On Generalized Semiderivations of Prime Near Rings

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Received 6 August 2015; Accepted 1 October 2015

1. Introduction

Throughout the paper, N denotes a zero-symmetric left near ring with multiplicative centre Z; and for any pair of elements x, y ∈ N, [x, y] denotes the commutator xy − yx while the symbol (x, y) denotes the additive commutator x + y − x − y. An element x of N is said to be distributive if (y + z)x = xy + zx, for all y, z ∈ N. A near ring N is called zero-symmetric if 0x = 0, for all x ∈ N (recall that left distributivity yields that x0 = 0). The near ring N is said to be 3-prime if xNy = [0] for x, y ∈ N implies that x = 0 or y = 0. A near ring N is called 2-torsion free if (N, +) has no element of order 2. An additive mapping f : N → N is said to be a right (resp., left) generalized derivation with associated derivation D if f(xy) = f(x)y + xD(y) (resp., f(xy) = D(x)y + xf(y)), for all x, y ∈ N, and f is said to be a generalized derivation with associated derivation D on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation D. Motivated by a definition given by Bergen [1] for rings, we define an additive mapping d : N → N to be a semiderivation on a near ring N if there exists a function g : N → N such that (i) d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y) and (ii) d(g(x)) = g(d(x)), for all x, y ∈ N. In case g is the identity map on N, d is of course just a derivation on N, so the notion of semiderivation generalizes that of derivation. But the generalization is not trivial; for example, take N = N1 ⊕ N2, where N1 is a zero-symmetric near ring and N2 is a ring. Then the map d : N → N defined by d((x, y)) = (0, y) is a semiderivation associated with function g : N → N such that g(x, y) = (x, 0). However d is not a derivation on N. An additive mapping F : N → N is said to be a generalized semiderivation on N if there exists a semiderivation d : N → N associated with a map g : N → N such that (i) F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y) and (ii) F(g(x)) = g(F(x)) for all x, y ∈ N. All semiderivations are generalized semiderivations. Moreover, if g is the identity map on N, then all generalized semiderivations are merely generalized derivations; again the notion of generalized semiderivation generalizes that of generalized derivation. Moreover, the generalization is not trivial as the following example shows.

Example 1. Let S be a 2-torsion free left near ring and let

\[ N = \begin{pmatrix} x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \]
Define maps $F, d, g : N \to N$ by

$$
F \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & xy & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix},
$$

$$
g \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

(2)

It can be verified that $N$ is a left near ring and $F$ is a generalized semiderivation with associated semiderivation $d$ and a map $g$ associated with $d$. However $F$ is not a generalized derivation on $N$.

### 2. Preliminary Results

We begin with the following Lemmas which are extensively used to prove our main theorems. Unless it is stated otherwise, it will be assumed that $N$ is a zero-symmetric 3-prime near ring.

**Lemma 2** (see [2, Lemma 1.2]). Let $N$ be a 3-prime near ring.

(i) If $z \in Z \setminus \{0\}$ and $xz \in Z$, then $x \in Z$.

(ii) If $x \in Z \setminus \{0\}$ then $x$ is not a zero divisor.

**Lemma 3** (see [2, Lemma 1.5]). If $N$ is a 3-prime near ring and $Z$ contains a nonzero left semigroup ideal, then $N$ is a commutative ring.

**Lemma 4** (see [3, Theorem 2.1]). Let $N$ be a 2-torsion free 3-prime near ring with a nonzero semiderivation $d$ associated with a map $g$. If $d(N) \subseteq Z$, then $N$ is a commutative ring.

**Lemma 5.** Let $N$ be a 3-prime near ring admitting a generalized semiderivation $F$ associated with a semiderivation $d$. If $g$ is the map associated with $d$ such that $g(xy) = g(x)g(y)$ for all $x, y \in N$, then $N$ satisfies the following partial distributive laws:

(i) $(F(x)y + g(x)d(y))z = F(xy)yz + g(x)d(y)z$ for all $x, y, z \in N$.

(ii) $(d(x)y + g(x)d(y))z = d(xy)yz + g(x)d(y)z$ for all $x, y, z \in N$.

**Proof.** (i) Let $x, y, z \in N$, and by defining $F$ we have

$$
F(xy)z = F(xy)z + g(xy)d(z)
$$

$$
= (F(x)y + g(x)d(y))z + g(x)g(y)d(z).
$$

(3)

On the other hand,

$$
F(xyz) = F(x)yz + g(x)d(yz)
$$

$$
= F(x)yz + g(x)(d(y)z + g(y)d(z))
$$

$$
= F(x)yz + g(x)d(y)z + g(x)g(y)d(z).
$$

Combining both expressions of $F(xyz)$, we obtain

$$
(F(x)y + g(x)d(y))z = F(x)yz + g(x)d(y)z,
$$

$\forall x, y, z \in N$.

(5)

(ii) With a simple calculation of $d((xy)z) = d(xy)z$, we obtain the required result.

**Lemma 6.** Let $N$ be a 2-torsion free zero-symmetric 3-prime near ring. If $d$ is a nonzero semiderivation of $N$ associated with a map $g$ which is onto, then $d^2 \neq 0$.

**Proof.** Suppose $d^2(N) = 0$. Then for $x, y \in N$, we may write

$$
0 = d^2(xy)
$$

$$
= d(d(xy))
$$

$$
= d(d(x)y + g(x)d(y)) \quad \forall x, y \in N
$$

$$
= d^2(x)y + d(x)d(y) + d(g(x))d(y)
$$

$$
+ g(x)d^2(y)
$$

$$
= d(x)d(y) + d(g(x))d(y).
$$

(6)

Note that $g(d(x)) = d(g(x))$ and $g$ is onto; we get

$$
2d(x)d(y) = 0, \quad \forall x, y \in N.
$$

(7)

Since $N$ is 2-torsion free, we get

$$
d(x)d(y) = 0, \quad \forall x, y \in N.
$$

(8)

Replacing $y$ by $ry$ in the above relation, we get

$$
d(x)d(ry) = 0, \quad \forall x, y, r \in N.
$$

(9)

This implies that

$$
d(x)g(r)d(y) = 0, \quad \forall x, y, r \in N.
$$

(10)

Thus we obtain that $d = 0$, a contradiction.

The following lemma extends results of Herstein [4, Theorem 2] and Bell and Mason [2, Theorem 3].
Lemma 7. Let \( N \) be a 2-torsion free 3-prime near ring admitting a nonzero semiderivation \( d \) and a map \( g \) associated with \( d \) such that \( g \) is onto and \( g(xy) = g(x)g(y) \) for all \( x, y \in N \). If \( [d(x), d(y)] = 0 \) for all \( x, y \in N \), then \( N \) is a commutative ring.

Proof. Suppose that
\[
d(x)d(y) = d(y)d(x), \quad \forall x, y \in N. \quad (11)
\]
Replacing \( y \) by \( yz \) in (11) and using Lemma 5(ii), we obtain
\[
d(x)d(y)z + d(x)g(y)d(z)
= d(y)zd(x) + g(y)d(z)d(x), \quad \forall x, y, z \in N. \quad (12)
\]
Substituting \( d(y) \) for \( y \) in (12) and using (11), we find that
\[
d(x)d^2(y)z = d^2(y)zd(x), \quad \forall x, y, z \in N. \quad (13)
\]
Taking \( zt \) instead of \( z \) in (13) after using (13), we arrive at
\[
d^2(y)z[d(x), t] = 0, \quad \forall x, y, t, z \in N. \quad (14)
\]
which can be rewritten as
\[
d^2(y)N[d(x), t] = [0], \quad \forall x, y, t \in N. \quad (15)
\]
In the light of the 3-primeness of \( N \), (15) implies that
\[
d^2 = 0 \quad \text{or} \quad d(N) \subseteq Z. \quad (16)
\]
But \( d^2 = 0 \) contradicts Lemma 6, so \( d(N) \) is contained in \( Z \) and \( N \) is a commutative ring by Lemma 4.

3. The Condition \([F(N),F(N)] = \{0\}\)

The theorems that we prove in this section are motivated by the results proved in [2, Theorem 2], [5, Theorem 2.1], [6, Theorem 2.1 and 4.1], and [3, Theorem 2.1].

Theorem 8. Let \( N \) be a 2-torsion free 3-prime near ring with a generalized semiderivation \( F \) associated with a nonzero semiderivation \( d \) and onto map \( g \) associated with \( d \) such that \( g(xy) = g(x)g(y) \) for all \( x, y \in N \). If \( F(N) \subseteq Z \), then \( N \) is a commutative ring.

Proof. Assume that
\[
F(x)F(y) = F(y)F(x), \quad \forall x, y \in N. \quad (23)
\]
Then
\[
F(x + z)F(y + z) = F(x)F(y + z) + F(z)F(x + y), \quad \forall x, y, z \in N. \quad (24)
\]
By (23), the last equation yields that
\[
F(y)F(x + z) + F(z)F(x + y)
= F(x)F(y + z) + F(x)F(y + z), \quad \forall x, y, z \in N. \quad (25)
\]
Hence \( F(x) = F(y) \) for all \( x, y \in N \), and \( N \) is a commutative ring by Lemma 3.

Corollary 9 (see [6, Theorem 3.2]). Let \( N \) be a 2-torsion free 3-prime near ring. If \( N \) admits a nonzero generalized derivation \( F \) such that \( F(N) \subseteq Z \), then \( N \) is a commutative ring.

Theorem 10. Let \( N \) be a 3-prime near ring admitting a generalized semiderivation \( F \) associated with a nonzero semiderivation \( d \) and onto map \( g \) associated with \( d \) such that \( g(xy) = g(x)g(y) \) for all \( x, y \in N \). If \( [F(N), F(N)] = 0 \), then \((N, +)\) is abelian.

Proof. Assume that
\[
F(x)F(y) = F(y)F(x), \quad \forall x, y \in N. \quad (23)
\]
Then
\[
F(x + z)F(y + z) = F(x)F(y + z) + F(z)F(x + y), \quad \forall x, y, z \in N. \quad (24)
\]
By (23), the last equation yields that
\[
F(y)F(x + z) + F(z)F(x + y)
= F(x)F(y + z) + F(x)F(y + z), \quad \forall x, y, z \in N. \quad (25)
\]
We conclude that \( N \) is a commutative ring by Lemma 7.
Hence,

\[ F(y) F(x) + F(y) F(x) + F(z) F(x) + F(z) F(x) \]
\[ = F(x) F(y) + F(x) F(z) + F(x) F(y) + F(x) F(z), \tag{26} \]

that is

\[ F(y) F(x) + F(z) F(x) \]
\[ = F(x) F(z) + F(x) F(y), \quad \forall x, y, z \in N, \]

which implies that

\[ F(y + z - y - z) F(x) = 0, \quad \forall x, y, z \in N. \tag{27} \]

Putting \( x r \) instead of \( x \) in (27), we get

\[ F(y, z) g(x) d(r) = 0, \quad \forall x, y, z, r \in N, \tag{28} \]

which can be rewritten as

\[ F(y, z) N d(r) = \{0\}, \quad \forall y, z, r \in N. \tag{29} \]

Since \( N \) is a 3-prime near ring and \( d \neq 0 \), we get

\[ F(y, z) = 0, \quad \forall y, z \in N. \tag{30} \]

Replacing \( y \) and \( z \) by \( r y \) and \( rz \), respectively, in (30), we obtain

\[ 0 = F(r y, z) \]
\[ = r F(y, z) + d(r) g(y, z) \tag{31} \]
\[ = d(r) g(y, z), \quad \forall y, z, r \in N. \]

Taking \( rt \) instead of \( r \) in the last equation and using Lemma 5(ii), we get

\[ d(r) t g(y, z) = 0, \quad \forall y, z, r, t \in N. \tag{32} \]

Thus,

\[ d(r) N g(y, z) = \{0\}, \quad \forall y, z, r \in N. \tag{33} \]

Again using the fact that \( N \) is 3-prime and \( d \neq 0 \), we find that \( g(y) + g(z) = g(z) + g(y) \) for all \( y, z \in N \). Since \( g \) is onto, \( (N, +) \) is abelian.

**Theorem 11.** Let \( N \) be a 2-torsion free 3-prime near ring admitting a nonzero generalized semiderivation \( F \) associated with a nonzero semiderivation \( d \) and onto map \( g \) associated with \( d \) such that \( g(xy) = g(x)g(y) \) for all \( x, y \in N \). If \([F(N), F(N)] = 0\), then \( N \) is a commutative ring.

**Proof.** By the hypothesis

\[ F(x) F(y) = F(y) F(x), \quad \forall x, y \in N. \tag{34} \]

Replace \( y \) by \( F(z) y \) in the above relation, and we get

\[ F(x) F(F(z) y) = F(F(z) y) F(x), \quad \forall x, y, z \in N. \tag{35} \]

This implies that

\[ F(x) (d(F(z)) g(y) + F(z) F(y)) \]
\[ = (d(F(z)) g(y) + F(z) F(y)) F(x), \quad \forall x, y, z \in N. \tag{36} \]

Using Lemma 5(i), we find that

\[ F(x) d(F(z)) g(y) = d(F(z)) g(y) F(x), \quad \forall x, y, z \in N. \tag{37} \]

Taking \( yw \) instead of \( y \) in (37) and using (37)

\[ d(F(z)) g(y) F(x) g(w) \]
\[ = d(F(z)) g(y) g(w) F(x), \quad \forall x, y, z, w \in N. \tag{38} \]

Since \( g \) is onto, we get

\[ d(F(z)) yF(x) w = d(F(z)) ywF(x), \quad \forall x, y, z, w \in N. \tag{39} \]

This implies that

\[ d(F(z)) N (F(x) w - wF(x)) = \{0\}, \quad \forall z, w \in N. \tag{40} \]

Since \( N \) is a 3-prime near ring, we have

\[ d(F(N)) = \{0\} \tag{41} \]
\[ \text{or } F(N) \subseteq Z. \]

If \( F(N) \) is contained in \( Z \), then \( N \) is a commutative ring by Theorem 8. On the other hand, we see that if \( d(F(N)) = 0 \), then

\[ d(F(xy)) = d(dx) g(y) + xF(y) = 0, \quad \forall x, y \in N. \tag{42} \]

Thus

\[ d^2(x) g(y) + d(x) d(g(y)) + d(x) F(y) \]
\[ + xd(F(y)) = 0, \quad \forall x, y \in N. \tag{43} \]

This implies that

\[ d^2(x) g(y) + d(x) d(g(y)) + d(x) F(y) = 0, \quad \forall x, y \in N. \tag{44} \]
Replacing \( y \) by \( yz \) and using the fact that \( g \) is onto, we get
\[
d^2(x)yz + d(x)F(yz) + d(x)F(yz) = 0,
\]
\[\forall x, y, z \in N.\]

Using (44) and noting that \( g \) is an automorphism, we get
\[
d(x)yz + d(x)(yFd(z) + d(y)g(z)) + d(x)(F(y)z + g(y)d(z)) = 0,
\]
\[\forall x, y, z \in N,\] (45)

Since \( N \) is 2-torsion free, using (44) we get
\[
d(x)yz = 0, \quad \forall x, y, z \in N.
\]
(46)

Thus we obtain that \( d = 0 \), a contradiction which completes the proof. \( \square \)

**Corollary 12** ([6, Theorem 4.1]). Let \( N \) be a 2-torsion free prime near ring. If \( N \) admits a generalized derivation \( F \) associated with a nonzero derivation \( d \) such that \( F(x, y) = [F(x), y] \) for all \( x, y \in N \), then \( N \) is a commutative ring.

**Theorem 13.** Let \( N \) be a 2-torsion free 3-prime near ring. If \( F \) is a generalized semiderivation of \( N \) associated with a nonzero semiderivation \( d \) and an automorphism \( g \) associated with \( d \), then the following assertions are equivalent:

(i) \( F([x, y]) = [F(x), y] \) for all \( x, y \in N \).

(ii) \( F([x, y]) = -[F(x), y] \) for all \( x, y \in N \).

(iii) \( N \) is a commutative ring.

**Proof.** Obviously, (iii) implies both (i) and (ii).

Now we prove that (i) \( \Rightarrow \) (iii). By hypothesis
\[
F([x, y]) = [F(x), y], \quad \forall x, y \in N.
\]
(47)

Taking \( xy \) instead of \( y \) in (47) and noting that \( [x, xy] = x[x, y] \), we get
\[
xF([x, y]) + d(x)g([x, y]) = F(x)xy - xyF(x),
\]
\[\forall x, y \in N.\] (48)

Using (47) and noting that \( F(x)x = xF(x) \) by (47), then the last equation yields
\[
d(x)g([x, y]) = 0, \quad \forall x, y \in N.\] (49)

Replacing \( yt \) in (50) and using (50), we arrive at
\[
d(x)F(x)t = 0, \quad \forall x, t \in N.\] (51)

This implies that
\[
d(x)Nt = 0, \quad \forall t \in N.\] (52)

3-primeness of \( N \) yields that either \( N \subseteq Z \) or \( d(N) = \{0\} \). In both the cases \( N \) is a commutative ring by Lemmas 3 and 4, respectively.

Using the similar techniques as above we can show that (ii) \( \Rightarrow \) (iii). \( \square \)

**Corollary 14** (see [7, Theorem 2.6]). Let \( N \) be a 3-prime near ring. If \( F \) is a generalized semiderivation of \( N \) associated with a nonzero derivation \( d \) such that \( F(x, y) = [F(x), y] \) for all \( x, y \in N \), then \( N \) is a commutative ring.

**Theorem 15.** Let \( N \) be a 2-torsion free 3-prime near ring. If \( F \) is a generalized semiderivation of \( N \) associated with a nonzero semiderivation \( d \) and an automorphism \( g \) associated with \( d \), then the following assertions are equivalent:

(i) \( F([x, y]) = [F(x), y] \) for all \( x, y \in N \).

(ii) \( F([x, y]) = -[F(x), y] \) for all \( x, y \in N \).

(iii) \( N \) is a commutative ring.

**Proof.** Obviously, (iii) implies both (i) and (ii).

Now we prove that (i) \( \Rightarrow \) (iii). By hypothesis
\[
F([x, y]) = [F(x), y], \quad \forall x, y \in N.
\]
(53)

Replacing \( x \) by \( xy \) in (53), we arrive at
\[
yF([x, y]) + d(y)g([x, y]) = yxF(y) - F(y)yx,
\]
\[\forall x, y \in N.\] (54)

Using (53) and noting that \( yF(y) = F(y)y \) by (53), we find that
\[
d(y)g([x, y]) = 0, \quad \forall x, y \in N.\] (55)

Arguing in the similar manner as in Theorem 13, we get the result.

Similarly we can prove that (ii) \( \Rightarrow \) (iii). \( \square \)

**Corollary 16** (see [7, Theorem 2.7]). Let \( N \) be 3-prime near ring. If \( N \) admits a generalized derivation \( F \) associated with a nonzero derivation \( d \) such that \( F(x, y) = [x, F(y)] \) for all \( x, y \in N \), then \( N \) is a commutative ring.

The following example shows that the conditions on the hypothesis of the above theorems are not superfluous.
Example 17. Let $S$ be a 2-torsion free left near ring and let

$$N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}.$$  \hfill (56)

Define $F, d, g : N \to N$ by

$$F(\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} ;$$

$$d(\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$g(\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} .$$ \hfill (57)

It can be checked that $N$ is a left near ring and $F$ is a generalized semiderivation of $N$ associated with a semiderivation $d$ and onto map $g$ associated with $d$ such that $g(xy) = g(x)g(y)$ for all $x, y \in N$. If $F$ acts as a homomorphism on $N$, then either $F$ is identity map or $F = 0$.

**Proof.** By the hypothesis

$$F(xy) = d(x)g(y) + xF(y) = F(x)F(y),$$

$\forall x, y \in N. \hfill (58)$

Replacing $y$ by $yz$ in the above relation, we get

$$F(xyz) = d(x)g(yz) + xF(yz),$$

$\forall x, y, z \in N. \hfill (59)$

This implies that

$$(d(x)g(y) + xF(y))F(z) = d(x)g(yz) + x(d(y)g(z) + yF(z)), \hfill (60)$$

$\forall x, y, z \in N.$

Using Lemma 5(ii), we obtain

$$d(x)g(y)F(z) + xF(y)F(z) = d(x)g(yz) + xzd(y)g(z) + xzdF(z), \hfill (61)$$

$\forall x, y, z \in N.$

This implies that

$$d(x)g(y)F(z) + xF(y)g(z) + xyF(z) = d(x)g(yz) + xzd(y)g(z) + xzdF(z),$$

$\forall x, y, z \in N.$

Thus

$$d(x)g(y)F(z) + xF(y)g(z) = d(x)g(yz) + xzd(y)g(z) + xzdF(z), \hfill (62)$$

$\forall x, y, z \in N.$

This is because

$$d(x)g(y)F(z) - d(x)g(y)g(z) = d(x)(g(yz) - g(y)g(z)), \hfill (63)$$

$\forall x, y, z \in N.$

Therefore, $d(N) = \{0\}$ or $F(z) = z$ for all $z \in N$.

In the later case $F$ is an identity map. On the other hand suppose that $d(N) = \{0\}$. Then $F(xy) = F(x)F(y)$; that is, $F(x)(y - F(y)) = 0$ for all $x, y \in N$. Replacing $y$ by $zy$, $z \in N$, and noting that $F(z) = zF(y)$, we have $F(x)N(y - F(y)) = \{0\}$ for all $x, y \in N$. Therefore, $F(N) = \{0\}$ or $F$ is an identity map.

**Theorem 18.** Let $N$ be a 3-prime near ring. Suppose that $F$ is a generalized semiderivation of $N$ associated with a semiderivation $d$ and onto map $g$ associated with $d$ such that $g(xy) = g(x)g(y)$ for all $x, y \in N$. If $F$ acts as a homomorphism on $N$, then either $F$ is identity map or $F = 0$.

**Proof.** By the hypothesis

$$F(xy) = d(x)g(y) + xF(y) = F(x)F(y),$$

$\forall x, y \in N. \hfill (58)$

Replacing $y$ by $yz$ in the above relation, we get

$$F(xyz) = d(x)g(yz) + xF(yz),$$

$\forall x, y, z \in N. \hfill (59)$

This implies that

$$(d(x)g(y) + xF(y))F(z) = d(x)g(yz) + x(d(y)g(z) + yF(z)), \hfill (60)$$

$\forall x, y, z \in N.$

Using Lemma 5(ii), we obtain

$$d(x)g(y)F(z) + xF(y)F(z) = d(x)g(yz) + xzd(y)g(z) + xzdF(z), \hfill (61)$$

$\forall x, y, z \in N.$

This implies that

$$d(x)g(y)F(z) + xF(y)g(z) + xyF(z) = d(x)g(yz) + xzd(y)g(z) + xzdF(z),$$

$\forall x, y, z \in N.$

Thus

$$d(x)g(y)F(z) = d(x)g(yz) + xzdF(z), \hfill (63)$$

$\forall x, y, z \in N.$

Therefore, $d(N) = \{0\}$ or $F(z) = z$ for all $z \in N$.

In the later case $F$ is an identity map. On the other hand suppose that $d(N) = \{0\}$. Then $F(xy) = F(x)F(y)$; that is, $F(x)(y - F(y)) = 0$ for all $x, y \in N$. Replacing $y$ by $zy$, $z \in N$, and noting that $F(z) = zF(y)$, we have $F(x)N(y - F(y)) = \{0\}$ for all $x, y \in N$. Therefore, $F(N) = \{0\}$ or $F$ is an identity map.

\hfill $\square$
Theorem 19. Let \( N \) be a 2-torsion free 3-prime near ring. Suppose that \( F \) is a generalized semiderivation of \( N \) associated with a semiderivation \( d \) and onto map \( g \) such that \( g(xy) = g(x)g(y) \) for all \( x, y \in N \). If \( F \) acts as antihomomorphism on \( N \), then \( F = 0 \) or \( F \) is the identity map on \( N \) and \( N \) is a commutative ring.

Proof. By the hypothesis
\[
F(xy) = d(x)g(y) + xF(y) = F(y)F(x),
\]
\( \forall x, y \in N. \) (64)

Thus
\[
F(y)F(x) = d(x)g(y) + xF(y), \quad \forall x, y \in N. \quad (65)
\]
Replacing \( y \) by \( xy \) in the above relation, we obtain
\[
F(xy)F(x) = d(x)g(xy) + xF(xy), \quad \forall x, y \in N,
\]
that is
\[
d(x)g(y)F(x) + xF(y)F(x) = d(x)g(xy) + xF(y)F(x), \quad \forall x, y \in N. \quad (66)
\]
By Lemma 5(ii), we have
\[
d(x)g(y)F(x) + xF(y)F(x) = d(x)g(xy) + xF(y), \quad \forall x, y \in N. \quad (67)
\]
This implies that
\[
d(x)g(y)F(x) = d(x)g(x)g(y), \quad \forall x, y \in N. \quad (68)
\]
Replacing \( y \) by \( yr \) in the above relation, we get
\[
d(x)g(y)g(r)F(x) = d(x)g(x)g(y)g(r), \quad \forall x, y, r \in N. \quad (69)
\]
Using (68) in the above relation, we get
\[
d(x)g(y)g(r)F(x) = d(x)g(y)F(x)g(r), \quad \forall x, y, r \in N. \quad (70)
\]
Since \( g \) is onto, we have
\[
d(x)N[F(x), r] = \{0\}, \quad \forall x, r \in N. \quad (71)
\]
Therefore, either \( d(N) = \{0\} \) or \( F(N) \subseteq Z \). Hence in either case \( F \) acts as a homomorphism by Lemma 4 and Theorem 8 which completes the proof. \( \square \)

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

References