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Research Article

Double Laplace Transform Method for Solving Space and Time Fractional Telegraph Equations

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Double Laplace transform method is applied to find exact solutions of linear/nonlinear space-time fractional telegraph equations in terms of Mittag-Leffler functions subject to initial and boundary conditions. Furthermore, we give illustrative examples to demonstrate the efficiency of the method.

1. Introduction

The telegraph equation developed by Oliver Heaviside in 1880 is widely used in Science and Engineering. Its applications arise in signal analysis for transmission and propagation of electrical signals and also modelling reaction diffusion.

In recent years, great interest has been developed in fractional differential equation because of its frequent appearance in fluid mechanics, mathematical biology, electrochemistry, and physics. A space-time fractional telegraph equation is obtained from the classical telegraph equation by replacing the time and space derivative terms by fractional derivatives.

There are various methods developed to solve fractional telegraph equations. Orsingher and Xuelei [1] and Orsingher and Beghin [2] considered the space and time fractional telegraph equations and obtained the Fourier transform of their fundamental solution. Momani [3] and Garg and Sharma [4] used Adomian decomposition method developed by Adomian in [5] for solving homogeneous and nonhomogeneous space-time fractional telegraph equation. Chen et al. [6] implemented separation of variables method for deriving the analytical solutions of the nonhomogeneous time fractional telegraph equation under Dirichlet, Neumann, and Robin boundary conditions. Huang [7] considered the combine Fourier-Laplace transform to solve time fractional telegraph equation.

Variational iteration method is proposed by He [8] and used by Sevimlican [9] for solving space and time fractional telegraph equations. Yildirim in [10] used a homotopy perturbation method and Das et al. in [11] used a homotopy analysis method to obtain approximate analytical solution of fractional telegraph equation. Garg et al. [12] and Galue [13] used generalised differential transform method to derive the solution of space-time fractional telegraph equations in terms of Mittag-Leffler functions. Srivastava et al. [14] applied reduced differential transform method to solve Caputo time fractional order hyperbolic telegraph equation.

In recent years, significant attention has been given by many authors towards the study of fractional telegraph equations by using single Laplace transform combined with variational iteration method, homotopy analysis method, and homotopy perturbation method. Khan et al. [15], Kumar et al. [16], and Prakash [17] applied a combination of single Laplace transform and homotopy perturbation method to obtain analytic and approximate solutions of the space-time fractional telegraph equations. Alawad et al. [18] used a combination of single Laplace transform and variational iteration method for finding exact solutions of space-time fractional telegraph equations in terms of Mittag-Leffler functions. Kumar in [19] coupled single Laplace transform and homotopy analysis method for the solution of space-fractional telegraph equation.

To our knowledge, solving fractional partial differential equations using the double Laplace transform is still seen in very little proportionate or no work is available in the literature. So, the main objective of this paper is to find the exact solutions of homogeneous and nonhomogeneous space-time fractional telegraph equations in terms of Mittag-Leffler functions subject to initial and boundary conditions, by means of double Laplace transform.

2. A Brief Introduction of Double Laplace Transform and Caputo Fractional Derivative

Let $f(x,t)$ be a function of two variables x and t defined in the positive quadrant of the xt-plane. The double Laplace transform of the function $f(x,t)$ as given by Sneddon [20] is defined by

$$\begin{aligned} L_x L_t \{f(x,t)\} &= \overline{f}(p,s) \\ &= \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)\, dt\, dx, \end{aligned} \tag{1}$$

whenever that integral exists. Here p and s are complex numbers.

From this definition we deduce

$$\begin{aligned} L_x L_t [f(x) g(t)] &= \overline{f}(p) \overline{g}(s) \\ &= L_x [f(x)] L_t [g(t)]. \end{aligned} \tag{2}$$

The inverse double Laplace transform $L_x^{-1} L_t^{-1} \{\overline{f}(p,s)\} = f(x,t)$ is defined as in [21, 22] by the complex double integral formula:

$$\begin{aligned} L_x^{-1} L_t^{-1} \{\overline{f}(p,s)\} &= f(x,t) \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} dp \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} \overline{f}(p,s)\, ds, \end{aligned} \tag{3}$$

where $\overline{f}(p,s)$ must be an analytic function for all p and s in the region defined by the inequalities $\operatorname{Re} p \geq c$ and $\operatorname{Re} s \geq d$, where c and d are real constants to be chosen suitably.

The double Laplace transform formulas for the partial derivatives of an arbitrary integer order as in [23] are

$$\begin{aligned} L_x L_t \left\{ \frac{\partial^m f(x,t)}{\partial x^m} \right\} &= p^m \overline{f}(p,s) \\ &\quad - \sum_{j=0}^{m-1} p^{m-1-j} L_t \left\{ \frac{\partial^j f(0,t)}{\partial x^j} \right\}, \\ L_x L_t \left\{ \frac{\partial^n f(x,t)}{\partial t^n} \right\} &= s^n \overline{f}(p,s) \\ &\quad - \sum_{k=0}^{n-1} s^{n-1-k} L_x \left\{ \frac{\partial^k f(x,0)}{\partial t^k} \right\}. \end{aligned} \tag{4}$$

Definition 1. The Caputo fractional derivative of function $u(x,t)$ is defined in [18] as

$$\begin{aligned} \frac{\partial^\alpha f(x,t)}{\partial x^\alpha} &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\xi)^{m-\alpha-1} \frac{\partial^m f(\xi,t)}{\partial \xi^m} d\xi, \\ &\qquad m-1 < \alpha \leq m,\ m \in N, \\ \frac{\partial^\beta f(x,t)}{\partial t^\beta} &= \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} \frac{\partial^n f(x,\tau)}{\partial \tau^n} d\tau, \\ &\qquad n-1 < \beta \leq n,\ n \in N. \end{aligned} \tag{5}$$

The double Laplace transform formulas for the partial fractional Caputo derivatives as in [23] are

$$\begin{aligned} L_x L_t \left\{ \frac{\partial^\alpha f(x,t)}{\partial x^\alpha} \right\} &= p^\alpha \overline{f}(p,s) \\ &\quad - \sum_{j=0}^{m-1} p^{\alpha-1-j} L_t \left\{ \frac{\partial^j f(0,t)}{\partial x^j} \right\}, \\ L_x L_t \left\{ \frac{\partial^\beta f(x,t)}{\partial t^\beta} \right\} &= s^\beta \overline{f}(p,s) \\ &\quad - \sum_{k=0}^{n-1} s^{\beta-1-k} L_x \left\{ \frac{\partial^k f(x,0)}{\partial t^k} \right\}. \end{aligned} \tag{6}$$

Definition 2. The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad t, \beta \in \mathbb{C},\ \Re(\alpha) > 0. \tag{7}$$

The single Laplace transform of the function $t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha)$ takes the form

$$L_t \left\{ t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) \right\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \quad \text{for } |\lambda| < |s^\alpha|. \tag{8}$$

3. Double Laplace Transform Method

Consider the following general multiterms fractional telegraph equation as in [18]:

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} &= a \frac{\partial^\beta u(x,t)}{\partial t^\beta} + b \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + c u(x,t) \\ &\quad + h(x,t), \\ &\qquad 1 < \alpha, \beta \leq 2,\ 0 < \gamma \leq 1,\ x, t \geq 0, \end{aligned} \tag{9}$$

with initial conditions,

$$\begin{aligned} u(x,0) &= f_1(x), \\ u_t(x,0) &= f_2(x), \end{aligned} \tag{10}$$

and boundary conditions,

$$\begin{aligned} u(0,t) &= g_1(t), \\ u_x(0,t) &= g_2(t). \end{aligned} \tag{11}$$

Here a, b, c are constants and $h(x,t)$ is given function.

Applying the double Laplace transform on both sides of (9), we get

$$\begin{aligned} p^{\alpha}\overline{u}(p,s) - p^{\alpha-1}\overline{u}(0,s) - p^{\alpha-2}\overline{u_x}(0,s) \\ = a\left[s^{\beta}\overline{u}(p,s) - s^{\beta-1}\overline{u}(p,0) - s^{\beta-2}\overline{u_t}(p,0)\right] \\ + b\left[s^{\gamma}\overline{u}(p,s) - s^{\gamma-1}\overline{u}(p,0)\right] + c\overline{u}(p,s) \\ + \overline{h}(p,s), \end{aligned} \tag{12}$$

where $\overline{h}(p,s) = L_x L_t\{h(x,t)\}$.

Further, applying single Laplace transform to initial (10) and boundary conditions (11), we get

$$\begin{aligned} \overline{u}(p,0) &= \overline{f_1}(p), \\ \overline{u_t}(p,0) &= \overline{f_2}(p), \\ \overline{u}(0,s) &= \overline{g_1}(s), \\ \overline{u_x}(0,s) &= \overline{g_2}(s). \end{aligned} \tag{13}$$

By substituting (13) in (12) and simplifying, we obtain

$$\begin{aligned} \overline{u}(p,s) = \frac{1}{(p^{\alpha} - as^{\beta} - bs^{\gamma} - c)}\left[p^{\alpha-1}\overline{g_1}(s)\right. \\ + p^{\alpha-2}\overline{g_2}(s) - as^{\beta-1}\overline{f_1}(p) - as^{\beta-2}\overline{f_2}(p) \\ \left. - bs^{\gamma-1}\overline{f_1}(p) + \overline{h}(p,s)\right]. \end{aligned} \tag{14}$$

Applying inverse double Laplace transform to (14), we obtain the solution of (9) in the form

$$\begin{aligned} u(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(p^{\alpha} - as^{\beta} - bs^{\gamma} - c)}\left[p^{\alpha-1}\overline{g_1}(s)\right.\right. \\ + p^{\alpha-2}\overline{g_2}(s) - as^{\beta-1}\overline{f_1}(p) - as^{\beta-2}\overline{f_2}(p) \\ \left.\left. - bs^{\gamma-1}\overline{f_1}(p) + \overline{h}(p,s)\right]\right]. \end{aligned} \tag{15}$$

Here we assume that the inverse double Laplace transform of each term in the right side of (15) exists.

4. Illustrative Examples

In this section, we demonstrate the applicability of the previous method by giving examples.

Example 1. By substituting $a = 1$, $b = 1$, $c = 1$, $\beta = 2$, $\gamma = 1$, and $h(x,t) = 0$ in (9),

$$\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t), \quad 1 < \alpha \le 2,\ x,t \ge 0, \tag{16}$$

subject to the initial and boundary conditions,

$$\begin{aligned} u(x,0) &= f_1(x) = E_{\alpha}(x^{\alpha}) + xE_{\alpha,2}(x^{\alpha}), \\ u_t(x,0) &= f_2(x) = -\left[E_{\alpha}(x^{\alpha}) + xE_{\alpha,2}(x^{\alpha})\right], \end{aligned} \tag{17}$$

$$\begin{aligned} u(0,t) &= g_1(t) = e^{-t}, \\ u_x(0,t) &= g_2(t) = e^{-t}, \end{aligned} \tag{18}$$

a homogeneous space-fractional telegraph equation.

Taking single Laplace transform to initial (17) and boundary conditions (18), we get

$$\begin{aligned} \overline{f_1}(p) &= \left(\frac{1}{p} + \frac{1}{p^2}\right)\frac{p^{\alpha}}{p^{\alpha}-1}, \\ \overline{f_2}(p) &= -\left(\frac{1}{p} + \frac{1}{p^2}\right)\frac{p^{\alpha}}{p^{\alpha}-1}, \\ \overline{g_1}(s) &= \overline{g_2}(s) = \frac{1}{s+1}. \end{aligned} \tag{19}$$

Substituting above in (15), we get solution of (16):

$$\begin{aligned} u(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(p^{\alpha} - s^2 - s - 1)}\left[p^{\alpha-1}\frac{1}{s+1}\right.\right. \\ + p^{\alpha-2}\frac{1}{s+1} - s\left(\frac{1}{p} + \frac{1}{p^2}\right)\frac{p^{\alpha}}{p^{\alpha}-1} \\ \left.\left. + \left(\frac{1}{p} + \frac{1}{p^2}\right)\frac{p^{\alpha}}{p^{\alpha}-1} - \left(\frac{1}{p} + \frac{1}{p^2}\right)\frac{p^{\alpha}}{p^{\alpha}-1}\right]\right]. \end{aligned} \tag{20}$$

Simplifying, we obtain

$$\begin{aligned} u(x,t) &= L_x^{-1}L_t^{-1}\left[\frac{1}{(s+1)}\left(\frac{1}{p} + \frac{1}{p^2}\right)\frac{p^{\alpha}}{p^{\alpha}-1}\right], \\ u(x,t) &= e^{-t}\left[E_{\alpha}(x^{\alpha}) + xE_{\alpha,2}(x^{\alpha})\right], \end{aligned} \tag{21}$$

which agrees with the solution already obtained in [18].

If we take $\alpha = 2$ then we get the exact solution of standard telegraph equation:

$$u(x,t) = e^{x-t}. \tag{22}$$

Example 2. By substituting $a = 1$, $b = 1$, $c = 1$, $\beta = 2$, $\gamma = 1$, and $h(x,t) = -x^2 - t + 1$ in (9),

$$\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) - x^2 - t + 1, \quad 1 < \alpha \le 2,\ x,t \ge 0, \tag{23}$$

subject to the initial and boundary conditions,

$$\begin{aligned} u(x,0) &= f_1(x) \\ &= x^2 - 2 - 2x^2E_{\alpha,3}(x^{\alpha}) + 2E_{\alpha,1}(x^{\alpha}), \\ u_t(x,0) &= f_2(x) = 1, \end{aligned} \tag{24}$$

$$\begin{aligned} u(0,t) &= g_1(t) = t, \\ u_x(0,t) &= g_2(t) = 0, \end{aligned} \tag{25}$$

a space-fractional nonhomogeneous telegraph equation.

Taking single Laplace transform to initial (24) and boundary conditions (25), we get

$$\begin{aligned}\overline{f_1}(p) &= \frac{2}{p^3} - \frac{2}{p} - 2\frac{p^{\alpha-3}}{p^\alpha - 1} + 2\frac{p^{\alpha-1}}{p^\alpha - 1} \\ &= \frac{2\left(p^2-1\right)}{p^3\left(p^\alpha - 1\right)}, \\ \overline{f_2}(p) &= \frac{1}{p}, \\ \overline{g_1}(s) &= \frac{1}{s^2}, \\ \overline{g_2}(s) &= 0.\end{aligned} \tag{26}$$

Taking double Laplace transform of $h(x,t)$, we have

$$\overline{h}(p,s) = \frac{-2}{p^3 s} - \frac{1}{ps^2} + \frac{1}{ps}. \tag{27}$$

Substituting above in (15), we get solution of (23):

$$\begin{aligned}u(x,t) = L_x^{-1}L_t^{-1}\Bigg[\frac{1}{\left(p^\alpha - s^2 - s - 1\right)}\Bigg[p^{\alpha-1}\frac{1}{s^2} \\ - s\frac{2\left(p^2-1\right)}{p^3\left(p^\alpha-1\right)} - \frac{1}{p} - \frac{2\left(p^2-1\right)}{p^3\left(p^\alpha-1\right)} - \frac{2}{p^3 s} - \frac{1}{ps^2} \\ + \frac{1}{ps}\Bigg]\Bigg].\end{aligned} \tag{28}$$

Rearranging, we have

$$\begin{aligned}u(x,t) = L_x^{-1}L_t^{-1}\Bigg[\frac{1}{\left(p^\alpha - s^2 - s - 1\right)}\Bigg[\Bigg[\frac{p^\alpha}{ps^2} - \frac{1}{p} \\ - \frac{1}{ps^2} - \frac{1}{ps}\Bigg] + \Bigg[\frac{-2s\left(p^2-1\right)}{p^3\left(p^\alpha-1\right)} - \frac{2\left(p^2-1\right)}{p^3\left(p^\alpha-1\right)} \\ - \frac{2}{p^3 s} + \frac{2}{ps}\Bigg]\Bigg]\Bigg].\end{aligned} \tag{29}$$

Simplifying, we obtain

$$\begin{aligned}u(x,t) &= L_x^{-1}L_t^{-1}\left[\frac{1}{ps^2} + \frac{2\left(p^2-1\right)}{p^3 s\left(p^\alpha-1\right)}\right], \\ u(x,t) &= t + x^2 - 2 - 2x^2 E_{\alpha,3}\left(x^\alpha\right) + 2E_{\alpha,1}\left(x^\alpha\right),\end{aligned} \tag{30}$$

which agrees with the solution already obtained in [18] for $\gamma = 1$.

For $\alpha = 2$, then $u(x,t) = t + x^2$.

Example 3. By substituting $a = 1$, $b = 1$, $c = 0$, $\alpha = 2$, $\gamma = 1$, and $h(x,t) = -2t(x^2 - x)(t^{1-\beta}/\Gamma(3-\beta) + 1) + 2t^2$ in (9),

$$\begin{aligned}\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial u(x,t)}{\partial t} \\ = \frac{\partial^2 u(x,t)}{\partial x^2} + 2t\left(x^2 - x\right)\left(\frac{t^{1-\beta}}{\Gamma(3-\beta)} + 1\right) - 2t^2, \\ 1 < \beta \le 2,\ x,t \ge 0,\end{aligned} \tag{31}$$

subject to the initial and boundary conditions,

$$\begin{aligned}u(x,0) &= f_1(x) = 0, \\ u_t(x,0) &= f_2(x) = 0,\end{aligned} \tag{32}$$

$$\begin{aligned}u(0,t) &= g_1(t) = 0, \\ u_x(0,t) &= g_2(t) = -t^2,\end{aligned} \tag{33}$$

a time fractional telegraph equation in [24].

Taking single Laplace transform to initial (32) and boundary conditions (33), we get

$$\begin{aligned}\overline{f_1}(p) = \overline{f_2}(p) = \overline{g_1}(s) = 0, \\ \overline{g_2}(s) = \frac{-2}{s^3}.\end{aligned} \tag{34}$$

Taking double Laplace transform of $h(x,t)$, we have

$$\overline{h}(p,s) = -2\left(\frac{2}{p^3} - \frac{1}{p^2}\right)\left(\frac{1}{s^{3-\beta}} + \frac{1}{s^2}\right) + 2\frac{2}{ps^3}. \tag{35}$$

Substituting above in (15), we get solution of (31):

$$\begin{aligned}u(x,t) = L_x^{-1}L_t^{-1}\Bigg[\frac{1}{\left(p^2 - s^\beta - s\right)}\Bigg[\frac{-2}{s^3} \\ - 2\left(\frac{2}{p^3} - \frac{1}{p^2}\right)\left(\frac{1}{s^{3-\beta}} + \frac{1}{s^2}\right) + 2\frac{2}{ps^3}\Bigg]\Bigg].\end{aligned} \tag{36}$$

Simplifying, we obtain

$$\begin{aligned}u(x,t) &= L_x^{-1}L_t^{-1}\left[\left(\frac{2}{p^3} - \frac{1}{p^2}\right)\frac{2}{s^3}\right], \\ u(x,t) &= \left(x^2 - x\right)t^2.\end{aligned} \tag{37}$$

Example 4. By substituting $a = 1$, $b = 1$, $c = 1$, $\alpha = 2$, and $h(x,t) = 0$ in (9),

$$\begin{aligned}\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \\ 1 < \beta \le 2,\ \frac{1}{2} < \gamma \le 1,\ x,t \ge 0,\end{aligned} \tag{38}$$

subject to the initial and boundary conditions,

$$\begin{aligned}u(x,0) &= f_1(x) = 0, \\ u_t(x,0) &= f_2(x) = e^x,\end{aligned} \tag{39}$$

$$\begin{aligned}u(0,t) &= g_1(t) = tE_{\beta-\gamma,2}\left(-t^{\beta-\gamma}\right), \\ u_x(0,t) &= g_2(t) = tE_{\beta-\gamma,2}\left(-t^{\beta-\gamma}\right),\end{aligned} \tag{40}$$

a homogeneous time fractional telegraph equation in [25].

Taking single Laplace transform to initial (39) and boundary conditions (40), we get

$$\overline{f_1}(p) = 0,$$
$$\overline{f_2}(p) = \frac{1}{p-1}, \tag{41}$$
$$\overline{g_1}(s) = \overline{g_2}(s) = \frac{s^{\beta-\gamma-2}}{s^{\beta-\gamma}+1}.$$

Substituting above in (15), we get solution of (38):

$$u(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(p^2-s^\beta-s^\gamma-1)}\left[p\frac{s^{\beta-\gamma-2}}{(s^{\beta-\gamma}+1)} + \frac{s^{\beta-\gamma-2}}{(s^{\beta-\gamma}+1)} - s^{\beta-2}\frac{1}{(p-1)}\right]\right]. \tag{42}$$

Simplifying, we obtain

$$u(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(p-1)}\frac{s^{\beta-\gamma-2}}{(s^{\beta-\gamma}+1)}\right],$$
$$u(x,t) = e^x t E_{\beta-\gamma,2}\left(-t^{\beta-\gamma}\right), \tag{43}$$

which agrees with the solution already obtained in [25].

Example 5. By substituting $a = b = c = 1, \alpha = 2$, and $h(x,t) = -\sinh x(t^n/\Gamma(n+1))$ in (9),

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial^{\beta-1}u(x,t)}{\partial t^{\beta-1}} + u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \sinh x\frac{t^n}{\Gamma(n+1)}, \quad 1 < \beta \le 2,\ x,t \ge 0, \tag{44}$$

subject to the initial and boundary conditions,

$$u(x,0) = f_1(x) = 0,$$
$$u_t(x,0) = f_2(x) = 0, \tag{45}$$

$$u(0,t) = g_1(t) = 0,$$
$$u_x(0,t) = g_2(t) = t^{n+\beta}E_{1,n+\beta+1}(-t), \tag{46}$$

a nonhomogeneous time fractional telegraph equation in [25].

Taking single Laplace transform to initial (45) and boundary conditions (46), we get

$$\overline{f_1}(p) = \overline{f_2}(p) = 0,$$
$$\overline{g_1}(s) = 0, \tag{47}$$
$$\overline{g_2}(s) = \frac{s^{-n-\beta}}{s+1}.$$

Taking double Laplace transform of $h(x,t)$, we have

$$\overline{h}(p,s) = \frac{-1}{(p^2-1)s^{n+1}}. \tag{48}$$

Substituting above in (15), we get solution of (44):

$$u(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(p^2-s^\beta-s^{\beta-1}-1)}\left[\frac{s^{-n-\beta}}{(s+1)} - \frac{1}{(p^2-1)s^{n+1}}\right]\right]. \tag{49}$$

Simplifying, we obtain

$$u(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(p^2-1)}\frac{s^{-n-\beta}}{(s+1)}\right],$$
$$u(x,t) = (\sinh x)\, t^{n+\beta}E_{1,n+\beta+1}(-t), \tag{50}$$

which agrees with the solution already obtained in [25].

Example 6. Consider the following space-fractional-order nonlinear telegraph equation:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u^2(x,t) - e^{-2t}\left(x-x^2\right)^2 - 2e^{-t}\frac{x^{2-\alpha}}{\Gamma(3-\alpha)}, \quad 1 < \alpha \le 2,\ x,t \ge 0, \tag{51}$$

under the initial conditions,

$$u(x,0) = -u_t(x,0) = x - x^2, \tag{52}$$

and boundary conditions,

$$u(0,t) = 0,$$
$$u_x(0,t) = e^{-t}. \tag{53}$$

Applying the double Laplace transform on both sides of (51), we get

$$p^\alpha\overline{u}(p,s) - p^{\alpha-1}\overline{u}(0,s) - p^{\alpha-2}\overline{u_x}(0,s) = s^2\overline{u}(p,s) - s\overline{u}(p,0) - \overline{u_t}(p,0) + s\overline{u}(p,s) - \overline{u}(p,0) - \frac{2}{(s+1)p^{3-\alpha}} + L_xL_t\left[u^2(x,t) - e^{-2t}\left(x-x^2\right)^2\right]. \tag{54}$$

Further, applying single Laplace transform to initial (52) and boundary conditions (53), we get

$$\overline{u}(p,0) = -\overline{u_t}(p,0) = \frac{1}{p^2} - \frac{2}{p^3},$$
$$\overline{u}(0,s) = 0, \tag{55}$$
$$\overline{u_x}(0,s) = \frac{1}{s+1}.$$

By substituting (55) in (54) and simplifying, we obtain

$$\begin{aligned}\left(p^{\alpha}-s^{2}-s\right)\overline{u}\left(p,s\right)&=\frac{p^{\alpha}}{p^{2}\left(s+1\right)}-s\left[\frac{1}{p^{2}}-\frac{2}{p^{3}}\right]+\left[\frac{1}{p^{2}}-\frac{2}{p^{3}}\right]\\&\quad-\left[\frac{1}{p^{2}}-\frac{2}{p^{3}}\right]-\frac{2p^{\alpha}}{\left(s+1\right)p^{3}}\\&\quad+L_{x}L_{t}\left[u^{2}\left(x,t\right)-e^{-2t}\left(x-x^{2}\right)^{2}\right],\end{aligned}\tag{56}$$

$$\begin{aligned}\left(p^{\alpha}-s^{2}-s\right)\overline{u}\left(p,s\right)&=\left[\frac{p^{\alpha}}{p^{2}\left(s+1\right)}-\frac{s}{p^{2}}\right]-\left[\frac{2p^{\alpha}}{\left(s+1\right)p^{3}}-\frac{2s}{p^{3}}\right]\\&\quad+L_{x}L_{t}\left[u^{2}\left(x,t\right)-e^{-2t}\left(x-x^{2}\right)^{2}\right],\end{aligned}\tag{57}$$

$$\begin{aligned}\overline{u}\left(p,s\right)&=\left[\frac{1}{p^{2}}-\frac{2}{p^{3}}\right]\frac{1}{\left(s+1\right)}\\&\quad+\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}L_{x}L_{t}\left[u^{2}\left(x,t\right)-e^{-2t}\left(x-x^{2}\right)^{2}\right].\end{aligned}$$

Applying inverse double Laplace transform of (57), we get

$$\begin{aligned}u\left(x,t\right)&=\left(x-x^{2}\right)e^{-t}+L_{x}^{-1}L_{t}^{-1}\left[\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}\right.\\&\quad\left.\cdot L_{x}L_{t}\left[u^{2}\left(x,t\right)-e^{-2t}\left(x-x^{2}\right)^{2}\right]\right].\end{aligned}\tag{58}$$

Now we apply the Iterative method as in [26];

$$u\left(x,t\right)=\sum_{i=0}^{\infty}u_{i}\left(x,t\right).\tag{59}$$

Substituting (59) in (58), we get

$$\begin{aligned}\sum_{i=0}^{\infty}u_{i}\left(x,t\right)&=\left(x-x^{2}\right)e^{-t}+L_{x}^{-1}L_{t}^{-1}\left[\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}\right.\\&\quad\left.\cdot L_{x}L_{t}\left[\left[\sum_{i=0}^{\infty}u_{i}\left(x,t\right)\right]^{2}-e^{-2t}\left(x-x^{2}\right)^{2}\right]\right].\end{aligned}\tag{60}$$

The nonlinear term N is decomposed as

$$\begin{aligned}&\left[\sum_{i=0}^{\infty}u_{i}\left(x,t\right)\right]^{2}\\&\quad=\left[u_{0}\left(x,t\right)\right]^{2}\\&\qquad+\sum_{i=1}^{\infty}\left[\left[\sum_{k=0}^{i}u_{k}\left(x,t\right)\right]^{2}-\left[\sum_{k=0}^{i-1}u_{k}\left(x,t\right)\right]^{2}\right].\end{aligned}\tag{61}$$

Substituting (61) in (60), we get

$$\begin{aligned}\sum_{i=0}^{\infty}u_{i}\left(x,t\right)&=\left(x-x^{2}\right)e^{-t}+L_{x}^{-1}L_{t}^{-1}\left[\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}\right.\\&\quad\left.\cdot L_{x}L_{t}\left[\left[u_{0}\left(x,t\right)\right]^{2}-e^{-2t}\left(x-x^{2}\right)^{2}\right]\right]\\&\quad+L_{x}^{-1}L_{t}^{-1}\left[\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}\right.\\&\quad\left.\cdot L_{x}L_{t}\left[\sum_{i=1}^{\infty}\left[\left[\sum_{k=0}^{i}u_{k}\left(x,t\right)\right]^{2}-\left[\sum_{k=0}^{i-1}u_{k}\left(x,t\right)\right]^{2}\right]\right]\right].\end{aligned}\tag{62}$$

Then we define the recurrence relations as

$$\begin{aligned}u_{0}\left(x,t\right)&=\left(x-x^{2}\right)e^{-t},\\u_{1}\left(x,t\right)&=L_{x}^{-1}L_{t}^{-1}\left[\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}L_{x}L_{t}\left[\left[u_{0}\left(x,t\right)\right]^{2}\right.\right.\\&\quad\left.\left.-e^{-2t}\left(x-x^{2}\right)^{2}\right]\right]=0,\\u_{2}\left(x,t\right)&=L_{x}^{-1}L_{t}^{-1}\left[\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}\right.\\&\quad\left.\cdot L_{x}L_{t}\left[\left[u_{0}\left(x,t\right)+u_{1}\left(x,t\right)\right]^{2}-\left[u_{0}\left(x,t\right)\right]^{2}\right]\right]=0,\\u_{3}\left(x,t\right)&=L_{x}^{-1}L_{t}^{-1}\left[\frac{1}{\left(p^{\alpha}-s^{2}-s\right)}\right.\\&\quad\cdot L_{x}L_{t}\left[\left[u_{0}\left(x,t\right)+u_{1}\left(x,t\right)+u_{2}\left(x,t\right)\right]^{2}\right.\\&\quad\left.\left.-\left[u_{0}\left(x,t\right)+u_{1}\left(x,t\right)\right]^{2}\right]\right]=0,\end{aligned}\tag{63}$$

and so on.

Therefore, we obtain the solution of (51) as follows:

$$u\left(x,t\right)=e^{x-2t}.\tag{64}$$

This is the required exact solution of (51).

5. Conclusion

We have applied double Laplace transform to obtain exact solutions of linear/nonlinear space-time fractional telegraph equations. All of the examples considered show that double Laplace transform method is capable of reducing the volume of computational work as compared to other methods. It may be concluded that DLT technique solves the problems without using Adomian polynomials, Lagrange multiplier value, He's polynomials, and small parameters.

Competing Interests

The authors declare no competing interests regarding the publication of this paper.

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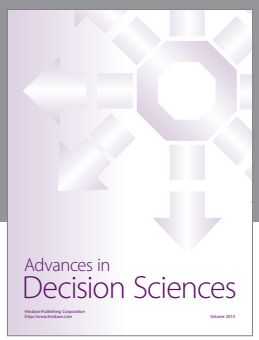

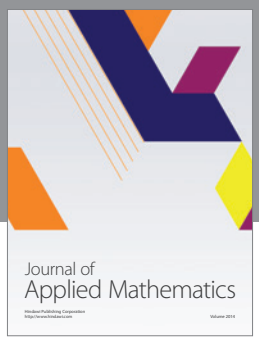

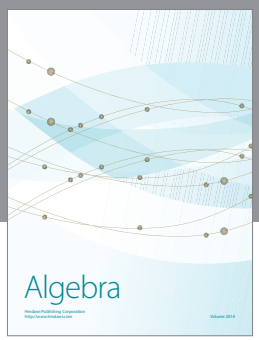

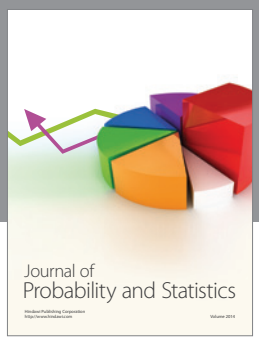

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