Research Article

Existence Results for Some Equilibrium Problems Involving α -Monotone Bifunction

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This paper deals with some existence results of equilibrium problems (EP_{Ψ}) on convex and closed sets (either bounded or unbounded) in Banach spaces. Moreover, an application to the existence of solution for a differential inclusion is given.

1. Introduction and Preliminaries

Equilibrium problems provide a unifying framework for many important problems, such as optimization problems, variational inequality problems, complementarity problems, minimax inequality problems, and fixed point problems. They have been widely applied to study real word applications, arising in economics, mechanics, and engineering science. In recent decades, many results concerning the existence of solutions for equilibrium problems and vector equilibrium problems have been established (see Ansari and Yao [1], Bianchi and Schaible [2], and Blum and Oettli [3]). Many problems of practical interest in optimization, economics, physics, mechanics, and engineering sciences involve equilibrium in their description. Because of their wide applicability, equilibrium problems and mixed equilibrium problems have been investigated and generalized by a number of authors (see [4–6]).

The generalized monotonicity plays an important role in the study of equilibrium problems. In recent years, a number of authors have proposed many essential generalizations of monotonicity, such as α -monotonicity, relaxed monotonicity, relaxed η - α monotonicity, and quasimonotonicity (see [7–10]).

Let K be a nonempty subset of a real reflexive Banach space. Let $F, \Psi, \alpha : K \times K \rightarrow \mathbb{R}$ be three real-valued functions; F is equilibrium function; that is, $F(x, x) = 0$.

We consider the following generalized equilibrium problem (for short, (EP_{Ψ})) is to find $x \in K$ such that

$$F(x, y) + \Psi(x, y) \geq 0 \quad \forall y \in K. \quad (1)$$

In particular cases, one can obtain the following:

- (1) If $\Psi \equiv 0$ then problem (1) is reduced to the classical equilibrium problem (for short, EP), which is to find $x \in K$ such that $F(x, y) \geq 0 \quad \forall y \in K$; see [3].
- (2) If $\Psi(x, y) = \Psi(y) - \Psi(x) \quad \forall y \in K$ then problem (1) is reduced to the mixed equilibrium problem (for short, MEP); see [11].
- (3) If $\Psi(x, y) = \langle Ax, y - x \rangle \quad \forall y \in K$ then problem (1) is reduced to the generalized equilibrium problem (for short, GEP); see [12].

Here, a generalization of monotone bifunction is introduced.

Definition 1. A bifunction $F : K \times K \rightarrow \mathbb{R}$ is called α -monotone if

$$F(x, y) + F(y, x) + \alpha(x, y) \leq 0 \quad \forall x, y \in K. \quad (2)$$

The notion of a KKM map and the well-known intersection lemma that is due to Fan [13] will be needed.

Definition 2 (see [14]). Let K be a nonempty subset of a Hausdorff topological vector space E . A mapping $\Lambda : K \rightarrow 2^E$ (by 2^E we understand the family of all the subsets of E) is said to be a KKM-mapping if, for any finite subset $\{u_1, u_2, \dots, u_n\}$ of K ; we have $\text{co}\{u_1, u_2, \dots, u_n\} \subset \bigcup_{i=1}^n \Lambda(u_i)$, where $\text{co}\{u_1, u_2, \dots, u_n\}$ denotes the convex hull of $\{u_1, \dots, u_n\}$.

Lemma 3 (see [15]). Let K be a nonempty subset of a Hausdorff topological vector space E and let $\Lambda : K \rightarrow 2^E$ be a KKM-mapping. If $\Lambda(x)$ is closed in E for every $x \in K$ and compact for some $u_0 \in K$, then $\bigcap_{u \in K} \Lambda(u) \neq \emptyset$.

Definition 4 (see [16]). A real-valued function, defined on a convex subset K of X , is said to be *hemicontinuous*, if

$$\lim_{t \rightarrow 0^+} g(tx + (1-t)y) = g(y) \quad \forall x, y \in K. \quad (3)$$

Definition 5 (see [17]). A bifunction $f : K \times K \rightarrow \mathbb{R}$ is called *vector 0-diagonally convex* if, for any finite subset $\{u_1, u_2, \dots, u_n\}$ of K and $\lambda_i \geq 0$ ($i = 1, 2, \dots, n$) with $x = \sum_{i=1}^n \lambda_i u_i$ and $\sum_{i=1}^n \lambda_i = 1$, one has

$$\sum_{i=1}^n \lambda_i f(x, x_i) \geq 0. \quad (4)$$

Definition 6. Let X be a Banach space. A mapping $\Lambda : X \rightarrow \mathbb{R}$ is said to be

(1) *lower semicontinuous* (for short, lsc) at $x_0 \in X$, if

$$\Lambda(x_0) \leq \liminf_n \Lambda(x_n), \quad (5)$$

(2) *upper semicontinuous* (for short, usc) at $x_0 \in X$, if

$$\Lambda(x_0) \geq \limsup_n \Lambda(x_n) \quad (6)$$

for any sequence x_n of X such that $x_n \rightarrow x_0$.

For technical reasons, the authors presented the following definition.

Definition 7. Assume that X is a Banach space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper function. One can say that $x^* \in X^*$ is α -subdifferential of f in $x \in \text{dom} f = \{x : f(x) < \infty\}$, if

$$\partial_\alpha f(x) = \left\{ x^* \in X^* : f(y) - \frac{\alpha(y, x)}{2} \geq f(x) + \langle x^*, y - x \rangle \quad (\forall y \in X) \right\}. \quad (7)$$

Throughout this paper, we consider $\forall r \in [0, 1]$:

$$\lim_{r \rightarrow 0} \frac{\alpha(x, x_r)}{r} = 0, \quad (8)$$

$$\alpha(x, y) \leq \lim_{r \rightarrow 0} \frac{r-1}{r} [\psi(x, x) + \alpha(x, x)].$$

In the present paper, the first goal is to study the equilibrium problem (EP_Ψ) for α -monotone operator in Banach and real

reflexive Banach spaces via the KKM technique. In addition, we provide sufficient conditions for existence solution for the case of unbounded sets. The results of this paper can be viewed as generalization of many known results (see [15, 18, 19]). Moreover, we recall an application of existence result for a differential inclusion.

2. Main Results

In this section we establish some existence results for equilibrium problem (EP_Ψ) . Surprisingly, through the results of this section, we prove the existence of a solution of the problem (EP_Ψ) without assumption of boundedness of the set K in the last theorem.

Theorem 8. Suppose that $F : K \times K \rightarrow \mathbb{R}$ is α -monotone bifunction, hemicontinuous in the first argument and convex in the second argument. Let $\Psi, \alpha : K \times K \rightarrow \mathbb{R}$ be convex in the second argument; then generalized equilibrium problem (EP_Ψ) is equivalent to the following problem.

Find $x \in K$ such that

$$F(y, x) + \alpha(x, y) \leq \Psi(x, y) \quad \forall y \in K. \quad (9)$$

Proof. Suppose that (EP_Ψ) has a solution. There exists $x \in K$ such that

$$F(x, y) + \Psi(x, y) \geq 0 \quad \forall y \in K. \quad (10)$$

Since F is α -monotone bifunction

$$F(x, y) + F(y, x) + \alpha(x, y) \leq 0 \quad \forall x, y \in K, \quad (11)$$

then

$$\begin{aligned} F(y, x) + \alpha(x, y) &\leq -F(x, y) \\ &\leq \Psi(x, y). \end{aligned} \quad (12)$$

Therefore, $x \in K$ is a solution of problem (9). Conversely, assume that $x \in K$ is a solution of problem (9) and fix $y \in K$.

Letting $x_r = ry + (1-r)x$, $r \in [0, 1]$, then $x_r \in K$, since K is convex, so

$$F(x_r, x) + \alpha(x, x_r) \leq \Psi(x, x_r). \quad (13)$$

Then

$$F(x_r, x) - \Psi(x, x_r) \leq -\alpha(x, x_r). \quad (14)$$

Since F is convex in the second argument

$$0 = F(x_r, x_r) \leq rF(x_r, y) + (1-r)F(x_r, x) \quad (15)$$

so

$$r[F(x_r, x) - F(x_r, y)] \leq F(x_r, x). \quad (16)$$

By the convexity of Ψ and α in the second argument

$$\begin{aligned} \alpha(x, x_r) &\leq r\alpha(x, y) + (1-r)\alpha(x, x), \\ \Psi(x, x_r) &\leq r\Psi(x, y) + (1-r)\Psi(x, x). \end{aligned} \quad (17)$$

Then,

$$\begin{aligned} r[\alpha(x, x) - \alpha(x, y)] &\leq \alpha(x, x) - \alpha(x, x_r), \\ r[\Psi(x, x) - \Psi(x, y)] &\leq \Psi(x, x) - \Psi(x, x_r). \end{aligned} \quad (18)$$

From (14), (16), and (18)

$$\begin{aligned} r[F(x_r, x) - F(x_r, y) + \alpha(x, x) - \alpha(x, y) + \Psi(x, x) \\ - \Psi(x, y)] &\leq F(x_r, x) - \alpha(x, x_r) - \Psi(x, x_r) \\ + \alpha(x, x) + \Psi(x, x) &\leq -2\alpha(x, x_r) + \alpha(x, x) \\ + \Psi(x, x). \end{aligned} \quad (19)$$

Since F is hemicontinuous in the first argument,

$$\begin{aligned} r[-F(x, y) - \Psi(x, y)] \\ \leq -2\alpha(x, x_r) + r\alpha(x, y) \\ + (1-r)[\Psi(x, x) + \alpha(x, x)], \end{aligned} \quad (20)$$

so

$$\begin{aligned} F(x, y) + \Psi(x, y) &\geq \frac{2\alpha(x, x_r)}{r} - \alpha(x, y) \\ &+ \frac{(r-1)}{r} [\Psi(x, x) + \alpha(x, x)]. \end{aligned} \quad (21)$$

From (8),

$$F(x, y) + \Psi(x, y) \geq 0 \quad \forall y \in K. \quad (22)$$

Therefore, (EP_Ψ) has a solution. $\square$

Theorem 9. Let K be a nonempty closed bounded convex subset of a real reflexive Banach space X . Assume that

- (1) $F : K \times K \rightarrow \mathbb{R}$ is α -monotone bifunction, 0-diagonal convex, hemicontinuous in first argument and lsc, convex in second argument;
- (2) $\Psi : K \times K \rightarrow \mathbb{R}$ is convex in second argument, usc in first argument, and 0-diagonal convex;
- (3) $\alpha : K \times K \rightarrow \mathbb{R}$ is lsc in first argument and convex in second argument.

Then (EP_Ψ) has a solution.

Proof. Define two set valued mappings $\eta, \Gamma : K \rightarrow 2^K$ as follows:

$$\begin{aligned} \eta(y) &= \{x \in K : F(x, y) + \Psi(x, y) \geq 0 \quad \forall y \in K\} \\ \Gamma(y) &= \{x \in K : F(y, x) + \alpha(x, y) \leq \Psi(x, y) \quad \forall y \in K\}. \end{aligned} \quad (23)$$

Then, $\forall x \in K$ the problem (EP_Ψ) has a solution iff $\bigcap_{y \in K} \eta(y) \neq \emptyset$. Now, we show that η is a KKM-mapping. On the contrary, η is not a KKM-mapping. Then, there exists a

finite subset $\{x_1, x_2, \dots, x_n\}$ of K and $\lambda_i \geq 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \lambda_i = 1$ such that $x_0 = \sum_{i=1}^n \lambda_i x_i \notin \bigcup_{i=1}^n \eta(y_i)$. Then

$$F(x_0, x_i) + \Psi(x_0, x_i) < 0 \quad \forall i = 1, 2, \dots, n,$$

$$\sum_{i=1}^n \lambda_i [F(x_0, x_i) + \Psi(x_0, x_i)] < 0 \quad \forall i = 1, 2, \dots, n \quad (24)$$

and this contradicts the 0-diagonal convex property of F and Ψ . Then η is a KKM-mapping. Next, we will show that $\eta(y) \subset \Gamma(y) \quad \forall y \in K$. Let $x \in \eta(y)$; then

$$F(x, y) + \Psi(x, y) \geq 0 \quad \forall y \in K. \quad (25)$$

From definition of α -monotone bifunction,

$$\begin{aligned} F(y, x) + \alpha(x, y) - \Psi(x, y) \\ \leq -F(x, y) - \alpha(x, y) + \alpha(x, y) - \Psi(x, y) \leq 0. \end{aligned} \quad (26)$$

This implies that $x \in \Gamma(y)$. Thus $\eta(y) \subset \Gamma(y) \quad \forall y \in K$. Therefore, $\Gamma(y)$ is a KKM-mapping. Since $\alpha(\cdot, y)$, $F(y, \cdot)$ are lsc and $\Psi(\cdot, y)$ is usc, then

$$\begin{aligned} F(y, x) + \alpha(x, y) &\leq \liminf_n (F(y, x_n) + \alpha(x_n, y)) \\ &\leq \limsup_n \Psi(x_n, y) \leq \Psi(x, y). \end{aligned} \quad (27)$$

Therefore, $\Gamma(y)$ is weakly closed $\forall y \in K$. Since K is nonempty, bounded, closed, and convex and X is real reflexive, it follows that K is weakly compact. Hence, $\Gamma(y)$ is weakly compact $\forall y \in K$. From Lemma 3 and Theorem 8

$$\bigcap_{y \in K} \eta(y) = \bigcap_{y \in K} \Gamma(y) \neq \emptyset. \quad (28)$$

Therefore, the problem (EP_Ψ) has a solution. $\square$

Corollary 10. Assume that $0 \in K$ is a nonempty convex subset of Banach space X and assumptions (1-3) in Theorem 9 hold. Moreover,

- (1) $\Psi(x, 0) = 0$;
- (2) $\zeta_1 = \{x \in K : \alpha(x, 0) \leq 0\}$ and $\zeta_2 = \{x \in K : F(0, x) \leq 0\}$ are relative compact sets.

Then (EP_Ψ) has a solution.

Proof. It suffices to prove $\Gamma(y)$ is compact for some $y \in K$. Since

$$\begin{aligned} \Gamma(y) &= \{x \in K : F(y, x) + \alpha(x, y) \leq \Psi(x, y) \quad \forall y \in K\}, \\ &\subset \left\{x \in K : F(y, x) \leq \frac{\Psi(x, y)}{2}\right\} \\ &\cup \left\{x \in K : \alpha(x, y) \leq \frac{\Psi(x, y)}{2}\right\}. \end{aligned} \quad (29)$$

Since $0 \in K$,

$$\Gamma(0) \subset \left\{ x \in K : F(0, x) \leq \frac{\Psi(x, 0)}{2} \right\} \cup \left\{ x \in K : \alpha(x, 0) \leq \frac{\Psi(x, 0)}{2} \right\}. \quad (30)$$

From conditions (1-2), $\Gamma(0)$ is a subset of relative compact set ζ_1 or ζ_2 . This implies that $\Gamma(y)$ is compact for some $y \in K$. $\square$

Now, we show that the problem (EP_Ψ) admits one solution without boundedness of K and under suitable conditions.

Theorem 11. Assume that the same hypotheses as in Theorem 9 hold without the assumption of boundedness of K . Suppose in addition the following conditions are fulfilled:

- (1) Consider $\Psi(x, x) = 0 \quad \forall x \in K$.
- (2) $\exists x_0 \in K$ such that $F(x, x_0) + \Psi(x, x_0) \leq 0$, where $x \in K$ and $\|x\|$ is large enough.

Then the problem (EP_Ψ) admits a solution.

Proof. Set $K_n = \{y \in K : \|y\| \leq n, n > 0\}$. Let us consider the problem. Find $x_n \in K_n$ such that

$$F(x_n, y) + \Psi(x_n, y) \geq 0 \quad \forall y \in K_n. \quad (31)$$

From the boundedness of K_n and Theorem 8, we obtain that the problem (EP_Ψ) has at least one solution.

Choose $\|x_0\| < n_0$. From (31), we get

$$F(x_{n_0}, x_0) + \Psi(x_{n_0}, x_0) \geq 0. \quad (32)$$

So $\|x_{n_0}\| \leq n_0$ because $x_{n_0} \in K_{n_0}$. Let us choose $\|x_{n_0}\| = n_0$ and n_0 large enough so that by condition 2 we have

$$F(x_{n_0}, x_0) + \Psi(x_{n_0}, x_0) < 0, \quad (33)$$

which contradicts (32). Hence, $\|x_{n_0}\| < n_0$. For each $y \in K$, one can take $t \in (0, 1)$ small enough such that $z = ty + (1-t)x_{n_0} \in K_{n_0}$. From (31), for each $y \in K$, one can obtain

$$\begin{aligned} 0 &\leq F(x_{n_0}, z) + \Psi(x_{n_0}, z) \\ &= F(x_{n_0}, ty + (1-t)x_{n_0}) \\ &\quad + \Psi(x_{n_0}, ty + (1-t)x_{n_0}) \\ &\leq tF(x_{n_0}, y) + (1-t)F(x_{n_0}, x_{n_0}) + t\Psi(x_{n_0}, y) \\ &\quad + (1-t)\Psi(x_{n_0}, x_{n_0}). \end{aligned} \quad (34)$$

Dividing by t the conclusion follows. $\square$

3. Application

In this section, we apply our main results, expressed in the previous section to a partial differential inclusion problem.

To do this, we will consider Sobolev space as $H^2(\Omega)$. So, let us consider the following partial differential inclusion problem:

$$\begin{aligned} -\Delta u &\in \partial_{-2\alpha} f(u), \quad \text{if } x \in \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (35)$$

where Ω is an open subset of $\mathbb{R}^n$, $f: K \rightarrow \mathbb{R}^n$ is a continuous and concave function, and K is a bounded convex subset of Sobolev space $H^2(\Omega)$.

Definition 12. One says that $u \in K$ has a K -weak solution of problem (35) if

$$\langle -\Delta u, u - v \rangle \leq f(u) - f(v) - \alpha(u, v) \quad \forall v \in K. \quad (36)$$

By the integral form of $\langle \cdot, \cdot \rangle$, one can get that

$$\int_{\Omega} \nabla u \nabla (u - v) dx + \alpha(u, v) \leq f(u) - f(v). \quad (37)$$

Set $F(u, v) := f(u) - f(v)$ and $\Psi(u, v) := \int_{\Omega} \nabla u \nabla (v - u) dx$. Therefore, we obtain

$$F(v, u) + \alpha(u, v) \leq \Psi(u, v) \quad \forall v \in K. \quad (38)$$

Considering $\alpha(u, v) = -1$, then F is α -monotone bifunction. In Theorem 8 we proved that (9) and (10) are equivalent under some assumptions. Hence, we must prove that F and Ψ hold all assumptions of Theorem 9. It is clear that the bifunction $\alpha(u, v) = -1$ satisfies all assumptions in Theorem 9.

Claim 1. Ψ is convex in second argument. Let $z = tv_1 + (1-t)v_2 \in K$, $0 \leq t \leq 1$, so

$$\begin{aligned} \Psi(u, z) &= \Psi(u, tv_1 + (1-t)v_2) \\ &= \int_{\Omega} \nabla u \nabla ((tv_1 + (1-t)v_2) - u) dx \\ &= \int_{\Omega} \nabla u \nabla t(v_1 - u) dx \\ &\quad + \int_{\Omega} \nabla u \nabla (1-t)(v_2 - u) dx \\ &= t\Psi(u, v_1) + (1-t)\Psi(u, v_2). \end{aligned} \quad (39)$$

Claim 2. Ψ is 0-diagonal convex, so $\sum_{i=1}^n \lambda_i \Psi(u, u_i) \geq 0$ is needed. For this,

$$\begin{aligned} \sum_{i=1}^n \lambda_i \Psi(u, u_i) &= \sum_{i=1}^n \lambda_i \int_{\Omega} \nabla u \nabla (u_i - u) dx \\ &= \int_{\Omega} \sum_{i=1}^n \lambda_i \nabla u \nabla u_i dx \\ &\quad - \int_{\Omega} \sum_{i=1}^n \lambda_i \nabla u \nabla u dx \\ &= \int_{\Omega} \nabla u \left(\sum_{i=1}^n \lambda_i \nabla u_i \right) dx \\ &\quad - \int_{\Omega} \left(\sum_{i=1}^n \lambda_i \right) \nabla u \nabla u dx = 0. \end{aligned} \quad (40)$$

In a similar way, we prove that $F(u, v)$ is 0 -diagonal convex:

$$\begin{aligned}\sum_{i=1}^n \lambda_i F(u, u_i) &= \sum_{i=1}^n \lambda_i (f(u) - f(u_i)) \\ &= \sum_{i=1}^n \lambda_i f(u) - \sum_{i=1}^n \lambda_i f(u_i) \geq 0.\end{aligned}\quad (41)$$

It is clear that F is *hemicontinuous* in first argument, *lsc* and *convex* in second argument, because f is *concave* and *continuous* function. Moreover, Ψ is *usc*, since ∇ is *continuous* on Sobolev space.

Therefore, all conditions are achieved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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