Research Article

The Peak of Noncentral Stirling Numbers of the First Kind

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We locate the peak of the distribution of noncentral Stirling numbers of the first kind by determining the value of the index corresponding to the maximum value of the distribution.

1. Introduction

In 1982, Koutras [1] introduced the noncentral Stirling numbers of the first and second kind as a natural extension of the definition of the classical Stirling numbers, namely, the expression of the factorial $(x)_n$ in terms of powers of x and vice versa. These numbers are, respectively, denoted by $s_a(n, k)$ and $S_a(n, k)$ which are defined by means of the following inverse relations:

$$(t)_n = \sum_{k=0}^n \frac{1}{k!} \left[\frac{d^k}{dt^k} (t)_v \right]_{t=a} (t-a)^k, \quad (1)$$

$$(t-a)^n = \sum_{k=0}^n \frac{1}{k!} \left[\Delta^k (t-a)^n \right]_{t=0} (t)_k, \quad (2)$$

where a, t are any real numbers, n is a nonnegative integer, and

$$s_a(n, k) = \frac{1}{k!} \left[\frac{d^k}{dt^k} (t)_v \right]_{t=a}, \quad (3)$$

$$S_a(n, k) = \frac{1}{k!} \left[\Delta^k (t-a)^n \right]_{t=0}. \quad (4)$$

The numbers satisfy the following recurrence relations:

$$s_a(n+1, k) = s_a(n, k-1) + (a-n) s_a(n, k), \quad (5)$$

$$S_a(n+1, k) = S_a(n, k-1) + (k-a) S_a(n, k) \quad (6)$$

and have initial conditions

$$s_a(0, 0) = 1, \quad s_a(n, 0) = (a)_n, \quad s_a(0, k) = 0, \quad n, k \neq 0,$$

$$S_a(0, 0) = 1, \quad S_a(n, 0) = (-a)^n, \quad S_a(0, k) = 0, \quad n, k \neq 0. \quad (7)$$

It is worth mentioning that for a given negative binomial distribution Y and the sum $X = X_1 + X_2 + \cdots + X_k$ of k independent random variables following the logarithmic distribution, the numbers $s_a(n, k)$ appeared in the distribution of the sum $W = X + Y$, while the numbers $S_a(n, k)$ appeared in the distribution of the sum $Z = \hat{X} + \hat{Y}$ where $\hat{X}$ is the sum of k independent random variables following the truncated Poisson distribution away from zero and $\hat{Y}$ is a Poisson random variable. More precisely, the probability distributions of W and Z are given, respectively, by

$$P[W = n] = \frac{k!}{(1-\theta)^{-s} (-\log(1-\theta))^k} \frac{\theta^n}{n!} (-1)^{n-k} s_{-s}(n, k),$$

$$P[Z = n] = \frac{k!}{e^m (e^l - 1)^k} \frac{l^n}{n!} (-1)^{n-k} S_{-m/l}(n, k). \quad (8)$$

For a more detailed discussion of noncentral Stirling numbers, one may see [1].

Determining the location of the maximum of Stirling numbers is an interesting problem to consider. In [2], Mezö

obtained results for the so-called r -Stirling numbers which are natural generalizations of Stirling numbers. He showed that the sequences of r -Stirling numbers of the first and second kinds are strictly log-concave. Using the theorem of Erdős and Stone [3] he was able to establish that the largest index for which the sequence of r -Stirling numbers of the first kind assumes its maximum is given by the approximation

$$K_{n,r}^{(1)} = r + \left[\log \left(\frac{n-1}{r-1} \right) - \frac{1}{r} + o(1) \right]. \quad (8)$$

Following the methods of Mezö, we establish strict log-concavity and hence unimodality of the sequence of noncentral Stirling numbers of the first kind and, eventually, obtain an estimating index at which the maximum element of the sequence of noncentral Stirling numbers of the first kind occurs.

2. Explicit Formula

In this section, we establish an explicit formula in symmetric function form which is necessary in locating the maximum of noncentral Stirling numbers of the first kind.

Let $f_i(x)$, $i = 1, 2, \dots, n$ be differentiable functions and let $F_n(x) = \prod_{i=1}^n f_i(x)$. It can easily be verified that, for all $n \geq 3$,

$$F'_n(x) = \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_{n-1} \leq n \\ i \in \mathbb{N}_n \setminus \{j_1, j_2, \dots, j_{n-1}\}}} f'_i(x) \prod_{k=1}^{n-1} f_{j_k}(x). \quad (9)$$

Now, consider the following derivative of $(\xi j + a)_n$ when $n = 1, 2$:

$$\begin{aligned} \frac{d}{d\xi} (\xi j + a)_1 &= j, \\ \frac{d}{d\xi} (\xi j + a)_2 &= (\xi j + a) j + (\xi j + a - 1) j. \end{aligned} \quad (10)$$

Then, for $n \geq 3$ and using (9), we get

$$\frac{1}{j^n} \frac{d}{d\xi} (\xi j + a)_n = \sum_{0 \leq j_1 < j_2 < \dots < j_{n-1} \leq n-1} \prod_{k=1}^{n-1} \left(\xi + \frac{a - j_k}{j} \right). \quad (11)$$

Then, we have the following lemma.

Lemma 1. For any nonnegative integers n and k , one has

$$\frac{1}{j^n} \frac{d^k}{d\xi^k} (\xi j + a)_n = \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \frac{k!}{j^{n-k}} \prod_{q=1}^{n-k} \left(\xi + \frac{a - j_q}{j} \right). \quad (12)$$

Proof. We prove by induction on k . For $k = 0$, (12) clearly holds. For $k = 1$, (12) can easily be verified using (11). Suppose for $m \geq 1$,

$$\frac{1}{j^n} \frac{d^m}{d\xi^m} (\xi j + a)_n = \sum_{0 \leq j_1 < j_2 < \dots < j_{n-m} \leq n-1} m! \prod_{q=1}^{n-m} \left(\xi + \frac{a - j_q}{j} \right). \quad (13)$$

Then,

$$\begin{aligned} \frac{1}{j^n} \frac{d^{m+1}}{d\xi^{m+1}} (\xi j + a)_n \\ = m! \sum_{0 \leq j_1 < j_2 < \dots < j_{n-m} \leq n-1} \frac{d}{d\xi} \prod_{q=1}^{n-m} \left(\xi + \frac{a - j_q}{j} \right), \end{aligned} \quad (14)$$

where the sum has $\binom{n}{n-m} = n(n-1)(n-2) \dots (n-m+1)/m!$ terms and its summand

$$\frac{d}{d\xi} \prod_{q=1}^{n-m} \left(\xi + \frac{a - j_q}{j} \right) = \sum_{i_1 < i_2 < \dots < i_{n-m-1}} \prod_{q=1}^{n-m-1} \left(\xi + \frac{a - i_q}{j} \right), \quad (15)$$

$i_q \in \{j_1, j_2, \dots, j_{n-m}\}$ has $\binom{n-m}{n-m-1} = (n-m)(n-m-1)/(n-m-1)! = n-m$ terms. Therefore, the expansion of $(d^{m+1}/d\xi^{m+1})(\xi j + a)_n$ has a total of $n(n-1) \dots (n-m+1)(n-m)/m!$ terms of the form $\prod_{q=1}^{n-m-1} (\xi + (a - j_q)/j)$. However, if the sum is evaluated over all possible combinations $j_1 j_2 \dots j_{n-m-1}$ such that $0 \leq j_1 < j_2 < \dots < j_{n-m-1} \leq n-1$, then the sum has $\binom{n-m}{n-m-1} = n(n-1) \dots (n-m)(n-m-1)/(m+1)!(n-m-1)! = (1/(m+1))(n(n-1) \dots (n-m)/m!)$ distinct terms. It follows that every term $\prod_{q=1}^{n-m-1} (\xi + (a - j_q)/j)$ appears $m+1$ times in the expansion of $(1/j^n)(d^{m+1}/d\xi^{m+1})(\xi j + a)_n$. Thus we have

$$\begin{aligned} \frac{1}{j^n} \frac{d^{m+1}}{d\xi^{m+1}} (\xi j + a)_n \\ = m! \sum_{0 \leq j_1 < j_2 < \dots < j_{n-m-1} \leq n-1} m+1 \prod_{q=1}^{n-m} \left(\xi + \frac{a - j_q}{j} \right) \\ = (m+1)! \sum_{0 \leq j_1 < j_2 < \dots < j_{n-m-1} \leq n-1} \prod_{q=1}^{n-m} \left(\xi + \frac{a - j_q}{j} \right). \end{aligned} \quad (16)$$

□

Lemma 2. Let $\widehat{s}(n, k; a) = (1/k!) \lim_{\xi \rightarrow 0} ((\sum_{j=0}^k (-1)^{k-j} \binom{k}{j}) (d^k/d\xi^k)(\xi j + a)_n)/k!$. Then

$$\widehat{s}(n, k; a) = \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (a - j_q). \quad (17)$$

Proof. Using Lemma 1,

$$\begin{aligned} \widehat{s}(n, k; a) \\ = \frac{1}{k!} \lim_{\xi \rightarrow 0} \left(\left(\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \right) j^n \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} k! \right. \\ \left. \times \prod_{q=1}^{n-k} \left(\xi + \frac{a - j_q}{j} \right) \right) (k!)^{-1}. \end{aligned} \quad (18)$$

Note that $\prod_{q=1}^{n-k} (\xi + (a - j_q)/j) = (1/j^{n-k}) \prod_{q=1}^{n-k} (\xi j + a - j_q)$. Hence, the expression at the right-hand side of (18) becomes

$$\begin{aligned} & \frac{1}{k!} \lim_{\xi \rightarrow 0} \left[\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^k \right. \\ & \quad \times \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (\xi j + a - j_q) \left. \right] \\ &= \frac{1}{k!} \left[\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^k \right. \\ & \quad \times \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (a - j_q) \left. \right], \end{aligned} \quad (19)$$

which boils down to

$$\sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (a - j_q), \quad (20)$$

since

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^k = S(k, k) = 1, \quad (21)$$

where $S(n, k)$ denote the Stirling numbers of the second kind. $\square$

Theorem 3. The noncentral Stirling numbers of the first kind equal

$$s_a(n, k) = \widehat{s}(n, k; a) = \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (a - j_q). \quad (22)$$

Proof. We know that

$$\sum_{0 \leq j_1 < j_2 < \dots < j_{n-k+1} \leq n} \prod_{q=1}^{n-k+1} (a - j_q) \quad (23)$$

is equal to the sum of the products $(a - j_1)(a - j_2) \dots (a - j_{n+1-k})$ where the sum is evaluated overall possible combinations $j_1 j_2 \dots j_{n+1-k}$, $j_i \in \{0, 1, 2, \dots, n\}$. These possible combinations can be divided into two: the combinations with $j_i = n$ for some $i \in \{1, 2, \dots, n - k + 1\}$ and the combinations with $j_i \neq n$ for all $i \in \{0, 1, 2, \dots, n - k + 1\}$. Thus

$$\sum_{0 \leq j_1 < j_2 < \dots < j_{n-k+1} \leq n} \prod_{q=1}^{n-k+1} (a - j_q) \quad (24)$$

is equal to

$$\begin{aligned} & \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k+1} \leq n-1} \prod_{q=1}^{n-k+1} (a - j_q) + (a - n) \\ & \times \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (a - j_q). \end{aligned} \quad (25)$$

This implies that

$$\widehat{s}(n+1, k; a) = \widehat{s}(n, k-1; a) + (a-n) \widehat{s}(n, k; a). \quad (26)$$

This is exactly the triangular recurrence relation in (4) for $s_a(n, k)$. This proves the theorem. $\square$

The explicit formula in Theorem 3 is necessary in locating the peak of the distribution of noncentral Stirling numbers of the first kind. Besides, this explicit formula can also be used to give certain combinatorial interpretation of $s_a(n, k)$.

A 0-1 *tableau*, as defined in [4] by de Médis and Leroux, is a pair $\varphi = (\lambda, f)$, where

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k) \quad (27)$$

is a partition of an integer m , and $f = (f_{ij})_{1 \leq j \leq \lambda_i}$ is a “filling” of the cells of corresponding Ferrers diagram of shape λ with 0's and 1's, such that there is exactly one 1 in each column. Using the partition $\lambda = (5, 3, 3, 2, 1)$ we can construct 60 distinct 0-1 tableaux. One of these 0-1 tableaux is given in the following figure with $f_{14} = f_{15} = f_{23} = f_{31} = f_{42} = 1$, $f_{ij} = 0$ elsewhere ($1 \leq j \leq \lambda_i$):

0	0	0	1	1
0	0	1		
1	0	0		
0	1			
0				

(28)

Also, as defined in [4], an *A-tableau* is a list ϕ of column c of a Ferrers diagram of a partition λ (by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A = (a_i)_{i \geq 0}$, $a_i \in \mathbb{Z}^+ \cup \{0\}$. If $Td^A(h, r)$ is the set of *A-tableaux* with exactly r distinct columns whose lengths are in the set $A = \{a_0, a_1, \dots, a_h\}$, then $|Td^A(h, r)| = \binom{h+1}{r}$. Now, transforming each column c of an *A-tableau* in $Td^A(n-1, n-k)$ into a column of length $\omega(|c|)$, we obtain a new tableau which is called *A_ω-tableau*. If $\omega(|c|) = |c|$, then the *A_ω-tableau* is simply the *A-tableau*. Now, we define an *A_ω(0, 1)-tableau* to be a 0-1 tableau which is constructed by filling up the cells of an *A_ω-tableau* with 0's and 1's such that there is only one 1 in each column. We use $Td^{A_\omega(0,1)}(n-1, n-k)$ to denote the set of such *A_ω(0, 1)-tableaux*.

It can easily be seen that every $(n-k)$ combination $j_1 j_2 \dots j_{n-k}$ of the set $\{0, 1, 2, \dots, n-1\}$ can be represented geometrically by an element ϕ in $Td^A(n-1, n-k)$ with j_i as the length of $(n-k-i+1)$ th column of ϕ where $A = \{0, 1, 2, \dots, n-1\}$. Hence, with $\omega(|c|) = a - |c|$, (22) may be written as

$$s_a(n, k) = \sum_{\phi \in Td^{A_\omega(0,1)}(n-1, n-k)} \prod_{c \in \phi} \omega(|c|). \quad (29)$$

Thus, using (29), we can easily prove the following theorem.

Theorem 4. The number of *A_ω(0, 1)-tableaux* in $Td^{A_\omega(0,1)}(n-1, n-k)$ where $A = \{0, 1, 2, \dots, n-1\}$ such that $\omega(|c|) = a - |c|$ is equal to $s_a(n, k)$.

Let ϕ be an A -tableau in $Td^A(n-1, n-k)$ with $A = \{0, 1, 2, \dots, n-1\}$, and

$$\begin{aligned}\omega_A(\phi) &= \prod_{c \in \phi} \omega(|c|) \\ &= \prod_{i=1}^{n-k} (a - j_i), \quad j_i \in \{0, 1, 2, \dots, n-1\}.\end{aligned}\quad (30)$$

If $a = a_1 + a_2$ for some a_1 and a_2 , then, with $\omega^*(j) = a_2 - j$,

$$\begin{aligned}\omega_A(\phi) &= \prod_{i=1}^{n-k} (a_1 + \omega^*(j_i)) \\ &= \sum_{r=0}^{n-k} a_1^{n-k-r} \sum_{j_1 \leq q_1 < q_2 < \dots < q_r \leq j_{n-k}} \prod_{i=1}^r \omega^*(q_i).\end{aligned}\quad (31)$$

Suppose B_ϕ is the set of all A -tableaux corresponding to ϕ such that for each $\psi \in B_\phi$ either

- ψ has no column whose weight is a_1 , or
- ψ has one column whose weight is a_1 , or
- $\vdots$
- ψ has $n-k$ columns whose weights are a_1 . Then, we may write

$$\omega_A(\phi) = \sum_{\psi \in B_\phi} \omega_A(\psi). \quad (32)$$

Now, if r columns in ψ have weights other than a_1 , then

$$\omega_A(\psi) = a_1^{n-k-r} \prod_{i=1}^r \omega^*(q_i), \quad (33)$$

where $q_1, q_2, \dots, q_r \in \{j_1, j_2, \dots, j_{n-k}\}$. Hence, (29) may be written as

$$s_a(n, k) = \sum_{\phi \in Td^A(n-1, n-k)} \sum_{\psi \in B_\phi} \omega_A(\psi). \quad (34)$$

Note that for each r , there correspond $\binom{n-k}{r}$ tableaux with r distinct columns having weights $\omega^*(q_i)$, $q_i \in \{j_1, j_2, \dots, j_{n-k}\}$. Since $Td^A(n-1, n-k)$ has $\binom{n}{k}$ elements, for each $\phi \in Td^A(n-1, n-k)$, the total number of A -tableaux ψ corresponding to ϕ is

$$\binom{n}{k} \binom{n-k}{r} \quad (35)$$

elements. However, only $\binom{n}{r}$ tableaux in B_ϕ with r distinct columns of weights other than a_1 are distinct. Hence, every distinct tableau ψ appears

$$\frac{\binom{n}{k} \binom{n-k}{r}}{\binom{n}{r}} = \binom{n-r}{k} \quad (36)$$

times in the collection. Consequently, we obtain

$$s_a(n, k) = \sum_{r=0}^{n-k} \binom{n-r}{k} a_1^{n-k-r} \sum_{\psi \in B_r} \prod_{c \in \psi} \omega^*(|c|), \quad (37)$$

where B_r denotes the set of all tableaux ψ having r distinct columns whose lengths are in the set $\{0, 1, 2, \dots, n-1\}$. Reindexing the double sum, we get

$$s_a(n, k) = \sum_{j=k}^n \binom{j}{k} a_1^{j-k} \sum_{\psi^* \in B_{n-j}} \prod_{c \in \psi^*} \omega^*(|c|). \quad (38)$$

Clearly, $B_{n-j} = Td^A(n-1, n-j)$. Thus, using (22), we obtain the following theorem.

Theorem 5. *The numbers $s_a(n, k)$ satisfy the following identity:*

$$s_a(n, k) = \sum_{j=k}^n \binom{j}{k} a_1^{j-k} s_{a_2}(n, j), \quad (39)$$

where $a = a_1 + a_2$ for some numbers a_1 and a_2 .

The next theorem contains certain convolution-type formula for $s_a(n, k)$ which will be proved using the combinatorics of A -tableau.

Theorem 6. *The numbers $s_a(n, k)$ have convolution formula*

$$s_a(m+j, n) = \sum_{k=0}^n s_a(m, k) s_{a-m}(j, n-k). \quad (40)$$

Proof. Suppose that ϕ_1 is a tableau with exactly $m-k$ distinct columns whose lengths are in the set $A_1 = \{0, 1, 2, \dots, m-1\}$ and ϕ_2 is a tableau with exactly $j-n+k$ distinct columns whose lengths are in the set $A_2 = \{m, m+1, m+2, \dots, m+j-1\}$. Then $\phi_1 \in Td^{A_1}(m-1, m-k)$ and $\phi_2 \in Td^{A_2}(j-1, j-n+k)$. Notice that by joining the columns of ϕ_1 and ϕ_2 , we obtain an A -tableau ϕ with $m+j-n$ distinct columns whose lengths are in the set $A = \{0, 1, 2, \dots, m+j-1\}$; that is, $\phi \in Td^A(m+j-1, m+j-n)$. Hence

$$\begin{aligned}& \sum_{\phi \in Td^A(m+j-1, m+j-n)} \omega_A(\phi) \\ &= \sum_{k=0}^n \left\{ \sum_{\phi_1 \in Td^{A_1}(m-1, m-k)} \omega_{A_1}(\phi_1) \right\} \\ & \quad \times \left\{ \sum_{\phi_2 \in Td^{A_2}(j-1, j-n+k)} \omega_{A_2}(\phi_2) \right\}.\end{aligned}\quad (41)$$

Note that

$$\begin{aligned}
 & \sum_{\phi_2 \in Td^{A_2}(j-1, j-n+k)} \omega_{A_2}(\phi_2) \\
 &= \sum_{m \leq g_1 < g_2 < \dots < g_{j-n+k} \leq m+j-1} \prod_{q=1}^{j-n+k} (a - g_q) \\
 &= \sum_{0 \leq g_1 < g_2 < \dots < g_{n-k-j} \leq j-1} \prod_{q=1}^{j-n+k} (a - (m + g_q)) \\
 &= s_{a-m}(j, n-k).
 \end{aligned} \quad (42)$$

Also, using (29), we have

$$\begin{aligned}
 & \sum_{\phi_1 \in Td^{A_1}(m-1, m-k)} \omega_{A_1}(\phi_1) = s_a(m, k) \\
 & \sum_{\phi \in Td^A(m+j-1, m+j-n)} \omega_A(\phi) = s_a(m+j, n).
 \end{aligned} \quad (43)$$

Thus,

$$s_a(m+j, n) = \sum_{k=0}^n s_a(m, k) s_{a-m}(j, n-k). \quad (44)$$

□

The following theorem gives another form of convolution formula.

Theorem 7. The numbers $s_a(n, k)$ satisfy the second form of convolution formula

$$s_a(n+1, m+j+1) = \sum_{k=0}^n s_a(k, n) s_{a-(k+1)}(n-k, j). \quad (45)$$

Proof. Let

ϕ_1 be a tableau with $k-m$ columns whose lengths are in $A_1 = \{0, 1, \dots, k-1\}$,

ϕ_2 be a tableau with $n-k-j$ columns whose lengths are in $A_2 = \{k+1, \dots, n\}$.

Then $\phi_1 \in Td^{A_1}(k-1, k-m)$; $\phi_2 \in Td^{A_2}(n-k-1, n-k-j)$. Using the same argument above, we can easily obtain the convolution formula. □

3. The Maximum of Noncentral Stirling Numbers of the First Kind

We are now ready to locate the maximum of $s_a(n, k)$. First, let us consider the following theorem on Newton's inequality [5] which is a good tool in proving log-concavity or unimodality of certain combinatorial sequences.

Theorem 8. If the polynomial $a_1x + a_2x^2 + \dots + a_nx^n$ has only real zeros then

$$a_k^2 \geq a_{k+1}a_{k-1} \frac{k}{k-1} \frac{n-k+1}{n-k} \quad (k = 2, \dots, n-1). \quad (46)$$

Now, consider the following polynomial:

$$\sum_{k=0}^n s_a(n, k) (t+a)^k. \quad (47)$$

This polynomial is just the expansion of the factorial $\langle t \rangle_n = t(t+1)(t+2)\dots(t+n-1)$ which has real roots $0, -1, -2, \dots, -n+1$. If we replace t by $t-a$, we see at once that the roots of the polynomial $\sum_{k=0}^n s_a(n, k)t^k$ are $a, a-1, \dots, a-n+1$. Applying Newton's Inequality completes the proof of the following theorem.

Theorem 9. The sequence $\{s_a(n, k)\}_{k=0}^n$ is strictly log-concave and, hence, unimodal.

By replacing t with $-t$, the relation in (1) may be written as

$$\langle t \rangle_n = \sum_{k=0}^n (-1)^{n-k} s_a(n, k) (t+a)^k, \quad (48)$$

where $\langle t \rangle_n = t(t+1)(t+2)\dots(t+n-1)$. Note that, from Theorem 3 with $a < 0$,

$$s_a(n, k) = (-1)^{n-k} \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (b + j_q), \quad (49)$$

where $b = -a > 0$. Now, we define the signless noncentral Stirling number of the first kind, denoted by $|s_a(n, k)|$, as

$$\begin{aligned}
 |s_a(n, k)| &= (-1)^{n-k} s_a(n, k) \\
 &= \sum_{0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1} \prod_{q=1}^{n-k} (b + j_q).
 \end{aligned} \quad (50)$$

To introduce the main result of this paper, we need to state first the following theorem of Erdős and Stone [3].

Theorem 10 (see [3]). Let $u_1 < u_2 < \dots$ be an infinite sequence of positive real numbers such that

$$\sum_{i=1}^{\infty} \frac{1}{u_i} = \infty, \quad \sum_{i=1}^{\infty} \frac{1}{u_i^2} < \infty. \quad (51)$$

Denote by $\sum_{n,k}$ the sum of the product of the first n of them taken k at a time and denote by K_n the largest value of k for which $\sum_{n,k}$ assumes its maximum value. Then

$$K_n = n - \left[\sum_{i=1}^n \frac{1}{u_i} - \sum_{i=1}^n \frac{1}{u_i^2} \left(1 + \frac{1}{u_i} \right)^{-1} + o(1) \right]. \quad (52)$$

We also need to recall the asymptotic expansion of harmonic numbers which is given by

$$\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} = \log n + \gamma + o(1), \quad (53)$$

where γ is the Euler-Mascheroni constant.

The following theorem contains a formula that determines the value of the index corresponding to the maximum of the sequence $\{|s_a(n, k)|\}_{k=0}^n$.

Theorem 11. The largest index for which the sequence $\{|s_a(n, k)|\}_{k=0}^n$ assumes its maximum is given by the approximation

$$k_{a,n} = \left\lceil \log \left(\frac{b+n}{b} \right) + o(1) \right\rceil, \quad (54)$$

where $[x]$ is the integer part of x and $b = -a$, $a < 0$.

Proof. Using Theorem 10 and by (50), we see that $|s_a(n, k)| = \sum_{n-1, n-k}^{n-k}$. Denoting by $k_{a,n}$ for which $\sum_{n, n-k}$ is maximum and with $u_1 = b+0, u_2 = b+1, \dots, u_{n-1} = b+n-1$ we have

$$\begin{aligned} k_{a,n} &= \left\lceil \sum_{i=0}^{n-1} \frac{1}{b+i} - \sum_{i=0}^{n-1} \frac{1}{(b+i)^2} \left(1 + \frac{1}{b+i} \right)^{-1} + o(1) \right\rceil \\ &= \left\lceil \sum_{i=0}^{n-1} \frac{1}{b+i} - \sum_{i=0}^{n-1} \frac{1}{(b+i)(b+i+1)} + o(1) \right\rceil \\ &= \left\lceil \sum_{i=0}^{n-1} \frac{1}{b+i+1} + o(1) \right\rceil. \end{aligned} \quad (55)$$

But using (53), we see that

$$\sum_{i=0}^{n-1} \frac{1}{b+i+1} = \log(b+n) - \log b. \quad (56)$$

From this we get

$$k_{a,n} = \left\lceil \log \left(\frac{b+n}{b} \right) + o(1) \right\rceil. \quad (57)$$

□

For the case in which $a > 0$ we will only consider the sequence of noncentral Stirling numbers of the first kind for which $a \geq n$.

Theorem 12. The maximizing index for which the maximum noncentral Stirling number occurs for $a \geq n$ is given by the approximation

$$k_{a,n} = \left\lceil \log \left(\frac{a+1}{a-n+1} \right) + o(1) \right\rceil. \quad (58)$$

Proof. From the definition, for $a \geq n, s_a(n, k) > 0$ and by Theorem 3, $s_a(n, k)$ is the sum of the products $(a-j_1)(a-j_2) \cdots (a-j_{n-k})$ where j_i 's are taken from the set $\{0, 1, 2, \dots, n-1\}$. By Theorem 10, $s_a(n, k) = \sum_{n, n-k}^{n-k}$. Thus with $u_1 = a, u_2 = a-1, \dots, u_{n-1} = a-n+1$ we have

$$\begin{aligned} k_{a,n} &= \left\lceil \sum_{i=0}^{n-1} \frac{1}{a-i} - \sum_{i=0}^{n-1} \frac{1}{(a-i)^2} \left(1 + \frac{1}{a-i} \right)^{-1} + o(1) \right\rceil \\ &= \left\lceil \sum_{i=0}^{n-1} \frac{1}{a-i} - \sum_{i=0}^{n-1} \frac{1}{(a-i)(a-i+1)} + o(1) \right\rceil \\ &= \left\lceil \sum_{i=0}^{n-1} \frac{1}{a-i+1} + o(1) \right\rceil. \end{aligned} \quad (59)$$

TABLE 1: Values of $s_{-1}(n, k)$.

n/k	0	1	2	3
0	1			
1	1			
2	2	1		
3	6	6	1	
4	24	35	10	1
5	120	225	85	15
6	720	1624	735	175
7	5040	13132	6769	1960
8	40320	109584	118124	67284
9	362880	1026576	1172700	723680
10	3628800	10628640	12753576	8409500

Again, using (53), we get

$$k_{a,n} = \left\lceil \log \left(\frac{a+1}{a-n+1} \right) + o(1) \right\rceil. \quad (60)$$

□

Example 13. The maximum element of the sequence $\{s_{-1}(9, k)\}_{k=0}^9$ occurs at (Table 1)

$$\begin{aligned} k_{-1,9} &= \left\lceil \log \left(\frac{1+8}{1} \right) + o(1) \right\rceil \\ &= \lceil \log 9 + o(1) \rceil \\ &\approx 2. \end{aligned} \quad (61)$$

Example 14. The maximum element of the sequence $\{s_{10}(10, k)\}_{k=0}^{10}$ occurs at (Table 2)

$$\begin{aligned} k_{10,10} &= \left\lceil \log \left(\frac{10+1}{10-10+1} \right) + o(1) \right\rceil \\ &= \lceil \log 11 + o(1) \rceil \\ &\approx 2. \end{aligned} \quad (62)$$

We know that the classical Stirling numbers of the first kind are special cases of $s_a(n, k)$ by taking $a = 0$. However, formulas in Theorems 11 and 12 do not hold when $a = 0$. Hence, these formulas are not applicable to determine the maximum of the classical Stirling numbers. Here, we derive a formula that determines the value of the index corresponding to the maximum of the signless Stirling numbers of the first kind.

The signless Stirling numbers of the first kind [6] are the sum of all products of $n-k$ different integers taken from $\{1, 2, 3, \dots, n-1\}$. That is,

$$|s(n, k)| = \sum_{1 \leq i_1 < i_2 < \dots < i_{n-k} \leq n-1} i_1 i_2 \cdots i_{n-k}. \quad (63)$$

TABLE 2: Values of $s_{10}(n, k)$.

n/k	0	1	2	3
0	1			
1	10	1		
2	90	19	1	
3	720	242	27	1
4	5040	2414	431	34
5	30240	19524	5000	635
6	151200	127860	44524	8175
7	604800	662640	305956	77224
8	1814400	2592720	1580508	537628
9	3628800	6999840	5753736	2655764
10	3628800	10628640	12753576	8409500

TABLE 3: Values of $|s(n, k)|$ for $0 \leq n \leq 10$.

$s(n, k)$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
6	0	120	274	225	85	15
7	0	720	1764	1624	735	175
8	0	540	13068	13132	6769	1960
9	0	40320	109584	118124	67284	22449
10	0	362880	1026576	1172700	723680	269325

Using Theorem 10, $|s(n, k)| = \sum_{n-1, n-k}$. We use k_n to denote the largest value of k for which $\sum_{n-1, n-k}$ is maximum. With $u_1 = 1, u_2 = 2, \dots, u_{n-1} = n-1$ we have

$$\begin{aligned}
 k_n &= 1 + \left[\sum_{i=1}^{n-1} \frac{1}{i} - \sum_{i=1}^{n-1} \frac{1}{i^2} \left(1 + \frac{1}{i} \right)^{-1} + o(1) \right] \\
 &= 1 + \left[\sum_{i=1}^{n-1} \frac{1}{i} - \sum_{i=1}^{n-1} \left(\frac{1}{i} - \frac{1}{i+1} \right) + o(1) \right] \\
 &= 1 + \left[\sum_{i=1}^{n-1} \frac{1}{i+1} + o(1) \right].
 \end{aligned} \quad (64)$$

Using (53), we see that

$$\sum_{i=1}^{n-1} \frac{1}{1+i} = \log n - \log 1 + \gamma + o(1). \quad (65)$$

Therefore, we have

$$k_n = [\log n + \gamma + o(1)]. \quad (66)$$

Example 15. It is shown in Table 3 that the maximum value of $|s(n, k)|$ when $n = 7$ occurs at $k = 2$. Using (66), it

can be verified that the maximum element of the sequence $\{|s(7, k)|\}_{k=0}^7$ occurs at

$$\begin{aligned}
 k_7 &= [\log 7 + \gamma + o(1)] \\
 &= [1.95 + 0.5772 + o(1)] \\
 &= [2.53 + o(1)] \\
 &\approx 2.
 \end{aligned} \quad (67)$$

Moreover, when $n = 10$, the maximum value occurs at

$$\begin{aligned}
 k_{10} &= [\log 10 + \gamma + o(1)] \\
 &= [2.30 + 0.5772 + o(1)] \\
 &= [2.8772 + o(1)] \\
 &\approx 3.
 \end{aligned} \quad (68)$$

Recently, a paper by Cakić et al. [7] established explicit formulas for multiparameter noncentral Stirling numbers which are expressible in symmetric function forms. One may then try to investigate the location of the maximum value of these numbers using the Erdős-Stone theorem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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