Research Article

A Tauberian Theorem for Double Cesàro Summability Method

Bidu Bhusan Jena, 1 Susanta Kumar Paikray, 1 and Umakanta Misra 2

1Department of Mathematics, VSS University of Technology, Burla 768018, India
2Department of Mathematics, NIST, Palur Hills, Golanthara 761008, India

Correspondence should be addressed to Susanta Kumar Paikray; skpaikray_math@vssut.ac.in

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1. Introduction

Let \( u = (u_{mn}) \) be a double real sequence. We define \( \Delta_n u_{mn} = u_{mn} - u_{m,n-1}, \) \( \Delta_m u_{mn} = u_{mn} - u_{m-1,n}, \) and \( \Delta_{mn} u_{mn} = u_{mn} - u_{m-1,n-1}. \)

A double sequence \( (u_{mn}) \) is said to be bounded if there exists a finite real number \( C > 0 \) such that \( |u_{mn}| \leq C, \forall m, n \in \mathbb{Z}^+ \). Let us write

\[
\sigma_{mn}^{(11)} (u) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} u_{pq} 
\]

(see [1]),

\[
\sigma_{mn}^{(10)} (u) = \frac{1}{m+1} \sum_{p=0}^{m} u_{pn},
\]

\[
\sigma_{mn}^{(01)} (u) = \frac{1}{n+1} \sum_{q=0}^{n} u_{mq}.
\]

Then, we say that a double sequence \( u = (u_{mn}) \) is \((C, 1, 1)\) summable to \( s \) if \( \sigma_{mn}^{(11)} (u) \) converges to \( s \), as \( m, n \to \infty \). Similarly, we say that it is \((C, 1, 0)\) summable to \( s \) if \( \sigma_{mn}^{(10)} (u) \) converges to \( s \) as \( m, n \to \infty \) and \((C, 0, 1)\) summable to \( s \) if \( \sigma_{mn}^{(01)} (u) \) converges to \( s \) as \( m, n \to \infty \).

For all nonnegative integers \( k \) and \( r \), we define \( \sigma_{mn}^{(kr)} (u) \) as follows:

\[
\sigma_{mn}^{(kr)} (u) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} a_{pq}^{(k-1,r-1)} (u) \text{ for } k, r \geq 1,
\]

\[
\sigma_{mn}^{(kr)} (u) = \frac{1}{m+1} \sum_{p=0}^{m} u_{pn} \text{ for } k = 0,
\]

\[
\sigma_{mn}^{(kr)} (u) = \frac{1}{n+1} \sum_{q=0}^{n} u_{mq} \text{ for } r = 0.
\]

A double sequence \( u = (u_{mn}) \) is said to be \((C, k, r)\) summable to \( s \) if \( \sigma_{mn}^{(kr)} (u) \) converges to \( s \). If \( k = 1 \) and \( r = 1 \), then \((C, k, r)\) summability reduces to \((C, 1, 1)\) summability. Again, if \( k \neq 0 \) and \( r = 0 \), then \((C, k, r)\) summability reduces to \((C, k, 0)\) summability. Further, if \( k = 0 \) and \( r \neq 0 \), then \((C, k, r)\) summability reduces to \((C, 0, r)\) summability. Here, Cesàro summability of \( u = (u_{mn}) \) refers to \((C, k, r)\) summability of \( u = (u_{mn}) \). It may also be noted that the convergence of a double sequence \( u = (u_{mn}) \) implies the Cesàro summability of \( u = (u_{mn}) \), but the converse is not generally true.

For example, consider a function \( f(x, y) = e^{2x} \sin(3y) \): the sequence \( (u_{mn}) \), which is the sequence of coefficients in the Taylor series expansion of the function \( f(x, y) \) about origin, is Cesàro summable but not convergent.

For the proof of the converse part, certain conditions are presented in terms of oscillatory behavior of double sequence \( u = (u_{mn}) \).
Let us define \((u_{mn})\) as
\[
u_{mn} = V_{mn}^{(1)} + \frac{m}{p+1} \sum_{p=1}^{m} \sum_{q=1}^{n} \frac{V_{pq}^{(1)}}{pq} + u_{00}, \quad (m, n = 1, 2, \ldots),
\]
where
\[
u_{mn} - V_{mn}^{(1)}(u) = V_{mn}^{(1)}(\Delta u)
= \frac{1}{(m + 1)(n + 1)} \sum_{p=0}^{m} \sum_{q=0}^{n} p q (\Delta_{pq} u_{pq})
\]
(see [2]).

Moreover, in analogy to Kronecker identity for a single sequence, we can write
\[
V_{mn}^{(1)}(\Delta u) = \frac{1}{(m + 1)} \sum_{p=0}^{m} p (\Delta_{p} u_{pn}),
V_{mn}^{(0)}(\Delta u) = \frac{1}{(n + 1)} \sum_{q=0}^{n} q (\Delta_{q} u_{mnq})
\]
as the \((C,1,0)\) mean of the sequence \(m \Delta_{m} u_{mn}\) and the \((C,0,1)\) mean of the sequence \(n \Delta_{n} u_{nm}\) respectively.

Further, as the sequence \(V_{mn}^{(1)}(\Delta_{mn} u_{mn})\) is the \((C,1,1)\) mean of the sequence \(mn(\Delta_{mn} u_{mn})\), the sequence \(mn(\Delta_{mn} u_{mn})\) is \((C,1,1)\) summable to \(s\) whenever \(V_{mn}^{(1)}(\Delta_{mn} u_{mn})\) converges to \(s\) as \(m, n \rightarrow \infty\). For all nonnegative integers \(k\) and \(r\), let us define \(V_{mn}^{(kr)}(\Delta u)\) as follows:
\[
V_{mn}^{(kr)}(\Delta u) = \begin{cases} \frac{1}{(m + 1)(n + 1)} \sum_{p=0}^{m} \sum_{q=0}^{n} V_{pq}^{(k-1,r-1)}(\Delta u) \text{ for } k, r \geq 1, \\ mn(\Delta_{mn} u_{mn}) \text{ for } k = 0, r = 0. \end{cases}
\]

The sequence \(mn(\Delta_{mn} u_{mn})\) is said to be \((C,k,r)\) summable to \(s\) if \(V_{mn}^{(kr)}(\Delta_{mn} u_{mn})\) converges to \(s\) as \(m, n \rightarrow \infty\). In particular, if \(k = 1\) and \(r = 1\), then \((C,k,r)\) summability reduces to \((C,1,1)\) summability. Again, if \(k \neq 0\) and \(r = 0\), then \((C,k,r)\) summability reduces to \((C,k,0)\) summability. Further, if \(k = 0\) and \(r \neq 0\), then \((C,k,r)\) summability reduces to \((C,0,r)\) summability.

Then, the de la Vallée-Poussin mean of double real sequence \((u_{mn})\) is defined by
\[
\tau_{mn}(u) = \frac{1}{(\lceil mn \rceil - m)(\lceil mn \rceil - n)} \sum_{j, k = \lfloor j \rceil = \lfloor k \rceil + 1}^{\lfloor mn \rceil} u_{ij},
\]
for sufficiently large nonnegative integers \(m, n\) and \(\lambda > 1\), and
\[
\tau_{mn}(u) = \frac{1}{(m - \lfloor mn \rceil)(n - \lfloor mn \rceil)} \sum_{j, k = \lfloor j \rceil = \lfloor k \rceil + 1}^{m, n} u_{ij},
\]
for sufficiently large nonnegative integers \(m, n\) and \(0 < \lambda < 1\).

A single sequence \(u = (u_{n})\) is slowly oscillating [3] if
\[
\lim_{n \to \infty} \sup_{|k| < \lceil mn \rceil} |u_{k} - u_{n}| = 0.
\]
A double sequence \(u = (u_{mn})\) is slowly oscillating [4] if
\[
\lim_{m, n \to \infty} \sup_{|k| < \lceil mn \rceil} \max_{m = m_{1} + 1, n = n_{1} + 1} \left| \sum_{x = m_{1} + 1}^{i} \sum_{y = n_{1} + 1}^{j} \Delta_{x,y} u_{xy} \right| = 0.
\]

In an earlier paper by Jena et al. [5], a proof of the generalized Littlewood Tauberian theorem by Cesàro summability method has been established. For a proof of Littlewood Tauberian theorem differently, the paper of Canak and Totur [6] and Canak [7–9] can be referred to. Also, a similar result was introduced earlier by Çanak [10] under the consideration of improper integral. Recently, Totur [4] has introduced Littlewood Tauberian theorem by \((C,1,1)\) mean for double real sequence.

In the proposed paper, with certain novelty, we have generalized it for \((C,k,r)\) summability of a double real sequence defined in (4).

2. Known Results

Theorem 1 (see [4]). If the sequence \((u_{mn})\) is \((C,1,1)\) summable to \(s\) and \((u_{mn})\) is slowly oscillating (in the sense of \((1, 1)\)), then \(\lim_{m,n \to \infty} u_{mn} = s\).

Corollary 2 (see [5]). If the sequence \((u_{n})\) is \((C,1)\) summable to \(s\) and \((u_{n})\) is slowly oscillating, then \(\lim_{n \to \infty} u_{n} = s\).

Theorem 3 (see [4]). If the sequence \(u = (u_{mn})\) is \((C,1,1)\) summable to \(s\) and \(V_{mn}^{(1)}(\Delta_{mn} u_{mn})\) is slowly oscillating, then \(\lim_{m,n \to \infty} u_{mn} = s\).

Corollary 4 (see [5]). If the sequence \((u_{n})\) is \((C,1)\) summable to \(s\) and \(V_{mn}^{(0)}(\Delta u)\) is slowly oscillating, then \(\lim_{n \to \infty} u_{n} = s\).

3. Main Result

Theorem 5. If \((u_{mn})\) is \((C,k,r)\) summable to \(s\) and \((u_{mn})\) is slowly oscillating, then \(\lim_{m,n \to \infty} u_{mn} = s\).

To prove the above theorem, we need the help of the following lemmas.

Lemma 6. A double sequence \(u = (u_{mn})\) is slowly oscillating if and only if \(V_{mn}^{(1)}\) is slowly oscillating and bounded.

Proof. Let \(u = (u_{mn})\) be slowly oscillating. Initially, let us show that \(V_{mn}^{(1)} = O(1)\).

We have by definition of slow oscillation, for \(\lambda > 1\),
\[
\lim_{m,n \to \infty} \sup_{|k| < \lceil mn \rceil} \max_{m = m_{1} + 1, n = n_{1} + 1} \left| \sum_{x = m_{1} + 1}^{i} \sum_{y = n_{1} + 1}^{j} \Delta_{x,y} u_{xy} \right| = 0,
\]
and let us rewrite the finite sum $\sum_{i=0}^{m} \sum_{j=1}^{n} i j \Delta u_{ij}$ as the series
\[ \sum_{i=0}^{m} \sum_{j=1}^{n} i j \Delta u_{ij} = \sum_{x=0}^{m} \sum_{y=1}^{n} \sum_{i=0}^{m} \sum_{j=1}^{n} i j \Delta u_{ij}. \]
Clearly,
\[ \sum_{x=0}^{m} \sum_{y=1}^{n} \sum_{i=0}^{m} \sum_{j=1}^{n} i j \Delta u_{ij} \leq \left( \sum_{x=0}^{m} \sum_{y=1}^{n} mn \right) \cdot \max_{m/2+1 \leq k \leq m/2+1, n/2+1 \leq \nu \leq n/2+1} \sum_{i=0}^{m} \sum_{j=1}^{n} \Delta_{ij} \]
\[ \leq m n C^* \left( \sum_{x=0}^{m} \sum_{y=1}^{n} \frac{1}{x y} \right) = m n C^*, \]
where $C^* > 0$.

Consequently, we have
\[ V_{mn}^{(11)} (\Delta u) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} p q (\Delta_{pq} u_{pq}) \]
\[ = O(1), \quad \text{as } m, n \to \infty. \]

Since $\sigma_{mn}^{(11)}(u) = \sum_{p=0}^{m} \sum_{q=0}^{n} (V_{pq}^{(11)} / p q) + u_{\infty}$ is slowly oscillating, $V_{mn}^{(11)}$ is slowly oscillating.

To prove the converse part, consider $V_{mn}^{(11)}$ to be bounded and slowly oscillating. Now, the boundedness of $V_{mn}^{(11)}$ implies that $\sigma_{mn}^{(11)}(u)$ is slowly oscillating. Further, $V_{mn}^{(11)}$ is slowly oscillating, so, by Kronecker identity (5), $(u_{mn})$ is slowly oscillating.

This completes the proof of Lemma 6. \qed

Next, we represent the difference $(u_{mn} - \sigma_{mn}^{(11)}(u))$ under two different cases in the following lemma.

**Lemma 7** (see [1]). Let $u = (u_{mn})$ be a sequence of real numbers with $m, n$ sufficiently large; then one has the following:

(i) For $\lambda > 1$,
\[ u_{mn} - \sigma_{mn}^{(11)}(u) = \frac{([\lambda m] + 1) ([\lambda n] + 1)}{([\lambda m] - m) ([\lambda n] - n)} \left( \sigma_{mn}^{(11)}(u) - \sigma_{mn}^{(11)}(u) \right) \]
\[ - \frac{[\lambda m] + 1}{[\lambda n] - m} \left( \sigma_{mn}^{(11)}(u) - \sigma_{mn}^{(11)}(u) \right) \]
\[ + \frac{[\lambda n] + 1}{[\lambda m] - m} \left( \sigma_{mn}^{(11)}(u) - \sigma_{mn}^{(11)}(u) \right) \]
\[ - \frac{1}{([\lambda m] - m) ([\lambda n] - n)} \sum_{i=m+1}^{\lambda m} \sum_{j=n+1}^{\lambda n} (u_{ij} - u_{mn}). \]

(ii) For $0 < \lambda < 1$,
\[ u_{mn} - \sigma_{mn}^{(11)}(u) = \frac{([\lambda m] + 1) ([\lambda n] + 1)}{m - [\lambda m]} \left( \sigma_{mn}^{(11)}(u) \right) \]
\[ - \frac{[\lambda m] + 1}{m - [\lambda m]} \left( \sigma_{mn}^{(11)}(u) - \sigma_{mn}^{(11)}(u) \right) \]
\[ + \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_{mn}^{(11)}(u) - \sigma_{mn}^{(11)}(u) \right) \]
\[ - \frac{1}{m - [\lambda m] + 1} \left( \sigma_{mn}^{(11)}(u) - \sigma_{mn}^{(11)}(u) \right). \]

**Proof of Theorem 5.** Let $u = (u_{mn})$ be slowly oscillating; then, $\sigma_{mn}^{(k,r)}(u)$ is slowly oscillating (by Lemma 6). Further, since $u = (u_{mn})$ is $(C, k, r)$ summable to $s$, by Theorem 1,
\[ \lim_{m,n \to \infty} \sigma_{mn}^{(k,r)}(u) = s. \]

Next, from the definition,
\[ \sigma_{mn}^{(k,r)}(u) = \sigma_{mn}^{(1)}(u) \left( \sigma_{mn}^{(k-1,r-1)}(u) \right). \]

Clearly, (17) and (18) imply that $u = (u_{mn})$ is $(C, k - 1, r - 1)$ summable to $s$. Again, $\sigma_{mn}^{(k-1,r-1)}(u)$ is also slowly oscillating (by Lemma 6).

Thus, by Theorem 1, we have
\[ \lim_{m,n \to \infty} \sigma_{mn}^{(k-1,r-1)}(u) = s. \]

Continuing in this way, we get
\[ \lim_{m,n \to \infty} (u_{mn}) = s. \]

**Remark 8.** If $k = 0$ and $r \neq 0$, then $(C, k, r)$ summability reduces to $(C, 0, r)$ summability. Again, for $k \neq 0$ and $r = 0$, $(C, k, r)$ summability reduces to $(C, k, 0)$ summability and, consequently, the following corollary is generated from the main result.

**Corollary 9** (see [5]). If $u = (u_n)$ is $(C, 0, r)$ or $(C, k, 0)$ summable to $s$ and $u = (u_n)$ is slowly oscillating, then $\lim_{n \to \infty} u_n = s$.

**Theorem 10.** If $(u_{mn})$ is $(C, k, r)$ summable to $s$ and $V_{mn}^{(11)}(\Delta u)$ is slowly oscillating, then $\lim_{m,n \to \infty} u_{mn} = s$.

**Proof.** As $V_{mn}^{(11)}(\Delta u)$ is slowly oscillating, setting $u = (u_{mn})$ in place of $V_{mn}^{(11)}(\Delta u)$, $\sigma_{mn}^{(k,r)}(V_{mn}^{(11)}(\Delta u))$ is slowly oscillating by Lemma 6. Again, as $V_{mn}^{(11)}(\Delta u)$ is $(C, k, r)$ summable to $s$, by Theorem 3, we have
\[ \lim_{m,n \to \infty} \sigma_{mn}^{(k,r)}(V_{mn}^{(11)}(\Delta u)) = s. \]
By definition,
\[
\sigma_{mn}^{(k,r)}(V_{mn}^{(1)}(\Delta u)) = \sigma_{mn}^{(1,1)}(V_{mn}^{(1)}(\Delta u))
\]
(21)

From (20) and (21), we have that \(V_{mn}^{(1)}(\Delta u)\) is \((C, k-1, r-1)\) summable to \(s\). Again, by Lemma 6, since \(\sigma_{mn}^{(k-1, r-1)}(V_{mn}^{(1)}(\Delta u))\) is slowly oscillating, we have \(\lim_{m,n \to \infty} \sigma_{mn}^{(k-1, r-1)}(V_{mn}^{(1)}(\Delta u)) = s\) (by Theorem 3).

Continuing in this way, we get \(\lim_{m,n \to \infty} (V_{mn}^{(1)}(\Delta u)) = s\).

Remark II. If \(k = 0\) and \(r \neq 0\), then \((C, k, r)\) summability reduces to \((C, 0, r)\) summability. Again, for \(k \neq 0\) and \(r = 0\), \((C, k, r)\) summability reduces to \((C, k, 0)\) summability and consequently the following corollaries are generated from the main result.

**Corollary 12** (see [5]). If \(u = (u_n)\) is \((C, 0, r)\) summable to \(s\) and \(V_{mn}^{(1,0)}(\Delta u)\) is slowly oscillating, then \(\lim_{n \to \infty} u_n = s\).

**Corollary 13** (see [5]). If \(u = (u_n)\) is \((C, k, 0)\) summable to \(s\) and \(V_{mn}^{(k,0)}(\Delta u)\) is slowly oscillating, then \(\lim_{n \to \infty} u_n = s\).

**Competing Interests**

The authors declare that there are no competing interests regarding publication of this paper.

**References**


