Research Article

The Pre-Schwarzian Norm Estimate for Analytic Concave Functions

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Let $\mathbb{D}$ denote the open unit disk and let $\mathcal{S}$ denote the class of normalized univalent functions which are analytic in $\mathbb{D}$. Let $\text{Co}(\alpha)$ be the class of concave functions $f \in \mathcal{S}$, which have the condition that the opening angle of $f(\mathbb{D})$ at infinity is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$. In this paper, we find a sufficient condition for the Gaussian hypergeometric functions to be in the class $\text{Co}(\alpha)$. And we define a class $\text{Co}(\alpha, A, B)$, $(-1 \leq B < A \leq 1)$, which is a subclass of $\text{Co}(\alpha)$ and we find the set of variabilities for the functional $(1 - |z|^2)(f''(z)/f'(z))$ for $f \in \text{Co}(\alpha, A, B)$. This gives sharp upper and lower estimates for the pre-Schwarzian norm of functions in $\text{Co}(\alpha, A, B)$. We also give a characterization for functions in $\text{Co}(\alpha, A, B)$ in terms of Hadamard product.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We denote the class of locally univalent functions by $\mathcal{LU}$. Let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ with normalization $f(0) = f'(0) - 1 = 0$ and let $\mathcal{S}$ be the class of functions in $\mathcal{A}$ that are univalent in $\mathbb{D}$. Also we define the subclass $\mathcal{K} \subset \mathcal{S}$ of convex functions whenever $f(\mathbb{D})$ is a convex domain.

A function $f : \mathbb{D} \rightarrow \mathbb{C}$ is said to belong to the family $\text{Co}(\alpha)$ if f satisfies the following conditions:

- (i) f is analytic in $\mathbb{D}$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition, it satisfies $f(1) = \infty$.
- (ii) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- (iii) The opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class $\text{Co}(\alpha)$ is referred to as the class of concave univalent functions. We note that for $f \in \text{Co}(\alpha)$, $\alpha \in (1, 2]$, the closed set $\mathbb{C} \setminus f(\mathbb{D})$ is convex and unbounded. We observe that $\text{Co}(2)$ contains the classes $\text{Co}(\alpha)$, $\alpha \in (1, 2]$.

Avkhadiev and Wirths [1] found the analytic characterization for functions in $\text{Co}(\alpha)$, $\alpha \in (1, 2]$: $f \in \text{Co}(\alpha)$ if and only if

$$\text{Re} \left\{ \frac{2}{\alpha - 1} \left\{ \frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)} \right\} \right\} > 0, \quad z \in \mathbb{D}. \quad (1)$$

For $f \in \mathcal{LU}$, the pre-Schwarzian derivative T_f is defined by $T_f = f''/f'$ and we define the norm of T_f by

$$\|T_f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2) |T_f(z)|. \quad (2)$$

It is well known that $\|T_f\| \leq 6$ for $f \in \mathcal{S}$ and $\|T_f\| \leq 4$ for $f \in \mathcal{K}$. In [2], Bhowmik et al. obtained the estimate of the pre-Schwarzian norm for functions $f \in \text{Co}(\alpha)$ as the following:

$$4 \leq \|T_f\| \leq 2\alpha + 2, \quad f \in \text{Co}(\alpha). \quad (3)$$

For more investigation of concave functions, we may refer to [3–7].

We say that f is subordinate to F in $\mathbb{D}$, written as $f < F$, if and only if $f(z) = F(w(z))$ for some Schwarz functions $w(z)$, $w(0) = 0$, and $|w(z)| < 1$, $z \in \mathbb{D}$. If $F(z)$ is univalent in $\mathbb{D}$,

then the subordination $f \prec F$ is equivalent to $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

By using the subordination, we define a subclass of concave functions as follows.

Definition 1. Let A and B be real numbers such that $-1 \leq B < A \leq 1$. The function $f \in \mathcal{A}$ belongs to the class $\text{Co}(\alpha, A, B)$ if f satisfies the following:

$$\frac{2}{\alpha - 1} \left\{ \frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)} \right\} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{D}). \quad (4)$$

Note that $\text{Co}(\alpha, 1, -1) \equiv \text{Co}(\alpha)$.

Let a, b , and c be complex numbers with $c \neq 0, -1, -2, \dots$. We define the Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ by

$${}_2F_1(a, b, c; z) := F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad (5)$$

where $(\gamma)_k$ is Pochhammer symbol defined, in terms of Gamma function Γ , by

$$\begin{aligned} (\gamma)_k &:= \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} \\ &= \begin{cases} 1, & (k = 0) \\ \gamma(\gamma + 1) \cdots (\gamma + k - 1) & (k \in \mathbb{N}). \end{cases} \end{aligned} \quad (6)$$

We note that the Gaussian hypergeometric function F satisfies the hypergeometric differential equation

$$z(1 - z)F''(z) + [c - (a + b + 1)z]F'(z) - abF(z) = 0. \quad (7)$$

The Gaussian hypergeometric function has been studied extensively by various authors [8–13]. In particular, univalence, close-to-convexity, starlikeness, convexity, and various other properties associated with these hypergeometric functions were investigated based on the conditions of a, b , and c in [14–17].

If $f \in \mathcal{H}$ and $g \in \mathcal{H}$ given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (8)$$

then Hadamard product (or convolution) $f * g \in \mathcal{H}$ of f and g is defined (as usual) by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \quad (9)$$

In this paper, we find a sufficient condition for the Gaussian hypergeometric functions to be in the class $\text{Co}(\alpha)$. And we find the set of variabilities for the functional $(1 - |z|^2)T_f(z)$ and as a consequence of this we derive upper and lower bounds for the pre-Schwarzian norm $\|T_f\|$, for functions f in $\text{Co}(\alpha, A, B)$. And we give a representation formula in terms of Hadamard product for functions in $\text{Co}(\alpha, A, B)$.

2. A Sufficient Condition for Functions to Be in $\text{Co}(\alpha)$

In this section, we investigate a sufficient condition for the Gaussian hypergeometric functions to be in the class $\text{Co}(\alpha)$. The proof of our result in this section is based upon the following lemmas.

Lemma 2 (see [18], Miller and Mocanu, p. 35). *Let Ω be a set in the complex plane $\mathbb{C}$ and let b be a complex number such that $\text{Re}(b) > 0$. Suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$ satisfies the condition*

$$\psi(ix, y; z) \notin \Omega \quad (10)$$

for all real $x, y \leq -|b - ix|^2 / (2 \text{Re}(b))$ and all $z \in \mathbb{D}$. If the function $p(z)$ defined by $p(z) = b + b_1 z + b_2 z^2 + \dots$ is analytic in $\mathbb{D}$ and if

$$\psi(p(z), zp'(z); z) \in \Omega, \quad (11)$$

then $\text{Re}\{p(z)\} > 0$ in $\mathbb{D}$.

Lemma 3 (see [18], Miller and Mocanu, p. 239). *If a, b , and c are real and satisfy*

$$-1 \leq b \leq c, \quad a \in [-2, 0] \cup [c - 1, c + 1], \quad (12)$$

then $F'(a, b, c; z) \neq 0$.

Theorem 4. *Let $\alpha \in (1, 2]$ and a, b , and c be real and satisfy (12) and*

$$\alpha + c - 1 > |-\alpha + a + b - 1|, \quad (13)$$

$$\text{Re}\{D(z)\overline{B(z)}\} \leq 0, \quad (14)$$

in $\mathbb{D}$, where

$$\begin{aligned} B(z) &= 2(\alpha - 1)(1 - z)\{(-\alpha + a + b - 1)z - (\alpha + c - 1)\}, \\ D(z) &= \{(\alpha + 3)(\alpha - 2a - 2b - 1) + 4(a + 1)(b + 1)\}z^2 \\ &\quad + \{2(\alpha + 3)(\alpha + c) - 2(\alpha - 1)(a + b + 1) \\ &\quad - 4(a + 1)(b + 1)\}z + (\alpha - 1)(\alpha - 1 + 2c). \end{aligned} \quad (15)$$

Then,

$$\frac{c}{ab} \{F(a, b, c; z) - 1\} \in \text{Co}(\alpha). \quad (16)$$

Proof. Let

$$f(z) = \frac{c}{ab} \{F(a, b, c; z) - 1\}. \quad (17)$$

Then, f satisfies $f(0) = 0, f'(0) = 1$, and

$$F(z) := F(a, b, c; z) = \frac{ab}{c} f(z) + 1. \quad (18)$$

From the hypergeometric differential equation (7), we have

$$z(1-z)f''(z) + [c - (a+b+1)z]f'(z) - abf(z) - c = 0. \quad (19)$$

From condition (12) and Lemma 3, we have

$$f'(z) = \frac{c}{ab} F'(a, b, c; z) \neq 0 \quad (20)$$

in $\mathbb{D}$. If we set

$$p(z) = \frac{2}{\alpha-1} \left\{ \frac{\alpha+1}{2} \frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)} \right\}, \quad (21)$$

then p is analytic in $\mathbb{D}$ and satisfies $p(0) = 1$. Furthermore, we have

$$\begin{aligned} &(\alpha-1)(1-z)f'(z)p(z) \\ &= (\alpha+1)(1+z)f'(z) - 2(1-z)f'(z) \\ &\quad - 2z(1-z)f''(z). \end{aligned} \quad (22)$$

Hence, we get

$$\begin{aligned} &2z(1-z)f''(z) \\ &= \{\alpha-1 + (\alpha+3)z - (\alpha-1)(1-z)p(z)\}f'(z). \end{aligned} \quad (23)$$

If we use this substitution in (19), then

$$\begin{aligned} &\{\alpha-1 + (\alpha+3)z - (\alpha-1)(1-z)p(z)\}f'(z) \\ &\quad + 2\{c - (a+b+1)z\}f'(z) - 2abf(z) - 2c \\ &= 0. \end{aligned} \quad (24)$$

Differentiating (24), we have

$$\begin{aligned} &\{\alpha+3 + (\alpha-1)p(z) - (\alpha-1)(1-z)p'(z)\}f'(z) \\ &\quad + \{\alpha-1 + (\alpha+3)z - (\alpha-1)(1-z)p(z)\}f''(z) \\ &\quad - 2(a+b+1)f'(z) + 2\{c - (a+b+1)z\}f''(z) \\ &\quad - 2abf'(z) = 0. \end{aligned} \quad (25)$$

Substituting (23) in (25), we have

$$\begin{aligned} &2\{\alpha+3 + (\alpha-1)p(z) - (\alpha-1)(1-z)p'(z)\}z(1-z) \\ &\quad + \{\alpha-1 + (\alpha+3)z - (\alpha-1)(1-z)p(z)\}^2 \\ &\quad - 4(a+b+1)z(1-z) + 2\{c - (a+b+1)z\} \\ &\quad \cdot \{\alpha-1 + (\alpha+3)z - (\alpha-1)(1-z)p(z)\} \\ &\quad - 4abz(1-z) = 0. \end{aligned} \quad (26)$$

And this equation leads us to the following first order differential equation:

$$A(z)p^2(z) + B(z)p(z) + C(z)zp'(z) + D(z) = 0, \quad (27)$$

where

$$\begin{aligned} A(z) &= (\alpha-1)^2(1-z)^2, \\ B(z) &= 2(\alpha-1)(1-z)\{(-\alpha+a+b-1)z - (\alpha+c-1)\}, \\ C(z) &= -2(\alpha-1)(1-z)^2, \\ D(z) &= \{(\alpha+3)(\alpha-2a-2b-1) + 4(a+1)(b+1)\}z^2 \\ &\quad + \{2(\alpha+3)(\alpha+c) - 2(\alpha-1)(a+b+1) \\ &\quad - 4(a+1)(b+1)\}z + (\alpha-1)(\alpha-1+2c). \end{aligned} \quad (28)$$

Since inequality (13) implies $B(z) \neq 0$ in $\mathbb{D}$, we can rewrite (27) in the form

$$J(z)p^2 + p(z) + K(z)zp'(z) + L(z) = 0, \quad (29)$$

where

$$J(z) = \frac{A(z)}{B(z)}, \quad K(z) = \frac{C(z)}{B(z)}, \quad L(z) = \frac{D(z)}{B(z)}. \quad (30)$$

A bilinear transformation $w(z) = (1+Az)/(1+Bz)$, with $-1 \leq A \leq 1$ and $-1 \leq B \leq 1$, has the property that $\operatorname{Re}\{w(z)\} > 0$, for $z \in \mathbb{D}$. From the condition $\alpha \in (1, 2]$ and (13), we have

$$\begin{aligned} &\operatorname{Re}\{K(z)\} \\ &= \frac{1}{\alpha+c-1} \operatorname{Re} \left\{ \frac{1-z}{1 + ((\alpha-a-b+1)/(\alpha+c-1))z} \right\} > 0 \end{aligned} \quad (31)$$

in $\mathbb{D}$ and

$$\begin{aligned} &\operatorname{Re} \left\{ J(z) + \frac{1}{2}K(z) \right\} \\ &= \frac{2-\alpha}{2(\alpha+c-1)} \operatorname{Re} \left\{ \frac{1-z}{1 + ((\alpha-a-b+1)/(\alpha+c-1))z} \right\} \\ &\geq 0 \end{aligned} \quad (32)$$

in $\mathbb{D}$. Now, we let $\Omega = \{0\}$ and define a function $\psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$ by

$$\psi(r, s; z) = J(z)r^2 + r + K(z)s + L(z). \quad (33)$$

Then, (27) becomes

$$\psi(p(z), zp'(z); z) \in \Omega. \quad (34)$$

By using (14), (31), and (32), we obtain

$$\begin{aligned}
 & \operatorname{Re} \{\psi(i\rho, \sigma; z)\} \\
 &= -\operatorname{Re}\{J(z)\} \rho^2 + \operatorname{Re}\{K(z)\} \sigma + \operatorname{Re}\{L(z)\} \\
 &\leq -\left(\operatorname{Re}\left\{J(z) + \frac{1}{2}K(z)\right\}\right) \rho^2 - \frac{1}{2} \operatorname{Re}\{K(z)\} + \operatorname{Re}\{L(z)\} \\
 &< \operatorname{Re}\{L(z)\} \\
 &= \frac{\operatorname{Re}\{D(z)\overline{B(z)}\}}{|B(z)|^2} \\
 &\leq 0,
 \end{aligned} \tag{35}$$

for all $z \in \mathbb{D}$ and all real ρ, σ with $\sigma \leq -(1 + \rho^2)/2$. By Lemma 2, we have $\operatorname{Re}\{p(z)\} > 0$ in $\mathbb{D}$, which shows that $f \in \operatorname{Co}(\alpha)$. $\square$

Example 5. If we take $\alpha = 2, a = 3, b = 1$, and $c = 2$, then we can easily check that conditions (12) and (13) are satisfied. Furthermore,

$$\operatorname{Re}\{\varphi_1(z)\} \leq 0 \quad (z \in \mathbb{D}), \tag{36}$$

where

$$\varphi_1(z) = D(z)\overline{B(z)} = 2\overline{(1-z)(-3+z)}(5-2z-3z^2) \tag{37}$$

(see Figure 1). Hence,

$$f_1(z) = \frac{2}{3}F(3, 1, 2; z) = \frac{2}{3}\left(-1 + \frac{1-z/2}{(1-z)^2}\right) \tag{38}$$

belongs to $\operatorname{Co}(2)$, as shown in Figure 2.

Example 6. If we take $\alpha = 3/2, a = 3/2, b = 1$, and $c = 1$, then we can easily check that conditions (12) and (13) are satisfied. Furthermore,

$$\operatorname{Re}\{\varphi_2(z)\} \leq 0 \quad (z \in \mathbb{D}), \tag{39}$$

where

$$\varphi_2(z) = D(z)\overline{B(z)} = \frac{3}{8}\overline{(1-z)}(z^2 + 4z - 5) \tag{40}$$

(see Figure 3). Hence,

$$f_2(z) = \frac{2}{3}F\left(\frac{3}{2}, 1, 1; z\right) = \frac{2}{3}\left(-1 + \frac{1-z/2}{(1-z)^{3/2}}\right) \tag{41}$$

belongs to $\operatorname{Co}(3/2)$, as shown in Figure 4.

3. The Pre-Schwarzian Norm Estimate for Functions in $\operatorname{Co}(\alpha, A, B)$

Now, for $f \in \operatorname{Co}(\alpha, A, B)$, we find the exact set of variabilities for the functional $(1 - |z|^2)T_f(z)$, which gives both sharp upper and lower bounds for the pre-Schwarzian norm $\|T_f\|$.

FIGURE 1: The image of $\varphi_1(z)$.

FIGURE 2: The image of $f_1(z)$.

Theorem 7. Let $\alpha \in (1, 2]$ and $-1 \leq B < A \leq 1$ be fixed and let $f \in \operatorname{Co}(\alpha, A, B)$. Then, the set of variabilities of the functional $(1 - |z|^2)T_f(z)$ is the closed disk with center

$$\left(\alpha + 1 - \frac{B}{2}(B - A)(\alpha - 1)\right)\bar{z} + (\alpha + 1)\frac{1 - \bar{z}}{1 - z} \tag{42}$$

and radius

$$\frac{1}{2}(A - B)(\alpha - 1). \tag{43}$$

The points on the boundary of this disk are attained if and only if f is one of the functions g_θ , where

$$g_\theta(z) = \int_0^z (1 + Be^{i\theta}t)^{(B-A)(\alpha-1)/(2B)}(1-t)^{-(1+\alpha)}dt. \tag{44}$$

FIGURE 3: The image of $\varphi_2(z)$. FIGURE 4: The image of $f_2(z)$.

Proof. We use the characterization (4) for functions in $\text{Co}(\alpha, A, B)$ and the representation

$$P_f(z) = \frac{1 + Azw(z)}{1 + Bzw(z)}, \quad (45)$$

where $w : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ is a unimodular bounded analytic function. It follows that

$$\begin{aligned} T_f(z) &= (2(\alpha + 1) + (B - A)(\alpha - 1)w(z) \\ &\quad + ((A + B)\alpha + 3B - A)zw(z)) \\ &\quad \cdot (2(1 - z)(1 + Bzw(z)))^{-1}. \end{aligned} \quad (46)$$

By a routine computation, one recognizes that

$$\begin{aligned} (1 - |z|^2)T_f(z) - \left(\alpha + 1 - \frac{B}{2}(B - A)(\alpha - 1)\right)\bar{z} \\ - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} = \frac{1}{2}(B - A)(\alpha - 1)\frac{B\bar{z} + w(z)}{1 + Bzw(z)}. \end{aligned} \quad (47)$$

Hence, the condition $|w(z)| \leq 1$ is equivalent to

$$\begin{aligned} \left| (1 - |z|^2)T_f(z) - \left(\alpha + 1 - \frac{B}{2}(B - A)(\alpha - 1)\right)\bar{z} \right. \\ \left. - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} \right| \leq \frac{1}{2}(A - B)(\alpha - 1). \end{aligned} \quad (48)$$

This proves the first part of the assertion in the theorem. The second part follows from the fact that $|w(z)| = 1$ if and only if $w(z) \equiv e^{i\theta}$, $\theta \in [0, 2\pi]$, and that the solution of the differential equation (4) in this case is given by $f(z) = g_\theta(z)$. The relation between boundary points of the above circle and the extremal function becomes clear from the identity

$$\begin{aligned} (1 - |z|^2)T_{g_\theta}(z) - \left(\alpha + 1 - \frac{B}{2}(B - A)(\alpha - 1)\right)\bar{z} \\ - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} = \frac{1}{2}(B - A)(\alpha - 1)\frac{B\bar{z} + e^{i\theta}}{1 + Be^{i\theta}z}. \end{aligned} \quad (49)$$

This completes the proof of the theorem. $\square$

Remark 8. If we put $A = 1$ and $B = -1$ in Theorem 7, then we can obtain the result in Bhowmik et al. ([2], Theorem 2.5).

From the inequality (48), we can have the following corollary.

Corollary 9. Let $f \in \text{Co}(\alpha, A, B)$, $\alpha \in [1, 2]$ and $-1 \leq B < A \leq 1$. Then,

$$\begin{aligned} -\frac{1}{2}(1 - B)(A - B)(\alpha - 1) + 2\alpha + 2 \\ \leq \|T_f\| \leq \frac{1}{2}(1 + B)(A - B)(\alpha - 1) + 2\alpha + 2. \end{aligned} \quad (50)$$

The equality holds in lower estimate for the function g_0 and in upper estimate for the function g_π which are described in Theorem 7.

As a consequence of Theorem 7, we can obtain a distortion theorem for the functions in $\text{Co}(\alpha, A, B)$.

Theorem 10. Let $\alpha \in (1, 2]$ and $-1 \leq B < A \leq 1$. Then, for each $f \in \text{Co}(\alpha, A, B)$, one has

$$\begin{aligned} \frac{(1 - r)^{(1/4)(1-B)(A-B)(\alpha-1)}}{(1 + r)^{(1/4)(1+B)(A-B)(\alpha-1)+(\alpha+1)}} \\ \leq |f'(z)| \leq \frac{(1 + r)^{(1/4)(1-B)(A-B)(\alpha-1)}}{(1 - r)^{(1/4)(1+B)(A-B)(\alpha-1)+(\alpha+1)}} \end{aligned} \quad (51)$$

with $|z| = r < 1$. For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if $f = g_\theta$, where $\theta \in [0, 2\pi]$.

Remark 11. If we put $A = 1$ and $B = -1$ in Theorem 10, then we can obtain the distortion theorem for the functions in $\text{Co}(\alpha)$ which is a result from Bhowmik et al. ([2], Theorem 2.8).

Finally, we present a characterization for functions in the class $\text{Co}(\alpha, A, B)$, in view of the Hadamard product.

Theorem 12. Let $1 < \alpha \leq 2$ and $-1 \leq B < A \leq 1$. Then, $f \in \text{Co}(\alpha, A, B)$ if and only if

$$\begin{aligned} & \frac{1}{z} \left[f(z) * \left(((-A + B + \alpha A - \alpha B)z \right. \right. \\ & \quad \left. \left. + (-4x + (A + 3B - \alpha A + \alpha B))z^2 \right) \right. \\ & \quad \left. \cdot ((1-z)^3)^{-1} \right] \\ & + \left[f(z) * \left(((2\alpha + 2)x + (A - 3B - \alpha A - \alpha B))z \right. \right. \\ & \quad \left. \left. + ((2 - 2\alpha)x + (-A - B + \alpha A + \alpha B))z^2 \right) \right. \\ & \quad \left. \cdot ((1-z)^3)^{-1} \right] \neq 0 \end{aligned} \quad (52)$$

for all $|z| < 1$ and for all x with $|x| = 1$. Equivalently, this holds if and only if

$$\sum_{n=0}^{\infty} E_n z^n \neq 0 \quad (z \in \mathbb{D}, |x| = 1), \quad (53)$$

where

$$\begin{aligned} E_0 &= -\frac{1}{2} (A - B - \alpha A + \alpha B), \\ E_n &= \left\{ x - \frac{1}{2} (A + B - \alpha A + \alpha B) \right. \\ & \quad \left. + (B - x)(n+1) \right\} (n+1) a_{n+1} \\ & \quad + \left\{ \alpha x - \frac{1}{2} (-A + B + \alpha A + \alpha B) \right. \\ & \quad \left. - (-x + B)n \right\} n a_n \quad (n \in \mathbb{N}) \end{aligned} \quad (54)$$

with

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (a_1 = 1). \quad (55)$$

Remark 13. If we put $A = 1$ and $B = -1$ in Theorem 12, then we can obtain the convolution result for the functions in $\text{Co}(\alpha)$ which is a result from Bhowmik et al. ([2], Theorem 3.1).

Conflict of Interests

The authors declare that they have no competing interests.

References

- [1] F. G. Avkhadiyev and K.-J. Wirths, "Concave schlicht functions with bounded opening angle at infinity," *Lobachevskii Journal of Mathematics*, vol. 17, pp. 3–10, 2005.
- [2] B. Bhowmik, S. Ponnusamy, and K.-J. Wirths, "Characterization and the pre-Schwarzian norm estimate for concave univalent functions," *Monatshefte für Mathematik*, vol. 161, no. 1, pp. 59–75, 2010.
- [3] F. G. Avkhadiyev, C. Pommerenke, and K.-J. Wirths, "Sharp inequalities for the coefficients of concave schlicht functions," *Commentarii Mathematici Helvetici*, vol. 81, no. 4, pp. 801–807, 2006.
- [4] F. G. Avkhadiyev and K.-J. Wirths, "Convex holes produce lower bounds for coefficients," *Complex Variables: Theory and Application*, vol. 47, no. 7, pp. 553–563, 2002.
- [5] B. Bhowmik, S. Ponnusamy, and K.-J. Wirths, "On the Fekete–Szegő problem for concave univalent functions," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 2, pp. 432–438, 2011.
- [6] L. Cruz and Ch. Pommerenke, "On concave univalent functions," *Complex Variables and Elliptic Equations*, vol. 52, no. 2–3, pp. 153–159, 2007.
- [7] K.-J. Wirths, "Julia's lemma and concave schlicht functions," *Quaestiones Mathematicae*, vol. 28, no. 1, pp. 95–103, 2005.
- [8] B. C. Carlson and D. B. Shaffer, "Starlike and prestarlike hypergeometric functions," *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.
- [9] E. P. Merkes and W. T. Scott, "Starlike hypergeometric functions," *Proceedings of the American Mathematical Society*, vol. 12, pp. 885–888, 1961.
- [10] S. Ponnusamy, "Hypergeometric transforms of functions with derivative in a half plane," *Journal of Computational and Applied Mathematics*, vol. 96, no. 1, pp. 35–49, 1998.
- [11] S. Ponnusamy, "Starlikeness properties for convolutions involving hypergeometric series," *Annales Universitatis Mariae Curie-Skłodowska Sectio A*, vol. 52, pp. 1–16, 1998.
- [12] H. Silverman, "Starlike and convexity properties for hypergeometric functions," *Journal of Mathematical Analysis and Applications*, vol. 172, no. 2, pp. 574–581, 1993.
- [13] A. Swaminathan, "Hypergeometric functions in the parabolic domain," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 20, no. 1, pp. 1–16, 2004.
- [14] Y. C. Kim and S. Ponnusamy, "Sufficiency for Gaussian hypergeometric functions to be uniformly convex," *International Journal of Mathematics and Mathematical Sciences*, vol. 22, no. 4, pp. 765–773, 1999.
- [15] S. Ponnusamy, "Close-to-convexity properties of Gaussian hypergeometric functions," *Journal of Computational and Applied Mathematics*, vol. 88, no. 2, pp. 327–337, 1998.
- [16] S. Ponnusamy and M. Vuorinen, "Univalence and convexity properties for Gaussian hypergeometric functions," *Rocky Mountain Journal of Mathematics*, vol. 31, no. 1, pp. 327–353, 2001.
- [17] S. Ruscheweyh and V. Singh, "On the order of starlikeness of hypergeometric functions," *Journal of Mathematical Analysis and Applications*, vol. 113, no. 1, pp. 1–11, 1986.
- [18] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Marcel Dekker, New York, NY, USA, 2000.