Research Article

On $(k, kn - k^2 - 2k - 1)$ -Choosability of n -Vertex Graphs

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A (k, t) -list assignment L of a graph G is a mapping which assigns a set of size k to each vertex v of G and $|\bigcup_{v \in V(G)} L(v)| = t$. A graph G is (k, t) -choosable if G has a proper coloring f such that $f(v) \in L(v)$ for each (k, t) -list assignment L . In 2011, Charoenpanitseri et al. gave a characterization of (k, t) -choosability of n -vertex graphs when $t \geq kn - k^2 - 2k + 1$ and left open problems when $t \leq kn - k^2 - 2k$. Recently, Ruksasakchai and Nakprasit obtain the results when $t = kn - k^2 - 2k$. In this paper, we extend the results to case $t = kn - k^2 - 2k - 1$.

1. Introduction

A graph G is called k -colorable if every vertex of G can be labeled by at most k colors and every adjacent vertex receives distinct colors. The smallest number t such that G is t -colorable is called the *chromatic number* of G , denoted by $\chi(G)$. A k -list assignment L of a graph G is a function which assigns a set of size k to each vertex v of G . A (k, t) -list assignment of a graph is a k -list assignment with $|\bigcup_{v \in V(G)} L(v)| = t$. Given a list assignment L , a proper coloring f of G is an L -coloring of G if $f(v)$ is chosen from $L(v)$ for each vertex v of G . A graph G is L -colorable if G has an L -coloring. Particularly, if L is a (k, k) -list assignment of G , then any L -coloring of G is a k -coloring of G . A graph G is (k, t) -choosable if G is L -colorable for every (k, t) -list assignment L . If a graph G is (k, t) -choosable for each positive number t then G is called k -choosable and the smallest number k satisfying this property is called the *list chromatic number* of G denoted by $\text{ch}(G)$.

List coloring is a well-known problem in the field of graph theory. It was first studied by Vizing [1] and by Erdős et al. [2]. They gave a characterization of 2-choosable graphs. For $k \geq 3$, there is no characterization of k -choosable graphs. There are only results for some classes of graphs. For example, all planar graphs are 5-choosable, while some planar graphs are 3-choosable. (See [3–9].) In order to simplify the problem, (k, t) -choosability is defined. It is a partial problem of k -choosability. Instead of proving that a graph can always be

colored for entire k -list assignments, we prove the graph can be colored for k -list assignments that have exactly t colors. In 2011, (k, t) -choosability of graphs was explored in [10]. They proved the following theorem.

Theorem 1 (see [10]). *For an n -vertex graph G , if $t \geq kn - k^2 + 1$, then G is (k, t) -choosable.*

Moreover, they showed that the bound is best possible by proving if $t \leq kn - k^2$, then an n -vertex graph containing K_{k+1} is not (k, t) -choosable. Furthermore, they keep investigating the (k, t) -choosability to obtain another interesting theorem.

Theorem 2 (see [10]). *Let $k \geq 3$. A K_{k+1} -free graph with n vertices is (k, t) -choosable for $t \geq kn - k^2 - 2k + 1$.*

Again, they showed that the bound is best possible for K_{k+1} -free graphs with n vertices by proving an n -vertex graph containing $C_5 \vee K_{k-2}$ is not (k, t) -choosable for $t \leq kn - k^2 - 2k$. In conclusion, they gave a characterization of (k, t) -choosability of n -vertex graphs when $t \geq kn - k^2 - 2k + 1$.

In 2013, Ruksasakchai and Nakprasit gave a characterization of $(k, kn - k^2 - 2k)$ -choosability of n -vertex graphs as shown in the following theorem.

Theorem 3 (see [11]). *Let G be a graph with n vertices and $k \geq 3$. If G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k, t) -choosable for $t = kn - k^2 - 2k$.*

Results on $(2, t)$ -choosability of n -vertex graphs are almost completed by [12, 13]. Here, we focus on $k \geq 3$. We will prove that if an n -vertex graph G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k, t) -choosable for $t = kn - k^2 - 2k - 1$.

2. Preliminaries

Throughout the paper, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of G . For $X \subseteq V(G)$, $G - X$ is the graph obtained from deleting all vertices of X from G . In case $X = \{v\}$, we write $G - v$ instead of $G - \{v\}$. The *subgraph induced by X* , denoted by $G[X]$, is the graph obtained from deleting all vertices of $V(G)$ outside X . The notation $d(v)$ stands for the degree of v in G . For a subgraph H of G , $d_H(v)$ stands for the degree of v in H .

Let $S \subseteq V(G)$. If L is a list assignment of G , we let $L|_S$ denote L restricted to S and $L(S)$ denote $\bigcup_{v \in S} L(v)$. For a color set A , let $L - A$ be the new list assignment obtained from L by deleting all colors in A from $L(v)$ for each $v \in V(G)$. When A has only one color a , we write $L - a$ instead of $L - \{a\}$.

Example 4. The cycle C_n is $(2, t)$ -choosable unless n is odd and $t = 2$.

Note that a graph G is $(2, 2)$ -choosable if and only if G is 2-colorable. Hence, C_n is $(2, 2)$ -choosable if and only if n is even. It remains to show that all of the cycles are $(2, t)$ -choosable for $t \geq 3$.

Let $t \geq 3$ and L be a $(2, t)$ -list assignment of C_n . Thus there are two adjacent vertices $v_1, v_n \in V(G)$ such that $L(v_1) \neq L(v_n)$. Let $v_2, v_3, \dots, v_{n-1}$ be remaining vertices along the cycle C_n , where v_i is adjacent to v_{i+1} for $i = 1, 2, \dots, n-1$. First we assign v_1 a color c in $L(v_1)$ which is not in $L(v_n)$ and then we assign vertex v_2 a color in $L(v_2)$ different from c and so on. This algorithm guarantees that each pair of adjacent vertices receives distinct colors.

Theorem 5 is a powerful tool when combined to Lemma 6. Everyone uses both of them to obtain a result on (k, t) -choosability of graphs.

Theorem 5 (see [14]). *Let L be a list assignment of a graph G and let $S \subseteq V(G)$ be a maximal nonempty subset such that $|L(S)| < |S|$. If $G[S]$ is $L|_S$ -colorable, then G is L -colorable.*

Lemma 6 (see [10]). *Let $A_1, A_2, \dots, A_n$ be k -sets and $J \subseteq \{1, 2, \dots, n\}$. If $|\bigcup_{i=1}^n A_i| \geq p$, then $|\bigcup_{i \in J} A_i| \geq p - (n - |J|)k$.*

The following statements appear in [10, 11]. They use the tools to obtain characterizations of (k, t) -choosability of n -vertex graphs. We also need the tools, as well.

Lemma 7 (see [10]). *Let G be an n -vertex graph. If $k \geq n - 2$ and G is K_{k+1} -free, then G is (k, t) -choosable for any positive integer k .*

Lemma 8 (see [10]). *If a $(k+3)$ -vertex graph is K_{k+1} -free, then it is (k, t) -choosable for $t \geq k + 1$.*

FIGURE 1: The graph for Lemma 10.

Lemma 9 (see [11]). *Let G be an n -vertex graph, where $n \geq 6$. If G does not contain K_{n-2} and $C_5 \vee K_{n-5}$, then G is $(n-3)$ -colorable.*

Lemma 10 (see [11]). *Let G be a graph in Figure 1. If each list has size 2 exactly two lists of size 3 and $|L(x_4)| = 2$ or $|L(x_5)| = 2$, then G is L -colorable.*

Lemma 11 (see [11]). *Let G be a graph with 7 vertices and $\chi(G) = 4$. If G does not contain K_4 and $C_5 \vee K_1$, then $\delta(G) \geq 3$ and $\Delta(G) = 4$.*

Corollary 12 (see [11]). *If G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4 , then G is $(3, 6)$ -choosable.*

Recently, Ohba's conjecture is proved by Noel as shown in Theorem 13.

Theorem 13 (see [15]). *If $|V(G)| \leq 2\chi(G) + 1$, then $ch(G) = \chi(G)$.*

The theorem is powerful because several interesting results can be obtained; for example, Corollaries 14 and 15.

Corollary 14. *Let G be an n -vertex graph with $n \leq 7$. If G is 3-colorable, then G is 3-choosable.*

Proof. Let G be an n -vertex graph with $n \leq 7$. Assume that G is 3-colorable. Then $\chi(G) \leq 3$. Then we add some edges to G to obtain a graph H such that $\chi(H) = 3$. Since $|V(H)| \leq 7 = 2\chi(H) + 1$, we obtain that $ch(H) = \chi(H) = 3$ by Theorem 13. Hence G is 3-choosable because G is a subgraph of H . $\square$

Corollary 15. *Let G be an n -vertex graph with $n \leq 9$. If G is 4-colorable, then G is 4-choosable.*

FIGURE 2: The 7-vertex graph with chromatic number 4 and having no K_4 and $C_5 \vee K_1$.

Proof. Let G be an n -vertex graph with $n \leq 9$. Assume that G is 4-colorable. Then $\chi(G) \leq 4$. Then we add some edges to G to obtain a graph H such that $\chi(H) = 4$. Since $|V(H)| \leq 9 = 2\chi(H) + 1$, we obtain that $\text{ch}(H) = \chi(H) = 4$ by Theorem 13. Hence G is 4-choosable because G is a subgraph of H . $\square$

Finally, we need one more theorem and one more lemma to prove our main results.

Theorem 16 (see [16]). *If G is a graph other than odd cycle and complete graph, then $\chi(G) \leq \Delta(G)$.*

Lemma 17 (see [11, 17]). *Let G be a graph with $\delta(G) \geq 3$ and $\Delta(G) = 4$. If G has no K_4 and $C_5 \vee K_1$, then G must be one of the 7 graphs in Figure 2.*

3. Main Results

In this section, the main result is in Theorem 24. In order to prove the main result, we need to prove Theorems 19 and 23. Lemma 18 is established for Theorem 19 while Lemmas 20, 21, and 22 are established for Theorem 23.

Lemma 18. *Let G be an 8-vertex graph with $\Delta(G) = 5$. If G has no K_5 and $C_5 \vee K_2$, then G is 4-colorable.*

Proof. Let G be an 8-vertex graph with $\Delta(G) = 5$. Assume that G has no K_5 and $C_5 \vee K_2$.

If there is a vertex v such that $d(v) \leq 3$, then $G - v$ is 4-colorable by Lemma 9; hence, G is 4-colorable, as well. Suppose that $\delta(G) \geq 4$. Let v_1 be a vertex with $d(v_1) = 5$ of G and let v_2 be a vertex which is not adjacent to v_1 . Notice that $d(v_2)$ is 4 or 5.

Case 1 ($G - v_1 - v_2$ contains $C_5 \vee K_1$). Under the condition $\Delta(G) = 5$, $G - v_1 - v_2$ must be $C_5 \vee K_1$. Then G must be a subgraph of the graph shown in Figure 3. According to the figure, G and its subgraphs are 4-colorable.

Case 2 ($G - v_1 - v_2$ contains K_4). Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$, $H = G - \{u_1, u_2, u_3, u_4\}$, and $V(H) = \{v_1, v_2, w_1, w_2\}$.

Case 2.1 ($d(v_2) = 5$). Notice that H has at most 5 edges because H has 4 vertices and v_1, v_2 are not adjacent. Then $d_H(v_1) + d_H(v_2) + d_H(w_1) + d_H(w_2) \leq 10$. However, $d(v_1) = d(v_2) = 5$ and $d(w_1), d(w_2) \geq 4$. Then $d(v_1) + d(v_2) + d(w_1) + d(w_2) \geq 18$. Hence, there are at least 8 edges between H and K_4 . Since $d_{K_4}(u_i) = 3$ and $d(u_i) \leq 5$ for $i = 1, 2, 3, 4$, there are at most 8 edges between K_4 and H .

Hence, there are exactly 8 edges between K_4 and H . Moreover, we obtain that H has exactly 5 edges, $d(w_1) = d(w_2) = 4$, and $d(u_1) = d(u_2) = d(u_3) = d(u_4) = 5$. Furthermore, each of w_1, w_2 is adjacent to exactly one vertex in $V(K_4)$.

If w_1 and w_2 are adjacent to different vertices in $V(K_4)$, then G must be the left graph in Figure 4. If w_1 and w_2 are

FIGURE 3: Case 1.

FIGURE 4: Case 2.1.

adjacent to the same vertex in $V(K_4)$, then G must be the right graph in Figure 4. According to the figure, G is 4-colorable.

Case 2.2 ($d(v_2) = 4$). Notice that H has at most 5 edges because H has 4 vertices and v_1, v_2 are not adjacent. Then $d_H(v_1) + d_H(v_2) + d_H(w_1) + d_H(w_2) \leq 10$. However, $d(v_1) = 5$, $d(v_2) = 4$, and $d(w_1), d(w_2) \geq 4$. Then $d(v_1) + d(v_2) + d(w_1) + d(w_2) \geq 17$. Hence, there are at least 7 edges between H and K_4 .

Since G does not contain K_5 , v_1 is not adjacent to at least one vertex from K_4 , for example, u_4 . If $d(u_4) = 5$, then the proof is done by Case 2.1 because v_1 and u_1 are nonadjacent vertices with degree 5. Suppose that $d(u_4) = 4$. Since $d_{K_4}(u_1) = d_{K_4}(u_2) = d_{K_4}(u_3) = d_{K_4}(u_4) = 3$ and $d(u_i) \leq 5$ for $i = 1, 2, 3$ and $d(u_4) = 4$ there are at most 7 edges between K_4 and H .

Hence, there are exactly 7 edges between K_4 and H . Moreover, we obtain that H has exactly 5 edges, $d(w_1) = d(w_2) = 4$ and $d(u_1) = d(u_2) = d(u_3) = 5$, and $d(u_4) = 4$.

FIGURE 5: Case 2.2.

Furthermore, each of w_1, w_2 is adjacent to exactly one vertex in $V(K_4)$.

If w_1 and w_2 are adjacent to the same vertex in $V(K_4)$, then G must be the left graph in Figure 5. If w_1 and w_2 are adjacent to different vertices in $V(K_4)$, then G must be the middle graph or the right graph in Figure 5. According to the figure, G is 4-colorable.

Case 3 ($G - v_1 - v_2$ contains neither K_4 nor $C_5 \vee K_1$). By Lemma 9, $G - v_1 - v_2$ is 3-colorable. Hence, all vertices from $G - v_1 - v_2$ are labeled by 3 colors and v_1, v_2 are labeled by the fourth color. Therefore, G is 4-colorable. $\square$

Theorem 19. *If G is an 8-vertex graph having no $C_5 \vee K_2$ and K_5 , then G is $(4, 7)$ -choosable.*

Proof. Assume that G is an 8-vertex graph having no $C_5 \vee K_2$ and K_5 . Then for every vertex v , $G[N(v)]$ have no $C_5 \vee K_1$ and K_4 .

Case 1 ($\Delta(G) \leq 4$). Then G is 4-colorable by Theorem 16. Hence, G is 4-choosable by Corollary 15.

Case 2 ($\Delta(G) = 5$). Then G is 4-colorable by Lemma 18. Hence, G is 4-choosable by Corollary 15.

Case 3 ($\Delta(G) = 6$). Let v be a vertex with $d(v) = 6$ and w is the vertex which is not adjacent to v . Since $|L(v)| = |L(w)| = 4$ and $|L(V(G))| = 7$, $L(v)$ and $L(w)$ have a common color, for example, c . Hence, we label v and w by color c .

By Lemma 9, $G - u - v$ is 3-colorable because it has no $C_5 \vee K_1$ and K_4 . By Corollary 14, $G - u - v$ is 3-choosable. That is, we can label the remaining vertices.

Case 4 ($\Delta(G) = 7$). Let v be a vertex with $d(v) = 7$.

Case 4.1 ($L(v) - L(V(G - v)) \neq \emptyset$). Then we can label v in order to obtain that each of the remaining 7 vertices still has 4 available colors. Since $G - v$ has no $C_5 \vee K_1$ and K_4 , it is 4-colorable by Lemma 9. Then, it is 4-choosable by Corollary 15. That is, we can label the remaining vertices.

Case 4.2 ($L(v) - L(V(G - v)) = \emptyset$). Then we label v in order to obtain that each of the remaining seven vertices has at least 3 available colors and the number of the remaining colors is 7. According to Corollary 12, $G - v$ is $(3, 6)$ -choosable. That is, we can label the remaining vertices. $\square$

Lemma 20. $K_{2,2,2}$ is L -colorable if $|L(V(K_{2,2,2}))| = 4$ and each list has size 3 except for one list of size 2.

Proof. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2\}$ be partite sets of $K_{2,2,2}$. Let L be a list assignment of $K_{2,2,2}$ such that $|L(V(K_{2,2,2}))| = 4$ and each list has size 3 except for one list of size 2. Without loss of generality, suppose that $|L(x_1)| = 2$ and $|L(v)| = 3$ for the remaining 5 vertices. Since $|L(V(K_{2,2,2}))| = 4$, there is a color $c \in L(x_1) \cap L(x_2)$. Then we label x_1 and x_2 by color c . Hence, each of the remaining 4 vertices has 2 available colors. Since C_4 is 2-choosable, the remaining vertices can be labeled. $\square$

Lemma 21. G_5 is (3, 5)-choosable.

Proof. Let L be a (3, 5)-list assignment of G_5 . Since $|L(V(G_5))| = 5$, we obtain that $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_4) \cup L(x_5)| \geq 4$ or $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_6) \cup L(x_7)| \geq 4$. Without loss of generality, suppose that $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_4) \cup L(x_5)| \geq 4$.

Since $|L(x_6)| = |L(x_7)| = 3$ but $|L(V(G_5))| = 5$, there is a color $c \in L(x_6) \cap L(x_7)$. Then x_6 and x_7 are labeled by color c . Let $L' = L - c$ be the list assignment of $G - x_6 - x_7$. Since $|L'(x_i)| \geq 2$ for $i = 1, 2, 3, 4, 5$ and $|\bigcup_{x \in V(G - x_6 - x_7)} L'(x)| \geq 3$, $G - x_6 - x_7$ is L' -colorable by Example 4. $\square$

Lemma 22. G_7 is (3, 5)-choosable.

Proof. Notice that x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \pmod{7}$. Let L be a (3, 5)-list assignment of G_7 .

Case 1. There is a color c that appears in exactly 1 list. Without loss of generality, suppose $c \in L(x_1)$ but $c \notin L(v)$ for the remaining vertices v . Then we label x_1 by color c . Each of the remaining six vertices has 3 available colors. Since $G_7 - x_1$ does not contain K_4 or $C_5 \vee K_1$, it is 3-colorable by Lemma 9. Hence, $G_7 - x_1$ is 3-choosable by Corollary 14. That is, the remaining vertices can be labeled.

Case 2. There is a color c that appears in exactly 2 lists. Without loss of generality, we may prove only 3 subcases because x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \pmod{7}$.

Case 2.1 ($c \in L(x_1) \cap L(x_2)$). Then we label x_1 and x_2 by color c . Hence, $G - x_1 - x_2$ is 3-colorable by Lemma 9. Consequently, it is 3-choosable by Corollary 14.

Case 2.2 ($c \in L(x_1) \cap L(x_3)$). Then we label x_1 by color c . Lemma 20 confirms that the remaining 6 vertices can be labeled.

Case 2.3 ($c \in L(x_1) \cap L(x_4)$). Then we label x_1 by color c . Again, Lemma 20 confirms that the remaining six vertices can be labeled.

Case 3. There is a color c that appears in exactly 4, 5, or 6 lists. According to the fact that x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \pmod{7}$, we may prove only the following cases.

If c appears in exactly 4 lists, we suppose that $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_6)$. Then we label x_1 and x_2 by color c and Lemma 10 confirms that the remaining five vertices can be labeled.

If c appears in exactly 5 lists, we suppose that $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_6)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5) \cap L(x_6)$. Again, we label x_1 and x_2 by color c and Lemma 10 confirms that the remaining five vertices can be labeled.

If c appears in exactly 6 lists, we suppose that $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_6) \cap L(x_7)$. Similarly, we label x_1 and x_2 by color c and Lemma 10 confirms that the remaining five vertices can be labeled.

Case 4. Each color appears in exactly 3 or 7 lists. Let x and y be the number that appears in exactly 3 or 7 lists, respectively. Then $x + y = 5$ and $3x + 7y = 21$. There are no integer solutions for this system. Hence, this case is impossible. $\square$

Theorem 23. If G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4 , then G is (3, 5)-choosable.

Proof. Assume that G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4 . Let L be a (3, 5)-list assignment of G . According to Lemma 9, G is 4-colorable. Then $\chi(G) \leq 4$. If $\chi(G) \leq 3$, we apply Corollary 14. Hence, G is 3-choosable. Otherwise, suppose that $\chi(G) = 4$. By Lemma 11, we obtain that $\delta(G) \geq 3$ and $\Delta(G) = 4$. By Lemma 17, G must be one of the seven graphs in Figure 2. By Lemma 21, G_1 , G_2 , G_4 , and G_5 are (3, 5)-choosable. By Lemma 22, G_3 , G_6 , and G_7 are (3, 5)-choosable. $\square$

Theorem 24. Let $k \geq 3$. If an n -vertex graph G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k, t) -choosable for $t = kn - k^2 - 2k - 1$.

Proof. Let G be a graph with n vertices and $k \geq 3$. Assume that G does not contain K_{k+1} and $C_5 \vee K_{k-2}$.

Let L be a (k, t) -list assignment such that $t = kn - k^2 - 2k - 1$. To apply Theorem 5, let $S \subseteq V(G)$ such that $|L(S)| < |S|$. We need to prove that $G[S]$ is $L|_S$ -colorable.

By Lemma 6, $|L(S)| \geq t - (n - |S|)k = (kn - k^2 - 2k - 1) - (kn - |S|k) = |S|k - k^2 - 2k - 1$. Then $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$. It follows that $|S| < k + 3 + 4/(k - 1)$.

Case 1 ($|S| \leq k + 2$). Then $G[S]$ is $L|_S$ -colorable by Lemma 7.

Case 2 ($|S| = k + 3$). Then $G[S]$ is $L|_S$ -colorable by Lemmas 8 and 9.

Case 3 ($|S| \geq k + 4$). Then $k + 4 \leq |S| < k + 3 + 4/(k - 1)$. Hence, $k = 3$ or $k = 4$.

Case 3.1 ($k = 3$). Then $3 + 4 \leq |S| < 3 + 3 + 4/(3 - 1)$. Consequently, $|S| = 7$. Since $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$, we obtain that $|L(S)| \geq 5$. If $|L(S)| = 6$, then $G[S]$ is $L|_S$ -colorable by Corollary 12. If $|L(S)| = 5$, then $G[S]$ is $L|_S$ -colorable by Theorem 23.

Case 3.2 ($k = 4$). Then $4 + 4 \leq |S| < 4 + 3 + 4/(4 - 1)$. Consequently, $|S| = 8$. Since $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$, we obtain that $|L(S)| = 7$. By Theorem 19, $G[S]$ is $L|_S$ -colorable. $\square$

4. Open Problem

Thanks to [10, 11], a characterization of (k, t) -choosability of n -vertex graphs when $k \geq 3$ and $t \geq kn - k^2 - 2k$ is obtained. According to [13], a $(2, t)$ -choosability of n -vertex graphs when $t \geq 2n - 8$ is obtained. In this paper, we give a characterization of (k, t) -choosability of n -vertex graphs when $k \geq 3$ and $t = kn - k^2 - 2k - 1$.

Conjecture 25. *Let $k \geq 3$. If an n -vertex graph has no K_{k+1} and $K_{k+1} \vee C_5 \vee K_{k-2}$, then it is (k, t) -choosable of $t = kn - k^2 - 2k - 2$.*

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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