Research Article

Explicit Formulas for Meixner Polynomials

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Using notions of composita and composition of generating functions, we show an easy way to obtain explicit formulas for some current polynomials. Particularly, we consider the Meixner polynomials of the first and second kinds.

1. Introduction

There are many authors who have studied polynomials and their properties (see [1–10]). The polynomials are applied in many areas of mathematics, for instance, continued fractions, operator theory, analytic functions, interpolation, approximation theory, numerical analysis, electrostatics, statistical quantum mechanics, special functions, number theory, combinatorics, stochastic processes, sorting, and data compression.

The research area of obtaining explicit formulas for polynomials has received much attention from Srivastava [11, 12], Cenkci [13], Boyadzhiev [14], and Kruchinin [15–17].

The main purpose of this paper is to obtain explicit formulas for the Meixner polynomials of the first and second kinds.

In this paper we use a method based on a notion of composita, which was presented in [18].

Definition 1. Suppose $F(t) = \sum_{n>0} f(n)t^n$ is the generating function, in which there is no free term $f(0) = 0$. From this generating function we can write the following condition:

$$[F(t)]^k = \sum_{n>0} F(n, k) t^n. \quad (1)$$

The expression $F(n, k)$ is *composita* [19] and it is denoted by $F^\Delta(n, k)$. Below we show some required rules and operations with compositae.

Theorem 2. Suppose $F(t) = \sum_{n>0} f(n)t^n$ is the generating function, and $F^\Delta(n, k)$ is composita of $F(t)$, and α is constant. For the generating function $A(t) = \alpha F(t)$ composita is equal to

$$A^\Delta(n, k) = \alpha^k F^\Delta(n, k). \quad (2)$$

Theorem 3. Suppose $F(t) = \sum_{n>0} f(n)t^n$ is the generating function, and $F^\Delta(n, k)$ is composita of $F(t)$, and α is constant. For the generating function $A(t) = F(\alpha t)$ composita is equal to

$$A^\Delta(n, k) = \alpha^n F^\Delta(n, k). \quad (3)$$

Theorem 4. Suppose $F(t) = \sum_{n>0} f(n)t^n$, $R(t) = \sum_{n>0} r(n)t^n$ are generating functions, and $F^\Delta(n, k)$, $R^\Delta(n, k)$ are their compositae. Then for the composition of generating functions $A(t) = R(F(t))$ composita is equal to

$$A^\Delta(n, k) = \sum_{m=k}^n F^\Delta(n, m) G^\Delta(m, k). \quad (4)$$

Theorem 5. Suppose $F(t) = \sum_{n>0} f(n)t^n$, $R(t) = \sum_{n\geq 0} r(n)t^n$ are generating functions, and $F^\Delta(n, k)$ is composita of $F(t)$. Then for the composition of generating functions $A(t) = R(F(t))$ coefficients of generating functions $A(t) = \sum_{n\geq 0} a(n)t^n$ are

$$a(n) = \sum_{k=1}^n F^\Delta(n, k) r(k), \quad (5)$$

$$a(0) = r(0).$$

Theorem 6. Suppose $F(t) = \sum_{n \geq 0} f(n)t^n$, $R(t) = \sum_{n \geq 0} r(n)t^n$ are generating functions. Then for the product of generating functions $A(t) = F(t)R(t)$ coefficients of generating functions $A(t) = \sum_{n \geq 0} a(n)t^n$ are

$$a(n) = \sum_{k=0}^n f(k) r(n-k). \quad (6)$$

In this paper we consider an application of this mathematical tool for the Bessel polynomials and the Meixner polynomials of the first and second kinds.

2. Bessel Polynomials

Krall and Frink [20] considered a new class of polynomials. Since the polynomials connected with the Bessel function, they called them the Bessel polynomials. The explicit formula for the Bessel polynomials is

$$y_n(x) = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k. \quad (7)$$

Then Carlitz [21] defined a class of polynomials associated with the Bessel polynomials by

$$p_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right). \quad (8)$$

The $p_n(x)$ is defined by the explicit formula [22]

$$p_n(x) = \sum_{k=1}^n \frac{(2n-k-1)!}{(k-1)!(n-k)!} \left(\frac{1}{2}\right)^{n-k} x^k \quad (9)$$

and by the following generating function:

$$\sum_{n=0}^{\infty} \frac{p_n(x)}{n!} t^n = e^{x(1-\sqrt{1-2t})}. \quad (10)$$

Using the notion of composita, we can obtain an explicit formula (9) from the generating function (10).

For the generating function $C(t) = (1 - \sqrt{1-4t})/2$ composita is given as follows (see [19]):

$$C^\Delta(n, k) = \frac{k}{n} \binom{2n-k-1}{n-1}. \quad (11)$$

We represent $e^{x(1-\sqrt{1-2t})}$ as the composition of generating functions $A(B(t))$, where

$$A(t) = x(1 - \sqrt{1-2t}) = 2xC\left(\frac{t}{2}\right), \quad (12)$$

$$B(t) = e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n.$$

Then, using rules (2) and (3), we obtain composita of $A(t)$:

$$A^\Delta(n, k) = (2x)^k \left(\frac{1}{2}\right)^n C^\Delta(n, k) = 2^{k-n} x^k \frac{k}{n} \binom{2n-k-1}{n-1}. \quad (13)$$

Coefficients of the generating function $B(t) = \sum_{n=0}^{\infty} b(n)t^n$ are equal to

$$b(n) = \frac{1}{n!}. \quad (14)$$

Then, using (5), we obtain the explicit formula

$$p_n(x) = n! \sum_{k=1}^n A^\Delta(n, k) b(k) = \sum_{k=1}^n \frac{(2n-k-1)!}{(k-1)!(n-k)!} 2^{k-n} x^k, \quad (15)$$

which coincides with the explicit formula (9).

3. Meixner Polynomials of the First Kind

The Meixner polynomials of the first kind are defined by the following recurrence relation [23, 24]:

$$m_{n+1}(x; \beta, c) = \frac{(c-1)^n}{c^n} \cdot ((x-b_n)m_n(x; \beta, c) - \lambda_n m_{n-1}(x; \beta, c)), \quad (16)$$

where

$$\begin{aligned} m_0(x; \beta, c) &= 1, \\ m_1(x; \beta, c) &= x - b_0, \\ b_n &= \frac{(1+c)n + \beta c}{1-c}, \\ \lambda_n &= \frac{cn(n + \beta - 1)}{(1-c)^2}. \end{aligned} \quad (17)$$

Using (16), we obtain the first few Meixner polynomials of the first kind:

$$\begin{aligned} m_0(x; \beta, c) &= 1; \\ m_1(x; \beta, c) &= \frac{x(c-1) + \beta c}{c}; \\ m_2(x; \beta, c) &= \frac{x^2(c-1)^2 + x((2\beta+1)c^2 - 2\beta c - 1) + (\beta+1)\beta c^2}{c^2}. \end{aligned} \quad (18)$$

The Meixner polynomials of the first kind are defined by the following generating function [22]:

$$\sum_{n=0}^{\infty} \frac{m_n(x; \beta, c)}{n!} t^n = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}. \quad (19)$$

Using the notion of composita, we can obtain an explicit formula $m_n(x; \beta, c)$ from the generating function (19).

First, we represent the generating function (19) as a product of generating functions $A(t) B(t)$, where the functions $A(t)$ and $B(t)$ are expanded by binomial theorem:

$$A(t) = \left(1 - \frac{t}{c}\right)^x = \sum_{n=0}^{\infty} \binom{x}{n} (-1)^n c^{-n} t^n, \quad (20)$$

$$B(t) = (1-t)^{-x-\beta} = \sum_{n=0}^{\infty} \binom{-x-\beta}{n} (-1)^n t^n.$$

Coefficients of the generating functions $A(t) = \sum_{n=0}^{\infty} a(n)t^n$ and $B(t) = \sum_{n=0}^{\infty} b(n)t^n$ are, respectively, given as follows:

$$a(n) = \binom{x}{n} (-1)^n c^{-n}, \quad (21)$$

$$b(n) = \binom{-x-\beta}{n} (-1)^n.$$

Then, using (6), we obtain a new explicit formula for the Meixner polynomials of the first kind:

$$\begin{aligned} m_n(x; \beta, c) &= n! \sum_{k=0}^n a(k) b(n-k) \\ &= (-1)^n n! \sum_{k=0}^n \binom{x}{k} \binom{-x-\beta}{n-k} c^{-k}. \end{aligned} \quad (22)$$

4. Meixner Polynomials of the Second Kind

The Meixner polynomials of the second kind are defined by the following recurrence relation [23, 24]:

$$M_{n+1}(x; \delta, \eta) = (x - b_n) M_n(x; \delta, \eta) - \lambda_n M_{n-1}(x; \delta, \eta), \quad (23)$$

where

$$\begin{aligned} M_0(x; \delta, \eta) &= 1, \\ M_1(x; \delta, \eta) &= x - b_0, \\ b_n &= (2n + \eta) \delta, \\ \lambda_n &= (\delta^2 + 1) n(n + \eta - 1). \end{aligned} \quad (24)$$

Using (23), we get the first few Meixner polynomials of the second kind:

$$\begin{aligned} M_0(x; \delta, \eta) &= 1; \\ M_1(x; \delta, \eta) &= x - \delta\eta; \\ M_2(x; \delta, \eta) &= x^2 - x2\delta(1 + \eta) + \eta((\eta + 1)\delta^2 - 1). \end{aligned} \quad (25)$$

The Meixner polynomials of the second kind are defined by the following generating function [22]:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{M_n(x; \delta, \eta)}{n!} t^n &= ((1 + \delta t)^2 + t^2)^{-\eta/2} \\ &\cdot \exp\left(x \tan^{-1}\left(\frac{t}{1 + \delta t}\right)\right). \end{aligned} \quad (26)$$

Using the notion of composita, we can obtain an explicit formula $M_n(x; \delta, \eta)$ from the generating function (26).

We represent the generating function (26) as a product of generating functions $A(t) B(t)$, where

$$\begin{aligned} A(t) &= ((1 + \delta t)^2 + t^2)^{-\eta/2} \\ &= (1 + 2\delta t + (\delta^2 + 1)t^2)^{-\eta/2}, \\ B(t) &= \exp\left(x \tan^{-1}\left(\frac{t}{1 + \delta t}\right)\right). \end{aligned} \quad (27)$$

Next we represent $A(t)$ as a composition of generating functions $A_1(A_2(t))$ and we expand $A_1(t)$ by binomial theorem:

$$A_1(t) = (1 + t)^{-\eta/2} = \sum_{n=0}^{\infty} \binom{-\eta/2}{n} t^n, \quad (28)$$

$$A_2(t) = 2\delta t + (\delta^2 + 1)t^2.$$

Coefficients of generating function $A_1(t) = \sum_{n=0}^{\infty} a_1(n)t^n$ are

$$a_1(n) = \binom{-\eta/2}{n}. \quad (29)$$

The composita for the generating function $ax + bx^2$ is given as follows (see [15]):

$$a^{2k-n} b^{n-k} \binom{k}{n-k}. \quad (30)$$

Then composita of the generating function $A_2(t)$ equals

$$A_2^\Delta(n, k) = (2\delta)^{2k-n} (\delta^2 + 1)^{n-k} \binom{k}{n-k}. \quad (31)$$

Using (5), we obtain coefficients of the generating function $A(t) = \sum_{n=0}^{\infty} a(n)t^n$:

$$\begin{aligned} a(0) &= a_1(0), \\ a(n) &= \sum_{k=1}^n A_2^\Delta(n, k) a_1(k). \end{aligned} \quad (32)$$

Therefore, we get the following expression:

$$\begin{aligned} a(0) &= 1, \\ a(n) &= \sum_{k=1}^n (2\delta)^{2k-n} (\delta^2 + 1)^{n-k} \binom{k}{n-k} \binom{-\eta/2}{k}. \end{aligned} \quad (33)$$

Next we represent $B(t)$ as a composition of generating functions $B_1(B_2(t))$, where

$$\begin{aligned} B_1(t) &= e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n, \\ B_2(t) &= x \tan^{-1}\left(\frac{t}{1 + \delta t}\right). \end{aligned} \quad (34)$$

We also represent $B_2(t)$ as a composition of generating functions $x C_1(C_2(t))$, where

$$\begin{aligned} C_1(t) &= \tan^{-1}(t), \\ C_2(t) &= \frac{t}{1 + \delta t}. \end{aligned} \quad (35)$$

The composita for the generating function $\tan^{-1}(t)$ is given as follows (see [19]):

$$\left((-1)^{(3n+k)/2} + (-1)^{(n-k)/2} \right) \frac{k!}{2^{k+1}} \sum_{j=k}^n \frac{2^j}{j!} \binom{n-1}{j-1} \left[\begin{matrix} j \\ k \end{matrix} \right], \quad (36)$$

where $\left[\begin{matrix} j \\ k \end{matrix} \right]$ is the Stirling number of the first kind.

Then composita of generating function $C_1(t)$ equals

$$\begin{aligned} C_1^\Delta(n, k) &= \left((-1)^{(3n+k)/2} + (-1)^{(n-k)/2} \right) \frac{k!}{2^{k+1}} \\ &\cdot \sum_{j=k}^n \frac{2^j}{j!} \binom{n-1}{j-1} \left[\begin{matrix} j \\ k \end{matrix} \right]. \end{aligned} \quad (37)$$

The composita for the generating function $at/(1-bt)$ is given as follows (see [19]):

$$\binom{n-1}{k-1} a^k b^{n-k}. \quad (38)$$

Therefore, composita of generating function $C_2(t)$ is equal to

$$C_2^\Delta(n, k) = \binom{n-1}{k-1} (-\delta)^{n-k}. \quad (39)$$

Using rules (4) and (2), we obtain composita of generating function $B_2(t)$:

$$B_2^\Delta(n, k) = x^k \sum_{m=k}^n C_2^\Delta(n, m) C_1^\Delta(m, k). \quad (40)$$

Coefficients of generating function $B_1(t) = \sum_{n=0}^{\infty} b_1(n)t^n$ are defined by

$$b_1(n) = \frac{1}{n!}. \quad (41)$$

Then, using rule (5), we obtain coefficients of generating function $B(t) = \sum_{n=0}^{\infty} b(n)t^n$:

$$\begin{aligned} b(0) &= b_1(0), \\ b(n) &= \sum_{k=1}^n B_2^\Delta(n, k) b_1(k). \end{aligned} \quad (42)$$

After some transformations, we obtain the following expression:

$$\begin{aligned} b(0) &= 1, \\ b(n) &= \sum_{k=1}^n \frac{x^k}{n} \sum_{m=k}^n \binom{n}{m} \delta^{n-m} \left(\frac{1 + (-1)^{m+k}}{(-1)^{(m+k)/2-n}} \right) \\ &\cdot \sum_{j=k}^m \frac{2^{j-k-1}}{(j-1)!} \binom{m}{j} \left[\begin{matrix} j \\ k \end{matrix} \right]. \end{aligned} \quad (43)$$

Therefore, using (6), we obtain a new explicit formula for the Meixner polynomials of the second kind:

$$M_n(x; \delta, \eta) = n! \sum_{k=0}^n a(k) b(n-k). \quad (44)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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