0 + \frac{\delta_1}{\varphi_1/\varphi_2}, \quad (53)$$

where

$$\delta_1 = \frac{z(1-aq)(1-bq)(c-abzq)}{(1-c)(1-cq)},$$

$$\rho_0 = \frac{[1-c-z\{a+b-ab(1+q)\}]}{1-c}. \quad (54)$$

To calculate φ_1/φ_2 , we set

$$\begin{aligned} &(1-cq)(1-cq^2)\varphi_1 \\ &\quad - (1-cq^2)[1-cq-zq\{a+b-abq(1+q)\}]\varphi_2 \\ &= zq(1-aq^2)(1-bq^2)(c-abzq^2)\varphi_3. \end{aligned} \quad (55)$$

Solving (55), we obtain

$$\frac{\varphi_1}{\varphi_2} = \rho_1 + \delta_2 \frac{\varphi_3}{\varphi_2} = \rho_1 + \delta_2 \frac{\varphi_3}{\varphi_2} = \rho_1 + \frac{\delta_2}{\varphi_2/\varphi_3}, \quad (56)$$

where

$$\begin{aligned} \rho_1 &= \frac{[1-cq-zq\{a+b-abq(1+q)\}]}{1-cq}, \\ \delta_2 &= \frac{zq(1-aq^2)(1-bq^2)(c-abzq^2)}{(1-cq)(1-cq^2)}. \end{aligned} \quad (57)$$

To calculate φ_i/φ_{i+1} , we set

$$\begin{aligned} &(1-cq^i)(1-cq^{i+1})\varphi_i - (1-cq^{i+1}) \\ &\quad \cdot [1-cq^i-zq^i\{a+b-abq^i(1+q)\}]\varphi_{i+1} \\ &= zq^i(1-aq^{1+i})(1-bq^{1+i})(c-abzq^{i+1})\varphi_{i+2}. \end{aligned} \quad (58)$$

Assume $i = k$; we rewrite (58) as

$$\frac{\varphi_k}{\varphi_{k+1}} = \delta_{k+1} \frac{\varphi_{k+2}}{\varphi_{k+1}} + \rho_k = \rho_k + \frac{\delta_{k+1}}{\varphi_{k+1}/\varphi_{k+2}}, \quad (59)$$

$(k \in \mathbb{N} \cup \{0\}),$

where δ_{k+1} is (37) and ρ_k is (38). Substituting (56)–(59) in (53), gives the continued fraction (36). $\square$

Proof of Theorem 13. To prove Theorem 13, we first reduce the partial denominator of the continued fraction (36) to unity, by setting

$$\left| \frac{\delta_{k+1}}{\rho_k \rho_{k+1}} \right| \leq (1-\mathfrak{s}) \sqrt{\frac{T_k}{Q_k}} \leq (1-g_k) g_{k+1}, \quad (60)$$

$$k \in \mathbb{N} \cup \{0\},$$

where

$$\mathfrak{s} = g_k = 1 - \frac{q^k \{1 + q^{k+1} [abq^{k+1} - (a+b)]\}}{\sqrt{(1 - cq^k) \{(1 - cq^k) - 2\operatorname{Re}\{z\} q^k [a + b - abq^k (q+1)] + z\bar{z} [a + b - abq^k (q+1)]^2\}}}, \quad (61)$$

$$g_{k+1} = +\sqrt{\frac{T_k}{Q_k}}, \quad 0 < T_k < Q_k,$$

with

$$T_k = c [c - 2ab\operatorname{Re}\{z\} q^{k+1}] + z\bar{z} a^2 b^2 q^{2(k+1)}, \quad (62)$$

and also

$$\begin{aligned} Q_k &= (1 - cq^{k+1}) [(1 - cq^{k+1}) \\ &\quad - 2\operatorname{Re}\{z\} q^{k+1} [a + b - abq^{k+1} (1 + q)]] \\ &\quad + z\bar{z} q^{2(k+1)} [a + b - abq^{k+1} (1 + q)]^2. \end{aligned} \quad (63)$$

Moreover, (36) can be rewritten as

$$\rho_0 + \frac{\delta_1/\rho_1}{1} + \frac{\delta_2/\rho_1\rho_2}{1} + \frac{\delta_3/\rho_2\rho_3}{1} + \dots \quad (64)$$

Assuming $g_0 = 0$, we transform (64) by replacing $|\delta_{k+1}/\rho_k\rho_{k+1}|$ with $(1 - g_k)g_{k+1}$, $i, k \in \mathbb{N} \cup \{0\}$, and obtain

$$\frac{{}_2\Phi_1(a, b; c; q, z)}{{}_2\Phi_1(aq, bq; cq; q, z)} = \rho_0 + \frac{g_1}{1} + \frac{(1 - g_1)g_2}{1} + \dots \quad (65)$$

Since $|z_k| \leq 1$, ($k \in \mathbb{N} \setminus \{1\}$) and $0 < g_k, g_{k+1} < 1$ (65) can be written as

$$\begin{aligned} \frac{{}_2\Phi_1(a, b; c; q, z)}{{}_2\Phi_1(aq, bq; cq; q, z)} &= \rho_0 + \frac{g_1}{1} + \frac{(1 - g_1)g_2z_2}{1} \\ &\quad + \dots \end{aligned} \quad (66)$$

Hence, by Remark 9, the g -fraction of ${}_2\Phi_1(a, b; c; q, z)/{}_2\Phi_1(aq, bq; cq; q, z)$ converges uniformly in $\mathcal{U}$. $\square$

Remark 20. The addition of the leading constant ρ_0 does not affect the convergence of the continued fraction ${}_2\Phi_1(a, b; c; q, z)/{}_2\Phi_1(aq, bq; cq; q, z)$. The continued fraction on the right-hand side of (65) converges to the left-hand side provided (a, b, c) belongs to the neighborhood of $(0, 0, 0)$ with $z \in \mathcal{U}$ and z is not a pole of the right-hand side. We also note that the continued fraction (65) is called the infinite g -fraction.

Proof of Theorem 14. By Lemma 7(iii), the coefficients in the power series expansion about $z = 0$, of the analytic function, on the left-hand side of the continued fraction (65), are Hausdorff moments of an increasing function, in the open interval $(0, 1)$ with infinitely many points of increase (with

total increase less than or equal to 1). Hence, there exists a function ϑ_0 mapping $[0, 1]$ to itself, satisfying $0 = \vartheta_0(0) \leq \vartheta_0(\xi) \leq \vartheta_0(\zeta) \leq \vartheta_0(1)$ for $0 < \xi < \zeta < 1$ and its range contains infinitely many points such that

$$\frac{{}_2\Phi_1(a, b; c; q, z)}{{}_2\Phi_1(aq, bq; cq; q, z)} = \rho_0 + \int_0^1 \frac{1}{1 - \zeta z} d\vartheta_0(\zeta), \quad (67)$$

$\{z \in \mathbb{C} \setminus [1, \infty)\},$

by analytic continuation, with $\vartheta_0(\zeta) = \vartheta_0(a, b; c; q, \zeta)$.

Furthermore, since $\int_0^1 \vartheta(\zeta) = 1$,

$$\frac{z {}_2\Phi_1(a, b; c; q, z)}{{}_2\Phi_1(aq, bq; cq; q, z)} = z\rho_0 + \int_0^1 \frac{z}{1 - \zeta z} d\vartheta_0(\zeta), \quad (68)$$

for all $z \in \mathbb{C} \setminus [1, \infty)$. $\square$

Proof of Theorem 15. By (35),

$$\begin{aligned} \frac{z [f(q^2z) - f(qz)]}{[f(qz) - f(z)]} &= \frac{q}{c - abqz} \left\{ z \right. \\ &\quad \left. - (a + b - ab)z^2 - (1 - c)z\rho_0 - \frac{(1 - c)z\varphi_0}{\varphi_1} \right\}. \end{aligned} \quad (69)$$

By Theorem 14, (69) can be written as

$$\begin{aligned} \frac{z [f(q^2z) - f(qz)]}{[f(qz) - f(z)]} &= \frac{q^2}{c - abqz} \left\{ z \right. \\ &\quad \left. - (a + b - ab)z^2 + (c - 1)z\rho_0 \right. \\ &\quad \left. + (c - 1) \int_0^1 \frac{z}{1 - \zeta z} d\vartheta(\zeta) \right\}. \end{aligned} \quad (70)$$

For $z\varphi_0$ to be in $\mathcal{PE}_q$, we show that

$$\begin{aligned} \left| \frac{q^2}{c - abqz} \left\{ 1 - (a + b - ab)z - (1 - c)\rho_0 \right. \right. \\ \left. \left. - (1 - c) \int_0^1 \frac{1}{1 - \zeta z} d\vartheta(\zeta) \right\} \right| \leq 1, \end{aligned} \quad (71)$$

where $\rho_0 = [1 - c - z\{a + b - ab(1 + q)\}]/(1 - c)$.

Applying the triangle inequality to the left-hand side of (71), with $|z| \leq r < 1$, we obtain

$$\begin{aligned} & (1-r) + (a+b-ab)r(1-r) \\ & + (1-r)[1-c-r\{a+b-ab(1+q)\}] \\ & + (1-c) \leq \frac{(c-abqr)(1-r)}{q^2}, \end{aligned} \quad (72)$$

$$\alpha r^2 + \beta r + \gamma \geq 0,$$

where

$$\begin{aligned} \alpha &= abq[1+q^2], \\ \beta &= -[c(1+q^2) + q^2(abq-1) + q(ab-q)] \\ \gamma &= c + q^2(2c-3). \end{aligned} \quad (73)$$

By hypothesis of Theorem 15 and $\alpha > 0$, we need to show that $4\alpha\gamma \geq \beta^2$. Hence, Theorem 15 is established. $\square$

Proof of Theorem 16. The calculations in Theorem 16 were established in [18]. $\square$

Proof of Theorem 17. We then need to reduce the partial denominators of (44) to unity, by setting

$$\left| \frac{a_k}{b_k b_{k+1}} \right| = q \cdot \sqrt{\frac{\bar{\omega}_k}{\varrho_k}} \sqrt{\frac{l_k}{\varrho_{k+1}}} \leq (1-g_{2,k})g_{2,k+1}, \quad (74)$$

where

$$\begin{aligned} \bar{\omega}_k &= z\bar{z}q^{2(k+1)} + 1 - 2\operatorname{Re}\{z\}q^{k+1}, \\ \varrho_k &= (q+c)\{q+c-\operatorname{Re}\{z\}(a+b)q^{k+1}\} \\ &+ z\bar{z}(a+b)^2q^{2(k+1)}, \end{aligned} \quad (75)$$

with

$$\begin{aligned} g_{2,k} &= 1 - q \cdot \sqrt{\frac{\bar{\omega}_k}{\varrho_k}}, \\ l_k &= z\bar{z}a^2b^2q^{2(k+1)} + c\{c-2ab\operatorname{Re}\{z\}q^{k+1}\}, \end{aligned} \quad (76)$$

$$\begin{aligned} \varrho_{k+1} &= (q+c)\{q+c-\operatorname{Re}\{z\}(a+b)q^{k+2}\} \\ &+ z\bar{z}(a+b)^2q^{2(k+2)}, \end{aligned}$$

with

$$g_{2,k+1} = \sqrt{\frac{l_k}{\varrho_{k+1}}}. \quad (77)$$

By Remark 9, (44) can be written as

$$\frac{\psi(qz)}{(1-z)\psi(z)} = \frac{1/b_0}{1} - \frac{a_1/b_0b_1}{1} - \frac{a_2/b_1b_2}{1} - \dots \quad (78)$$

Hence (78) can be written as

$$\frac{\psi(qz)}{(1-z)\psi(z)} = \frac{g_{2,1}}{1} + \frac{(1-g_{2,1})g_{2,2}z_2}{1} + \dots \quad (79)$$

$\square$

The proofs of Theorems 18 and 19 are the same procedures as those of Theorems 14 and 15 and hence are omitted in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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