Research Article

Asymptotic Theory in Model Diagnostic for General Multivariate Spatial Regression

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We establish an asymptotic approach for checking the appropriateness of an assumed multivariate spatial regression model by considering the set-indexed partial sums process of the least squares residuals of the vector of observations. In this work, we assume that the components of the observation, whose mean is generated by a certain basis, are correlated. By this reason we need more effort in deriving the results. To get the limit process we apply the multivariate analog of the well-known Prohorov's theorem. To test the hypothesis we define tests which are given by Kolmogorov-Smirnov (KS) and Cramér-von Mises (CvM) functionals of the partial sums processes. The calibration of the probability distribution of the tests is conducted by proposing bootstrap resampling technique based on the residuals. We studied the finite sample size performance of the KS and CvM tests by simulation. The application of the proposed test procedure to real data is also discussed.

1. Introduction

As mentioned in the literatures of model checks for multivariate regression, the appropriateness of an assumed model is mostly verified by analyzing the least squares residual of the observations; see, for example, Zellner [1], Christensen [2], pp. 1–22, Anderson [3], pp. 187–191, and Johnson and Wichern [4], pp. 395–398. A common feature of these works is the comparison between the length of the matrix of the residuals under the null hypothesis and that of the residuals under a proposed alternative.

Instead of considering the residuals directly MacNeill [5] and MacNeill [6] proposed a method in model check for univariate polynomial regression based on the partial sums process of the residuals. These popular approaches are generalized to the spatial case by MacNeill and Jandhyala [7] for ordinary partial sums and Xie and MacNeill [8] for set-indexed partial sums process of the residuals. Bischoff and Somayasa [9] and Somayasa et al. [10] derived the limit process in the spatial case by a geometric method generalizing a univariate approach due to Bischoff [11] and Bischoff [12]. These results

can be used to establish asymptotic test of Cramér-von Mises and Kolmogorov-Smirnov type for model checks and change-point problems. Model checks for univariate regression with random design using the empirical process of the explanatory variable marked by the residuals was established in Stute [13] and Stute et al. [14]. In the papers mentioned above the limit processes were explicitly expressed as complicated functions of the univariate Brownian motion (sheet).

The purpose of the present article is to study the application of set-indexed partial sums technique to simultaneously check the goodness-of-fit of a multivariate spatial linear regression defined on high-dimensional compact rectangle. In contrast to the normal multivariate model studied in the standard literatures such as in Christensen [2], Anderson [3], and Johnson and Wichern [4] or in the references of model selection such as in Bedrick and Tsai [15] and Fujikoshi and Satoh [16], in this paper we will consider a multivariate regression model in which the components of the mean of the response vector are assumed to lie in different spaces and the underlying distribution model of the vector of random errors is unknown.

To see the problem in more detail let $n_1 \geq 1, \dots, n_d \geq 1$ be fixed. Let a p -dimensional random vector $\mathbf{Z} := (Z_i)_{i=1}^p$ be observed independently over an experimental design given by a regular lattice:

$$\Xi_{n_1 \dots n_d} := \left\{ \left(\frac{j_k}{n_k} \right)_{k=1}^d \in \mathbf{I}^d : 1 \leq j_k \leq n_k, k = 1, \dots, d \right\}, \quad (1)$$

where $\mathbf{I}^d := \prod_{k=1}^d [0, 1]$ is the d -dimensional unit cube. Let $\mathbf{g} := (g_i)_{i=1}^p$ be the true but unknown $\mathbb{R}^p$ -valued regression function on $\mathbf{I}^d$ which represents the mean function of the observations. Let $\mathbf{Z}_{j_1 \dots j_d} := (Z_{i,j_1 \dots j_d})_{i=1}^p$ and $\mathbf{g} := (g_{i,j_1 \dots j_d})_{i=1}^p$ be the observation and the corresponding mean in the experimental condition $(j_k/n_k)_{k=1}^d$. Under the null hypothesis H_0 we assume that $\mathbf{Z}_{j_1 \dots j_d}$ follows a multivariate linear model. That is, we assume a model

$$H_0 : \mathbf{Z}_{j_1 \dots j_d} = \left(\sum_{w=1}^{d_i} \beta_{iw} f_{iw} \left(\frac{j_k}{n_k} \right)_{k=1}^d \right)_{i=1}^p + \mathcal{E}_{j_1 \dots j_d}, \quad (2)$$

$$1 \leq j_1 \leq n_1, \dots, 1 \leq j_d \leq n_d,$$

where, for $i = 1, \dots, p$, $\beta_i := (\beta_{iw})_{w=1}^{d_i} \in \mathbb{R}^{d_i}$ is a d_i -dimensional vector of unknown parameters; $\mathbf{f}_i := (f_{iw})_{w=1}^{d_i}$ is a d_i -dimensional vector of known regression functions whose components are assumed to be square integrable with respect to the Lebesgue measure λ_1^d on $\mathbf{I}^d$, that is, $f_{iw} \in L_2(\lambda_1^d)$, for all i and w . $\mathcal{E}_{j_1 \dots j_d} := (\varepsilon_{i,j_1 \dots j_d})_{i=1}^p$ is the mutually independent p -dimensional vector of random errors defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that, for all $1 \leq j_1 \leq n_1, \dots, 1 \leq j_d \leq n_d$, $E(\mathcal{E}_{j_1 \dots j_d}) = \mathbf{0} \in \mathbb{R}^p$, and $\text{Cov}(\mathcal{E}_{j_1 \dots j_d}) = \Sigma = (\sigma_{uv})_{u,v=1}^p$. Let $\mathbf{Z}_{n_1 \dots n_d} := (\mathbf{Z}_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$ be the p -dimensional pyramidal array of random observations and let $\mathcal{E}_{n_1 \dots n_d} := (\mathcal{E}_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$ be the p -dimensional pyramidal array of random errors taking values in the Euclidean space $\prod_{i=1}^p \mathbb{R}^{n_1 \times \dots \times n_d}$. Then under H_0 the observations can be represented by

$$\mathbf{Z}_{n_1 \dots n_d} = \left(\sum_{w=1}^{d_i} \beta_{iw} f_{iw} \left(\Xi_{n_1 \dots n_d} \right) \right)_{i=1}^p + \mathcal{E}_{n_1 \dots n_d}, \quad (3)$$

where $f_{iw}(\Xi_{n_1 \dots n_d}) := (f_{iw}((j_k/n_k)_{k=1}^d))_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$. Under the alternative H_1 a multivariate nonparametric regression model

$$\mathbf{Z}_{n_1 \dots n_d} = \mathbf{g}(\Xi_{n_1 \dots n_d}) + \mathcal{E}_{n_1 \dots n_d} \quad (4)$$

is assumed, where $\mathbf{g}(\Xi_{n_1 \dots n_d}) := (\mathbf{g}((j_k/n_k)_{k=1}^d))_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \in \prod_{i=1}^p \mathbb{R}^{n_1 \times \dots \times n_d}$. By applying the similar argument as in Christensen [2] and Johnson and Wichern [4], the p -dimensional array of the least squares residuals of the observations is given by the following component-wise projection:

$$\mathbf{R}_{n_1 \dots n_d} := (\mathbf{r}_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \quad (5)$$

$$:= \mathbf{Z}_{n_1 \dots n_d} - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i,n_1 \dots n_d}} \mathbf{Z}_{n_1 \dots n_d},$$

with $\mathbf{r}_{j_1 \dots j_d} := (r_{i,j_1 \dots j_d})_{i=1}^p$, for $1 \leq j_k \leq n_k$, and $k = 1, \dots, d$. Thereby, for $i = 1, \dots, p$, we define $\mathbf{W}_{i,n_1 \dots n_d} := [f_{i1}(\Xi_{n_1 \dots n_d}), \dots, f_{id_i}(\Xi_{n_1 \dots n_d})]$ as the subspace of $\mathbb{R}^{n_1 \times \dots \times n_d}$ spanned by the arrays $\{f_{i1}(\Xi_{n_1 \dots n_d}), \dots, f_{id_i}(\Xi_{n_1 \dots n_d})\}$. It is worth mentioning that the Euclidean space $\mathbb{R}^{n_1 \times \dots \times n_d}$ is furnished with the inner product denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n_1 \times \dots \times n_d}}$ and defined by

$$\langle \mathbf{A}_{n_1 \dots n_d}, \mathbf{B}_{n_1 \dots n_d} \rangle_{\mathbb{R}^{n_1 \times \dots \times n_d}} := \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} a_{j_1 \dots j_d} b_{j_1 \dots j_d}, \quad (6)$$

for every $\mathbf{A}_{n_1 \dots n_d} := (a_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$, and $\mathbf{B}_{n_1 \dots n_d} := (b_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \in \mathbb{R}^{n_1 \times \dots \times n_d}$.

Next we define the set-indexed partial sums operator. Let $\mathcal{A}$ be the family of convex subset of $\mathbf{I}^d$, and let $\eta_{\lambda_1^d}$ be the Lebesgue pseudometric on $\mathcal{A}$ defined by $\eta_{\lambda_1^d}(A_1, A_2) := \lambda_1^d(A_1 \Delta A_2)$, for $A_1, A_2 \in \mathcal{A}$. Let $\mathcal{C}(\mathcal{A})$ be the set of continuous functions on $\mathcal{A}$ under $\eta_{\lambda_1^d}$. We embed the array of the residual $\mathbf{R}_{n_1 \dots n_d}$ into a p -dimensional stochastic process indexed by $\mathcal{A}$ by using the component-wise set-indexed partial sums operator

$$\mathbf{V}_{n_1 \dots n_d} : \prod_{i=1}^p \mathbb{R}^{n_1 \times \dots \times n_d} \mapsto \mathcal{C}^p(\mathcal{A}) := \prod_{i=1}^p \mathcal{C}(\mathcal{A}), \quad (7)$$

such that, for any $B \in \mathcal{A}$,

$$\mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d})(B) := \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} \sqrt{n_1 \dots n_d} \lambda_1^d(B \cap C_{j_1 \dots j_d}) \mathbf{r}_{j_1 \dots j_d}, \quad (8)$$

where, for $1 \leq j_1 \leq n_1, \dots, 1 \leq j_d \leq n_d$, $C_{j_1 \dots j_d} := \prod_{k=1}^d ((j_k - 1)/n_k, j_k/n_k]$. Let us call this process the p -dimensional set-indexed least squares residual partial sums process. The space $\mathcal{C}^p(\mathcal{A})$ is furnished with the uniform topology induced by the metric φ defined by

$$\varphi(\mathbf{u}, \mathbf{w}) := \sum_{i=1}^p \|u_i - w_i\|_{\mathcal{A}} = \sum_{i=1}^p \sup_{A \in \mathcal{A}} |u_i(A) - w_i(A)|, \quad (9)$$

for $\mathbf{u} := (u_i)_{i=1}^p$ and $\mathbf{w} := (w_i)_{i=1}^p \in \mathcal{C}^p(\mathcal{A})$.

We notice that, in the works of Bischoff and Somayasa [9], Bischoff and Gegg [17], and Somayasa and Adhi Wibawa [18], the limit process of the partial sums process of the least squares residuals has been investigated by applying the existing geometric method of Bischoff [11, 12]. However, the method becomes not applicable anymore in deriving the limit process of $\mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d})$ as the dimension of $\mathbf{W}_{i,n_1 \dots n_d}$ varies. Therefore, in this work, we attempt to adopt the vectorial analog of Prohorov's theorem; see, for example, Theorem 6.1 in Billingsley [19] for obtaining the limit process. For our result we need to extend the ordinary partial sums formula to p -dimensional case defined on $\mathbf{I}^d$ as follows. Let $K := \{1, 2, \dots, d\}$ and C_k^K be the set of all k -combinations of the set K , with $k = 1, \dots, d$. For a chosen value of k , we denote

the i th element of C_k^K by a k -tuple $(i(k_1), i(k_2), \dots, i(k_k))$, for $1 \leq i \leq |C_k^K|$, where $|C_k^K|$ is the number of elements of C_k^K which is clearly given by

$$|C_k^K| = \frac{d(d-1)(d-2) \cdots (d-k+1)}{k(k-1)(k-2) \cdots 1}. \quad (10)$$

For example, let $K = \{1, 2, 3\}$. Then, for $k = 1$, we denote the elements of C_1^K as $1(k_1) := 1$, $2(k_1) := 2$, and $3(k_1) := 3$. In a similar way, we denote the elements of C_2^K which consists of $\{(1, 2), (1, 3), (2, 3)\}$, respectively, by $(1(k_1), 1(k_2)) := (1, 2)$, $(2(k_1), 2(k_2)) := (1, 3)$, and $(3(k_1), 3(k_2)) := (2, 3)$. Finally the element of $C_3^K = \{(1, 2, 3)\}$ is sufficiently written by $(1(k_1), 1(k_2), 1(k_3))$. Hence the p -dimensional ordinary partial sums operator transforms any p -dimensional array $\mathbf{A}_{n_1 \cdots n_d} = (\mathbf{a}_{j_1 \cdots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \in \prod_{i=1}^p \mathbb{R}^{n_1 \times \cdots \times n_d}$ to a continuous function on $\mathbf{I}^d$ defined by

$$\begin{aligned} \mathbf{T}_{n_1 \cdots n_d}(\mathbf{A}_{n_1 \cdots n_d})(\mathbf{t}) &:= \frac{1}{\sqrt{n_1 \cdots n_d}} \sum_{j_1=1}^{[n_1 t_1]} \cdots \sum_{j_d=1}^{[n_d t_d]} \mathbf{a}_{j_1 \cdots j_d} \\ &+ \sum_{k=1}^d \sum_{i=1}^{|C_k^K|} \left(\prod_{u=1}^k \left(t_{i(k_u)} - \frac{[n_{i(k_u)} t_{i(k_u)}]}{n_{i(k_u)}} \right) \right) \\ &\cdot \frac{\sqrt{n_1 \cdots n_d}}{(n_1 \cdots n_d) / (n_{i(k_1)} \cdots n_{i(k_d)})} \\ &\cdot \sum_{\substack{[n_1 t_1], \dots, [n_d t_d] \\ \{j_1=1, \dots, j_d=1\} / \{j_{i(k_1)}, \dots, j_{i(k_d)}\}}}^{[n_1 t_1], \dots, [n_d t_d]} \mathbf{a}_{j_1 \cdots j_d} \cdot \frac{1}{[n_{i(k_1)} t_{i(k_1)}] + 1, \dots, [n_{i(k_d)} t_{i(k_d)}] + 1}, \end{aligned} \quad (11)$$

for every $\mathbf{t} := (t_1, \dots, t_d)^\top \in \mathbf{I}^d$, where for positive integers $b_u, b_{u+1}, \dots, b_{u+m} \in \mathcal{Z}_+$ we define a notation

$$\begin{aligned} \mathbf{a}_{j_1 \cdots j_d} \mid j_u = b_u, \dots, j_{u+m} = b_{u+m} \\ := \mathbf{a}_{j_1, \dots, j_{u-1}, b_u, \dots, b_{u+m}, j_{u+m+1}, \dots, j_d}. \end{aligned} \quad (12)$$

It is clear that the partial sums process of the residuals obtained using (11) is a special case of (8) since for every $\mathbf{t} := (t_1, \dots, t_d)^\top \in \mathbf{I}^d$ it holds

$$\mathbf{T}_{n_1 \cdots n_d}(\mathbf{A}_{n_1 \cdots n_d})(\mathbf{t}) = \mathbf{V}_{n_1 \cdots n_d}(\mathbf{A}_{n_1 \cdots n_d}) \left(\prod_{k=1}^d [0, t_k] \right). \quad (13)$$

It is worth noting that the extension of the study from univariate to multivariate model and also the expansion of the dimension of the lattice points are strongly motivated by the prediction problem in mining industry and geosciences. As for an example recently Tahir [20] presented data provided by PT Antam Tbk (a mining industry in Southeast Sulawesi). The data consist of a joint measurement of the percentage of several chemical elements and substances such as Ni, Co, Fe, MgO, SiO₂, and CaO which are recorded in every point of a three-dimensional lattice defined over the exploration region of the company. Hence, by the inherent existence of the correlation among the variables, the statistical analysis for the involved variables must be conducted simultaneously.

There have been many methods proposed in the literatures for testing H_0 . Most of them have been derived for the case of $\mathbf{W}_1 = \cdots = \mathbf{W}_p$ under normally distributed random error. Generalized likelihood ratio test which has been leading to Wilk's lambda statistic or variant of it can be found in Zellner [1], Christensen [2], pp. 1–22, Anderson [3], pp. 187–191, and Johnson and Wichern [4], pp. 395–398. Mardia and Goodall [21] derived the maximum likelihood estimation procedure for the parameters of the general normal spatial multivariate model with stationary observations. This approach can be straightforwardly extended for obtaining the associated likelihood ratio test in model check for the model. Unfortunately, in the practice especially when dealing with mining data, normal distribution is sometimes found to be not suitable for describing the distribution model of the observations, so that the test procedures mentioned above are consequently no longer applicable. Asymptotic method established in Arnold [22] for multivariate regression with $\mathbf{W}_1 = \cdots = \mathbf{W}_p$ can be generalized in such a way that it is valid for the general model. As a topic in statistics it must be well known. However, we cannot find literatures where the topic has been studied.

The rest of the paper is organized as follows. In Section 2 we show that when H_0 is true $\Sigma^{-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d})$ converges weakly to a projection of the p -dimensional set-indexed Brownian sheet. The limit process is shown to be useful for testing H_0 asymptotically based on the Kolmogorov-Smirnov (KS-test) and Cramér-von Mises (CvM-test) functionals of the set-indexed p -dimensional least squares residual partial sum processes, defined, respectively, by

$$\begin{aligned} \text{KS}_{n_1 \cdots n_d, \mathcal{A}} &:= \sup_{A \in \mathcal{A}} \left\| \Sigma^{-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d})(A) \right\|_{\mathbb{R}^p} \\ \text{CvM}_{n_1 \cdots n_d, \mathcal{A}} &:= \frac{1}{n_1 \cdots n_d} \sum_{A \in \mathcal{A}} \left\| \Sigma^{-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d})(A) \right\|_{\mathbb{R}^p}^2. \end{aligned} \quad (14)$$

For both tests the rejection of H_0 is for large value of the KS and CvM statistics, respectively. Under localized alternative the above sequence of random processes converges weakly to the above limit process with an additional deterministic trend (see Section 3). In Section 4, we define a consistent estimator for Σ . In Section 5 we investigate a residual based bootstrap method for the calibration of the tests. Monte Carlo simulation for the purpose of studying the finite sample behavior of the KS and CvM tests is reported in Section 6. Application of the test procedure in real data is presented in Section 7. The paper is closed in Section 8 with conclusion and some remarks for future research. Auxiliary results needed for deriving the limit process are presented in the appendix. We note that all convergence results derived throughout this paper which hold for $n_1, \dots, n_d$ simultaneously go to infinity, that is, for $n_k \rightarrow \infty$, for all $k = 1, \dots, d$; otherwise they will be stated in some way. The notion of convergence in distribution and convergence in probability will be conventionally denoted by $\xrightarrow{\mathcal{D}}$ and $\xrightarrow{\mathcal{P}}$, respectively.

2. The Limit of $\mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d})$ under H_0

Let $B := \{B(A) : A \in \mathcal{A}\}$ be the one-dimensional set-indexed Brownian sheet having sample path in $\mathcal{C}(\mathcal{A})$. We refer the reader to Pyke [23], Bass and Pyke [24], and Alexander and Pyke [25] for the definition and the existence of B . Let us consider a subspace of $\mathcal{C}(\mathcal{A})$ which is closely related to B , defined by

$$\begin{aligned} \mathcal{H}_B &:= \left\{ h_v : \mathcal{A} \longrightarrow \mathbb{R}, \exists v \in L_2(\lambda_1^d), \text{ s.t. } h_v(A) \right. \\ &:= \left. \int_A v(\mathbf{t}) \lambda^d(\mathbf{t}) \right\}. \end{aligned} \quad (15)$$

Under the inner product and the norm defined by

$$\begin{aligned} \langle h_v, h_w \rangle_{\mathcal{H}_B} &:= \int_{\mathbf{I}^d} v(\mathbf{t}) w(\mathbf{t}) \lambda_1^d(\mathbf{t}), \\ \|h_v\|^2 &:= \int_{\mathbf{I}^d} v^2(\mathbf{t}) \lambda_1^d(\mathbf{t}), \end{aligned} \quad (16)$$

it is clear that $\mathcal{H}_B$ and $L_2(\lambda_1^d)$ are isometric. For our result we need to define subspaces $\mathbf{W}_i$ and $\mathbf{W}_{i\mathcal{H}_B}$ associated with the regression functions $f_{i1}, \dots, f_{id_i}$, where $\mathbf{W}_i := [f_{i1}, \dots, f_{id_i}] \subset L_2(\lambda_1^d)$ and $\mathbf{W}_{i\mathcal{H}_B} := [h_{f_{i1}}, \dots, h_{f_{id_i}}] \subset \mathcal{H}_B$, for $i = 1, \dots, p$.

Now we are ready to state the limit process of the sequence of p -dimensional set-indexed residual partial sums processes for the model specified under H_0 .

Theorem 1. For $i = 1, \dots, p$, let $\{f_{i1}, \dots, f_{id_i}\}$ be an orthonormal basis (ONB) of $\mathbf{W}_i$. We assume that $\mathbf{W}_i \subseteq \mathbf{W}_{i+1}$, for $i = 1, \dots, p-1$. Let $\mathbf{B}_p := \{(B_i(A))_{i=1}^p : A \in \mathcal{A}\}$ be the p -dimensional set-indexed Brownian sheet with the covariance function $\text{Cov}(\mathbf{B}_p(A), \mathbf{B}_p(A')) := \lambda_1^d(A \cap A') \mathbf{I}_p$, for $A, A' \in \mathcal{A}$, where $\mathbf{I}_p$ is the $p \times p$ identity matrix. Suppose $\{f_{i1}, \dots, f_{id_i}\}$ are in $\mathcal{C}(\mathbf{I}^d) \cap \text{BV}_H(\mathbf{I}^d)$, where $\mathcal{C}(\mathbf{I}^d)$ is the space of continuous functions on $\mathbf{I}^d$ (see Definition A.4 for the definition of $\text{BV}_H(\mathbf{I}^d)$). Then under H_0 it holds that

$$\Sigma^{-1/2} \mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d}) \xrightarrow{\mathcal{D}} \mathbf{B}_{p,f}^{H_0} := \mathbf{B}_p - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i\mathcal{H}_B}}^* \mathbf{B}_p, \quad (17)$$

where

$$\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i\mathcal{H}_B}}^* \mathbf{B}_p := \left(\text{pr}_{\mathbf{W}_{i\mathcal{H}_B}}^* B_i \right)_{i=1}^p. \quad (18)$$

Thereby $\text{pr}_{\mathbf{W}_{i\mathcal{H}_B}}^*$ is a projector such that, for every $u \in \mathcal{C}(\mathcal{A})$ and $A \in \mathcal{A}$,

$$\left(\text{pr}_{\mathbf{W}_{i\mathcal{H}_B}}^* u \right)(A) := \sum_{j=1}^{d_i} \langle h_{f_{ij}}, u \rangle h_{f_{ij}}(A), \quad (19)$$

$$\text{where } \langle h_{f_{ij}}, u \rangle := \int_{\mathbf{I}^d} f_{ij}(\mathbf{t}) du(\mathbf{t}).$$

For $\mathbf{t} := (t_k)_{k=1}^d \in \mathbf{I}^d$, we set $u(\mathbf{t})$ for $u(\prod_{k=1}^d [0, t_k])$, and $\int_{\mathbf{I}^d}^{(R)}$ stands for the integral in the sense of Riemann-Stieltjes.

Moreover, the limit process $\mathbf{B}_{p,f}^{H_0}$ is a centered Gaussian process with the covariance function given by

$$\begin{aligned} K(A, C) &:= \text{diag} \left(\lambda(A \cap C) \right. \\ &- \sum_{j=1}^{d_1} h_{f_{1j}}(A) h_{f_{1j}}(C), \dots, \lambda(A \cap C) \\ &- \sum_{j=1}^{d_p} h_{f_{pj}}(A) h_{f_{pj}}(C) \left. \right). \end{aligned} \quad (20)$$

Proof. By applying the linear property of $\mathbf{V}_{n_1 \dots n_d}$ and Lemma C.2, we have under H_0 ,

$$\begin{aligned} \mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d}) &= \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d}) \\ &- \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d} \mathcal{H}_B} \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d}). \end{aligned} \quad (21)$$

It can be shown by extending the uniform central limit theorem studied in Pyke [23], Bass and Pyke [24], and Alexander and Pyke [25] to its vectorial analog that the term $\Sigma^{-1/2} \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d})$ on the right-hand side of (21) converges weakly to $\mathbf{B}_p$. Therefore we only need to show that the second term satisfies the weak convergence:

$$\begin{aligned} \mathcal{Z}_{n_1 \dots n_d} &:= \Sigma^{-1/2} \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d} \mathcal{H}_B} \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d}) \xrightarrow{\mathcal{D}} \\ U &:= \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i\mathcal{H}_B}}^* \mathbf{B}_p, \end{aligned} \quad (22)$$

where U is a p -dimensional centered Gaussian process with the covariance matrix given by

$$\begin{aligned} K_U(A_1, A_2) &= \text{diag} \left(\sum_{j=1}^{d_1} h_{f_{1j}}(A_1) h_{f_{1j}}(A_2), \dots, \right. \\ &\left. \sum_{j=1}^{d_p} h_{f_{pj}}(A_1) h_{f_{pj}}(A_2) \right), \text{ for } A_1, A_2 \in \mathcal{A}. \end{aligned} \quad (23)$$

By Prohorov's theorem it is sufficient to show that, for any finite collection of convex sets $A_1, \dots, A_r$ in $\mathcal{A}$ and real numbers $c_1, \dots, c_r$, with $r \geq 1$, it holds that

$$\sum_{k=1}^r c_k \mathcal{Z}_{n_1 \dots n_d}(A_k) \xrightarrow{\mathcal{D}} \sum_{k=1}^r c_k U(A_k), \quad (24)$$

where the left-hand side has the covariance which can be expressed as

$$\begin{aligned} \text{Cov} \left(\sum_{k=1}^r c_k \mathcal{Z}_{n_1 \dots n_d}(A_k) \right) &= \sum_{k=1}^r \sum_{\ell=1}^r c_k c_\ell \Sigma^{-1/2} \left(E(B_i(A_k) B_j(A_\ell)) \right)_{i,j=1}^p \Sigma^{-1/2}, \end{aligned} \quad (25)$$

where $B_i(A_k) := \text{pr}_{\mathbf{W}_{i,n_1 \dots n_d} \mathcal{R}_B} \mathbf{V}_{n_1 \dots n_d}(\varepsilon_{i,n_1 \dots n_d})(A_k)$, for $i = 1, \dots, p$ and $k = 1, \dots, r$. Let k and ℓ be fixed. Then by a simultaneous application of the definition of $\text{pr}_{\mathbf{W}_{i,n_1 \dots n_d} \mathcal{R}_B}$ (see Lemma C.3), (11), the definition of the Riemann-Stieltjes integral (cf. Stroock [26], pp. 7–17), and the independence of the array $\{\varepsilon_{i,j_1 \dots j_d} : 1 \leq j_k \leq n_k\}$, we further get

$$\begin{aligned} E(B_i(A_k) B_j(A_\ell)) &= \sum_{w=1}^{d_i} \sum_{w'=1}^{d_j} \int_{A_k} \tilde{s}_{iw}^{(n_1 \dots n_d)}(\mathbf{t}) \lambda_1^d(d\mathbf{t}) \\ &\quad \cdot \int_{A_\ell} \tilde{s}_{jw'}^{(n_1 \dots n_d)}(\mathbf{t}) \lambda_1^d(d\mathbf{t}) E \left(\int_{\mathbf{I}^d} \tilde{s}_{iw}^{(n_1 \dots n_d)}(\mathbf{t}) d\mathbf{V}_{n_1 \dots n_d} \right. \\ &\quad \cdot (\varepsilon_{i,n_1 \dots n_d})(\mathbf{t}) \int_{\mathbf{I}^d} \tilde{s}_{jw'}^{(n_1 \dots n_d)}(\mathbf{t}) d\mathbf{V}_{n_1 \dots n_d}(\varepsilon_{j,n_1 \dots n_d})(\mathbf{t}) \Big) \\ &= \sum_{w=1}^{d_i} \sum_{w'=1}^{d_j} \int_{A_k} \tilde{s}_{iw}^{(n_1 \dots n_d)}(\mathbf{t}) \lambda_1^d(d\mathbf{t}) \int_{A_\ell} \tilde{s}_{jw'}^{(n_1 \dots n_d)}(\mathbf{t}) \\ &\quad \cdot \lambda_1^d(d\mathbf{t}) \frac{\sigma_{ij}}{n_1 \dots n_d} \sum_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \tilde{s}_{iw}^{(n_1 \dots n_d)}\left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d}\right) \\ &\quad \cdot \tilde{s}_{jw'}^{(n_1 \dots n_d)}\left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d}\right). \end{aligned} \quad (26)$$

By recalling Lemma C.3 the last expression clearly converges to

$$\begin{aligned} &\sigma_{ij} \sum_{w=1}^{d_i} \sum_{w'=1}^{d_j} \langle f_{iw}, f_{jw'} \rangle_{L_2(\lambda_1^d)} h_{f_{iw}}(A_k) h_{f_{jw'}}(A_\ell) \\ &= \sigma_{ij} \sum_{w=1}^{d_i} h_{f_{iw}}(A_k) h_{f_{jw}}(A_\ell). \end{aligned} \quad (27)$$

Hence the matrix $(E(B_i(A_k) B_j(A_\ell)))_{i,j=1}^p$ converges element wise to the symmetric matrix which can be represented as $\Sigma \odot \mathbf{D}$, for a matrix $\mathbf{D}$ defined by

$$\mathbf{D} := \begin{pmatrix} \kappa_{11}(A_k, A_\ell) & \kappa_{12}(A_k, A_\ell) & \cdots & \kappa_{1p}(A_k, A_\ell) \\ \kappa_{21}(A_k, A_\ell) & \kappa_{22}(A_k, A_\ell) & \cdots & \kappa_{2p}(A_k, A_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{p1}(A_k, A_\ell) & \kappa_{p2}(A_k, A_\ell) & \cdots & \kappa_{pp}(A_k, A_\ell) \end{pmatrix} \quad (28)$$

with $\kappa_{ij}(A_k, A_\ell) := \sum_{w=1}^{d_{\min(i,j)}} h_{f_{iw}}(A_k) h_{f_{jw}}(A_\ell)$, for $i, j = 1, \dots, p$. Thereby $\odot$ denotes the Hadarmard product defined, for example, in Magnus and Neudecker [27], pp. 53–54. Since $\Sigma^{-1/2}(\Sigma \odot \mathbf{D})\Sigma^{-1/2} = \mathbf{D} \odot \mathbf{I}_p$, we successfully have shown that $\text{Cov}(\sum_{k=1}^r c_k \mathcal{Z}_{n_1 \dots n_d}(A_k))$ converges to the following general linear combination:

$$\begin{aligned} &\sum_{k=1}^r \sum_{\ell=1}^r c_k c_\ell \\ &\quad \cdot \text{diag} \left(\sum_{w=1}^{d_1} h_{f_{1w}}(A_k) h_{f_{1w}}(A_\ell), \dots, \sum_{w=1}^{d_p} h_{f_{pw}}(A_k) h_{f_{pw}}(A_\ell) \right), \end{aligned} \quad (29)$$

which is actually the covariance of $\sum_{k=1}^r c_k U(A_k)$. Next we observe that, by applying the definition of $\mathbf{V}_{n_1 \dots n_d}$ and the definition of the Riemann-Stieltjes integral, we can also write $\sum_{k=1}^r c_k \mathcal{Z}_{n_1 \dots n_d}(A_k)$ as follows:

$$\begin{aligned} &\sum_{k=1}^r c_k \mathcal{Z}_{n_1 \dots n_d}(A_k) \\ &= \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} \Sigma^{-1/2} \left(\sum_{k=1}^r \Lambda_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d}; A_k \right) \right)_{i=1}^p, \end{aligned} \quad (30)$$

where

$$\begin{aligned} \Lambda_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d}; A_k \right) &:= \sum_{w=1}^{d_i} \frac{c_k}{\sqrt{n_1 \dots n_d}} \\ &\quad \cdot \tilde{s}_{iw}^{(n_1 \dots n_d)} \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \varepsilon_{i,j_1 \dots j_d} h_{\tilde{s}_{iw}^{(n_1 \dots n_d)}}(A_k). \end{aligned} \quad (31)$$

Let $\Gamma_f := \max_{1 \leq w \leq d_i; 1 \leq i \leq p} \|f_{iw}\|_\infty$, $M := \max_{1 \leq k \leq r} |c_k|$, and $\|\Sigma^{-1/2}\|$ be the Euclidean norm of $\Sigma^{-1/2}$. Then by considering the stochastic independence of the array of the p -vector of the random errors, it holds that

$$\begin{aligned} 0 &\leq \mathbb{L}_{\mathcal{Z}_{n_1 \dots n_d}}(\epsilon) := \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} E \\ &\quad \cdot \left(\left\| \left(\sum_{k=1}^r \Lambda_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d}; A_k \right) \right)_{i=1}^p \right\|_{\mathbb{R}^p}^2 \right. \\ &\quad \cdot \mathbf{1} \left\{ \left\| \left(\sum_{k=1}^r \Lambda_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d}; A_k \right) \right)_{i=1}^p \right\|_{\mathbb{R}^p}^2 \geq \epsilon \right\} \Big) \\ &\leq (rM)^2 (d_p \Gamma_f^2)^2 \left\| \Sigma^{-1/2} \right\|^2 E \left(\left\| \mathcal{E}_{j_1 \dots j_d} \right\|_{\mathbb{R}^p}^2 \right. \\ &\quad \cdot \mathbf{1} \left\{ \left\| \mathcal{E}_{j_1 \dots j_d} \right\|_{\mathbb{R}^p} \geq \frac{\epsilon \sqrt{n_1 \dots n_d}}{(rM)(d_p \Gamma_f^2)} \right\} \Big) \quad \forall \epsilon > 0, \end{aligned} \quad (32)$$

in which by the well-known bounded convergence theorem (cf. Athreya and Lahiri [28], pp. 57–58) the last term converges to zero. Thus the Lindeberg condition is satisfied. Therefore by the Lindeberg-Levy multivariate central limit theorem studied, for example, in van der Vaart [29], pp. 16, it can be concluded that $\sum_{k=1}^r c_k \mathcal{Z}_{n_1 \dots n_d}(A_k)$ converges in distribution to $\sum_{k=1}^r c_k U(A_k)$, where $\sum_{k=1}^r c_k U(A_k)$ has the p -variate normal distribution with mean zero and the covariance given by (29).

The tightness of $\mathcal{Z}_{n_1 \dots n_d}$ can be shown as follows. By the definition $\mathcal{Z}_{n_1 \dots n_d}$ can also be expressed as

$$\begin{aligned} &\mathcal{Z}_{n_1 \dots n_d} \\ &= \Sigma^{-1/2} \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i,n_1 \dots n_d} \mathcal{R}_B} \Sigma^{1/2} \Sigma^{-1/2} \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d}). \end{aligned} \quad (33)$$

Since $\Sigma^{-1/2} \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d})$ is tight, hence by recalling Lemma 1 in Billingsley [19], pp. 38–40, it is sufficient to show that the mapping $\Sigma^{-1/2} \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{B}}} \Sigma^{1/2}$ is continuous on $\mathcal{E}^p(\mathcal{A})$ for every $n_1 \geq 1, \dots, n_d \geq 1$. Proposition C.4 finishes the proof. $\square$

Corollary 2. By Theorem 1 and the well-known continuous mapping theorem (cf. Theorem 5.1 in Billingsley [19]) the distribution of the statistics $KS_{n_1 \dots n_d, \mathcal{A}}$ and $CvM_{n_1 \dots n_d, \mathcal{A}}$ can be approximated, respectively, by those of

$$\begin{aligned} KS &:= \sup_{A \in \mathcal{A}} \left\| \left(\mathbf{B}_p - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, \mathcal{B}}}^* \mathbf{B}_p \right) (A) \right\|_{\mathbb{R}^p} \\ CvM &:= \int_{\mathbf{I}^d} \left\| \left(\mathbf{B}_p - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, \mathcal{B}}}^* \mathbf{B}_p \right) (A) \right\|_{\mathbb{R}^p}^2 dA. \end{aligned} \quad (34)$$

Let $\tilde{\tau}_{1-\alpha}$ and $\tilde{c}_{1-\alpha}$ be the $(1-\alpha)$ th quantile of the distributions of KS and CvM , respectively. When $KS_{n_1 \dots n_d, \mathcal{A}}$ is used, H_0 will be rejected at level α if and only if $KS_{n_1 \dots n_d, \mathcal{A}} \geq \tilde{\tau}_{1-\alpha}$. Likewise if $CvM_{n_1 \dots n_d, \mathcal{A}}$ is used, then H_0 will be rejected at level α if and only if $CvM_{n_1 \dots n_d, \mathcal{A}} \geq \tilde{c}_{1-\alpha}$.

3. The Limit of $\mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d})$ under H_1

The test procedures derived above are consistent in the sense of Definition 11.1.3 in Lehmann and Romano [30]. That is, the probability of rejection of H_0 under the competing alternative converges to 1. As an immediate consequence we cannot observe the performance of the tests when the model moves away from H_0 . Therefore, to be able to investigate the behavior of the tests, we consider the localized model defined as follows:

$$\mathbf{Z}_{n_1 \dots n_d} = \frac{1}{\sqrt{n_1 \dots n_d}} \mathbf{g}(\Xi_{n_1 \dots n_d}) + \mathcal{E}_{n_1 \dots n_d}. \quad (35)$$

When H_0 is true we get the similar least squares residuals as given in Section 1. Therefore, observing Model (35), the test problem will not be altered.

In the following theorem, we present the limit process of the p -dimensional set-indexed partial sums process of the residuals under H_1 associated with Model (35).

Theorem 3. For $i = 1, \dots, p$, let $\{f_{i1}, \dots, f_{id_i}\}$ be an ONB of $\mathbf{W}_i$ with $\mathbf{W}_i \subseteq \mathbf{W}_{i+1}$, for $i = 1, \dots, p-1$. If $\mathbf{g} = (g_i)_{i=1}^p \in \prod_{i=1}^p \text{BVV}(\mathbf{I}^d)$ and $\{f_{i1}, \dots, f_{id_i}\} \in \mathcal{E}(\mathbf{I}^d) \cap \text{BV}_H(\mathbf{I}^d)$ (see Definition A.3 for the notion of $\text{BVV}(\mathbf{I}^d)$), then, observing (35), we have under H_1 that

$$\Sigma^{-1/2} \mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d}) \xrightarrow{\mathcal{D}} \Sigma^{-1/2} \left(\mathbf{I}_p - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, \mathcal{B}}}^* \right) \mathbf{h}_{\mathbf{g}} + \mathbf{B}_{p, \mathbf{f}}^{H_0}, \quad (36)$$

where $\mathbf{h}_{\mathbf{g}} := (h_{g_i})_{i=1}^p : \mathcal{A} \rightarrow \mathcal{E}^p(\mathcal{A})$, with $h_{g_i}(A) := \int_A g_i(\mathbf{t}) \lambda_{\mathbf{I}^d}^d(d\mathbf{t})$.

Proof. Considering the linearity of $\mathbf{V}_{n_1 \dots n_d}$ and Lemma C.2, when H_1 is true we have

$$\begin{aligned} \Sigma^{-1/2} \mathbf{V}_{n_1 \dots n_d}(\mathbf{R}_{n_1 \dots n_d}) &= \Sigma^{-1/2} \frac{1}{\sqrt{n_1 \dots n_d}} \\ &\cdot \mathbf{V}_{n_1 \dots n_d}(\mathbf{g}(\Xi_{n_1 \dots n_d})) - \Sigma^{-1/2} \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{B}}}^* \\ &\cdot \frac{1}{\sqrt{n_1 \dots n_d}} \mathbf{V}_{n_1 \dots n_d}(\mathbf{g}(\Xi_{n_1 \dots n_d})) \\ &+ \Sigma^{-1/2} \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d}) \\ &- \Sigma^{-1/2} \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{B}}}^* \mathbf{V}_{n_1 \dots n_d}(\mathcal{E}_{n_1 \dots n_d}). \end{aligned} \quad (37)$$

Since, for $i = 1, \dots, p$, g_i is in $\text{BVV}(\mathbf{I}^d)$, it can be shown that $(1/\sqrt{n_1 \dots n_d}) \mathbf{V}_{n_1 \dots n_d}(\mathbf{g}(\Xi_{n_1 \dots n_d}))(B)$ converges uniformly to $\mathbf{h}_{\mathbf{g}}(B)$, for every $B \in \mathcal{A}$. Also the last two terms on the right-hand side of the preceding equation converge in distribution to $\mathbf{B}_{p, \mathbf{f}}^{H_0}$ by Theorem 1. Thus to the rest we only need to show that $\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{B}}}^* (1/\sqrt{n_1 \dots n_d}) \mathbf{V}_{n_1 \dots n_d}(\mathbf{g}_{n_1 \dots n_d})$ converges to $\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, \mathcal{B}}}^* \mathbf{h}_{\mathbf{g}}$. By the definition of component-wise projection we have

$$\begin{aligned} \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{B}}}^* \frac{1}{\sqrt{n_1 \dots n_d}} \mathbf{V}_{n_1 \dots n_d}(\mathbf{g}(\Xi_{n_1 \dots n_d})) &= \left(\text{pr}_{\mathbf{W}_{i, n_1 \dots n_d, \mathcal{B}}}^* \frac{1}{\sqrt{n_1 \dots n_d}} \mathbf{V}_{n_1 \dots n_d}^{(i)}(\mathbf{g}_i(\Xi_{n_1 \dots n_d})) \right)_{i=1}^p \\ &= \left(\sum_{j=1}^{d_i} \left(\int_{\mathbf{I}^d} \tilde{s}_{ij}^{(n_1 \dots n_d)}(\mathbf{t}) d \frac{1}{\sqrt{n_1 \dots n_d}} \mathbf{V}_{n_1 \dots n_d}^{(i)}(\mathbf{g}_i(\Xi_{n_1 \dots n_d}))(\mathbf{t}) h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}} \right) \right)_{i=1}^p \\ &= \left(\sum_{j=1}^{d_i} \left(\frac{1}{n_1 \dots n_d} \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} \tilde{s}_{ij}^{(n_1 \dots n_d)} \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) g_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \right) h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}} \right)_{i=1}^p. \end{aligned} \quad (38)$$

The right-hand side of the last expression is obtained directly from the definition of the ordinary partial sums (11) and

the definition of the Riemann-Stieltjes integral on $\mathbf{I}^d$. Since $\tilde{s}_{ij}^{(n_1 \dots n_d)}$ converges uniformly to f_{ij} and g_i has bounded variation

on $\mathbf{I}^d$ for all i and j , then the last expression clearly converges component-wise to $(\sum_{j=1}^{d_i} (\int_{\mathbf{I}^d}^{(R)} f_{ij}(\mathbf{t}) d\mathbf{h}_{g_i}(\mathbf{t}) h_{f_{ij}})_p^p)_{i=1}^p$, which can be written as $\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i\mathbb{R}_B}}^* \mathbf{h}_g$. We are done. $\square$

Corollary 4. By Theorem 3 the power function of the KS and CvM tests at a level α can now be approximated by computing the probabilities of the form

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{A \in \mathcal{A}} \left\| \Sigma^{-1/2} \left(\mathbf{h}_g - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i\mathbb{R}_B}}^* \mathbf{h}_g \right) + \mathbf{B}_{p,f}^{H_0} \right\|_{\mathbb{R}^p} \right. \\ & \quad \left. \geq \tilde{t}_{1-\alpha} \right\} \\ & \mathbb{P} \left\{ \int_{\mathbf{I}} \left\| \Sigma^{-1/2} \left(\mathbf{h}_g - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i\mathbb{R}_B}}^* \mathbf{h}_g \right) + \mathbf{B}_{p,f}^{H_0} \right\|_{\mathbb{R}^p}^2 dA \right. \\ & \quad \left. \geq \tilde{c}_{1-\alpha} \right\}, \end{aligned} \quad (39)$$

for a fixed $\mathbf{g} \in \prod_{i=1}^p \text{BV}(\mathbf{I}^d)$, respectively. In Section 5 we investigate the empirical power functions of the KS and CvM tests by simulation.

4. Estimating the Population Covariance Matrix

If the covariance matrix Σ is unknown, as it usually is, it is impossible to use $\text{KS}_{n_1 \cdots n_d, \mathcal{A}}$ and $\text{CvM}_{n_1 \cdots n_d, \mathcal{A}}$ in practice. What we propose to do is to employ a consistent estimate of Σ . We need some further notations for expressing the residuals of the model. For $i = 1, \dots, p$, let $Z_i^{(n_1 \cdots n_d)}$, $g_i^{(n_1 \cdots n_d)}$, and $\varepsilon_i^{(n_1 \cdots n_d)}$ be $(n_1 \cdots n_d)$ -dimensional column vectors defined by

$$\begin{aligned} Z_i^{(n_1 \cdots n_d)} &:= \left(Z_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \right)_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \in \mathbb{R}^{n_1 \cdots n_d} \\ g_i^{(n_1 \cdots n_d)} &:= \left(g_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \right)_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \in \mathbb{R}^{n_1 \cdots n_d} \\ \varepsilon_i^{(n_1 \cdots n_d)} &:= \left(\varepsilon_i \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \right)_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \in \mathbb{R}^{n_1 \cdots n_d}. \end{aligned} \quad (40)$$

Furthermore, let $\mathbf{Z}^{(n_1 \cdots n_d)}$, $\mathbf{g}^{(n_1 \cdots n_d)}$, and $\mathcal{E}^{(n_1 \cdots n_d)}$ be $(n_1 \cdots n_d) \times p$ -dimensional matrices whose i th column is given, respectively, by the column vectors $Z_i^{(n_1 \cdots n_d)}$, $g_i^{(n_1 \cdots n_d)}$, and $\varepsilon_i^{(n_1 \cdots n_d)}$, $i = 1, \dots, p$. Then Model (35) can also be represented as follows:

$$\mathbf{Z}^{(n_1 \cdots n_d)} = \frac{1}{\sqrt{n_1 \cdots n_d}} \mathbf{g}^{(n_1 \cdots n_d)} + \mathcal{E}^{(n_1 \cdots n_d)}, \quad (41)$$

where, for $u, v = 1, \dots, p$, $\text{Cov}(\varepsilon_u^{(n_1 \cdots n_d)}, \varepsilon_v^{(n_1 \cdots n_d)}) = \sigma_{uv} \mathbf{I}_{n_1 \cdots n_d}$, with $\mathbf{I}_{n_1 \cdots n_d}$ being the identity matrix in $\mathbb{R}^{(n_1 \cdots n_d) \times (n_1 \cdots n_d)}$.

Associated with the subspace $\mathbf{W}_{i, n_1 \cdots n_d}$ we define the design matrix $\mathbf{X}_i^{(n_1 \cdots n_d)}$ as an element of $\mathbb{R}^{(n_1 \cdots n_d) \times d_i}$ whose

u th column is given by the $(n_1 \cdots n_d)$ -dimensional column vector:

$$\begin{aligned} & \left(f_{iu} \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \right)_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d} \in \mathbb{R}^{n_1 \cdots n_d}, \\ & u = 1, \dots, d_i, \quad i = 1, \dots, p. \end{aligned} \quad (42)$$

We denote the column space of $\mathbf{X}_i^{(n_1 \cdots n_d)}$ by $\mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)}) \subset \mathbb{R}^{(n_1 \cdots n_d) \times d_i}$ for the sake of brevity. We also define the column-wise projection of any matrix $\mathbf{U}^{(n_1 \cdots n_d)} := (U_1^{(n_1 \cdots n_d)}, \dots, U_p^{(n_1 \cdots n_d)})$ in $\mathbb{R}^{n_1 \cdots n_d \times p}$ into the product space $\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})$ by

$$\begin{aligned} & \text{pr}_{\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})} \mathbf{U}^{(n_1 \cdots n_d)} \\ & := \left(\text{pr}_{\mathcal{E}(\mathbf{X}_1^{(n_1 \cdots n_d)})} U_1^{(n_1 \cdots n_d)}, \dots, \text{pr}_{\mathcal{E}(\mathbf{X}_p^{(n_1 \cdots n_d)})} U_p^{(n_1 \cdots n_d)} \right). \end{aligned} \quad (43)$$

A reasonable estimator of the covariance matrix Σ is denoted by $\hat{\Sigma}_{n_1 \cdots n_d}$, defined by

$$\begin{aligned} \hat{\Sigma}_{n_1 \cdots n_d} &:= \frac{1}{n_1 \cdots n_d} \left(\text{pr}_{\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})^\perp} \mathbf{Z}^{(n_1 \cdots n_d)} \right)^\top \\ & \cdot \left(\text{pr}_{\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})^\perp} \mathbf{Z}^{(n_1 \cdots n_d)} \right), \end{aligned} \quad (44)$$

where

$$\text{pr}_{\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})^\perp} := \mathbf{I}_{n_1 \cdots n_d} - \text{pr}_{\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})} \quad (45)$$

constitutes the component-wise orthogonal projector into the orthogonal complement of the product space $\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})$.

Zellner [1] and Arnold [22] investigated the consistency of $\hat{\Sigma}_{n_1 \cdots n_d}$ toward Σ in the case of the multivariate regression model with $\mathbf{W}_1 = \dots = \mathbf{W}_p$. Some difficulties appear when the situation is extended to the case of $\mathbf{W}_1 \neq \dots \neq \mathbf{W}_p$, since it involves the problem of finding the limit of matrices with the components given by inner products of two vectors.

Theorem 5. Suppose the localized model (41) is observed. If H_0 is true, then, under the conditions of Theorem 1, we have $\hat{\Sigma}_{n_1 \cdots n_d} \xrightarrow{\mathcal{P}} \Sigma$.

Proof. If H_0 is true, it can be easily shown that

$$\begin{aligned} (n_1 \cdots n_d) \hat{\Sigma}_{n_1 \cdots n_d} &= \left(\text{pr}_{\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})^\perp} \mathcal{E}^{(n_1 \cdots n_d)} \right)^\top \\ & \cdot \left(\text{pr}_{\prod_{i=1}^p \mathcal{E}(\mathbf{X}_i^{(n_1 \cdots n_d)})^\perp} \mathcal{E}^{(n_1 \cdots n_d)} \right). \end{aligned} \quad (46)$$

For technical reason we assume without lost of generality that $\mathbf{X}_i^{(n_1 \cdots n_d)}$ is an orthogonal matrix, for $i = 1, \dots, p$. Hence we further get the representation

$$\begin{aligned} & (n_1 \cdots n_d) \widehat{\Sigma}_{n_1 \cdots n_d} \\ &= \left(\varepsilon_u^{(n_1 \cdots n_d)^\top} \varepsilon_v^{(n_1 \cdots n_d)} \right)_{u,v=1}^{p,p} \\ & \quad - \left(\left(\mathbf{X}_v^{(n_1 \cdots n_d)^\top} \varepsilon_u^{(n_1 \cdots n_d)} \right)^\top \left(\mathbf{X}_v^{(n_1 \cdots n_d)^\top} \varepsilon_v^{(n_1 \cdots n_d)} \right) \right)_{u,v=1}^{p,p} \quad (47) \\ & \quad - \left(\left(\mathbf{X}_u^{(n_1 \cdots n_d)^\top} \varepsilon_u^{(n_1 \cdots n_d)} \right)^\top \left(\mathbf{X}_u^{(n_1 \cdots n_d)^\top} \varepsilon_v^{(n_1 \cdots n_d)} \right) \right)_{u,v=1}^{p,p} \\ & \quad + \left(\left(\mathbf{X}_u^{(n_1 \cdots n_d)^\top} \varepsilon_u^{(n_1 \cdots n_d)} \right)^\top \left(\mathbf{X}_v^{(n_1 \cdots n_d)^\top} \varepsilon_v^{(n_1 \cdots n_d)} \right) \right)_{u,v=1}^{p,p}. \end{aligned}$$

Since $(\varepsilon_u(j_1/n_1, \dots, j_d/n_d) \varepsilon_v(j_1/n_1, \dots, j_d/n_d))_{u,v=1}^{p,p}$ are independent and identically distributed random matrices with mean Σ , by the well-known weak law of large numbers, we get

$$\begin{aligned} & \frac{1}{n_1 \cdots n_d} \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} \left(\varepsilon_u \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \right. \\ & \quad \left. \cdot \varepsilon_v \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \right)_{u,v=1}^{p,p} \xrightarrow{\mathcal{P}} \Sigma. \end{aligned} \quad (48)$$

Note that in the practice we consider the polynomial regression model. Hence, for every $v = 1, \dots, p$, the design matrix $\mathbf{X}_v^{(n_1 \cdots n_d)}$ satisfies the so-called Huber condition (cf. Pruscha [31], pp. 115–117). By this reason, for the rest of the terms, we can immediately apply the technique proposed in Arnold [22] to show $\mathbf{X}_v^{(n_1 \cdots n_d)^\top} \varepsilon_u^{(n_1 \cdots n_d)} \xrightarrow{\mathcal{D}} N_{d_v}(\mathbf{0}, \sigma_{uu} \mathbf{I}_{d_v})$, for all $u, v = 1, \dots, p$. Therefore, we finally get the following component-wise convergence:

$$\begin{aligned} & \frac{1}{n_1 \cdots n_d} \left(\left(\mathbf{X}_v^{(n_1 \cdots n_d)^\top} \varepsilon_u^{(n_1 \cdots n_d)} \right)^\top \right. \\ & \quad \left. \cdot \left(\mathbf{X}_v^{(n_1 \cdots n_d)^\top} \varepsilon_v^{(n_1 \cdots n_d)} \right) \right)_{u,v=1}^{p,p} \xrightarrow{\mathcal{P}} \mathbf{O}_{p \times p} \\ & \frac{1}{n_1 \cdots n_d} \left(\left(\mathbf{X}_u^{(n_1 \cdots n_d)^\top} \varepsilon_u^{(n_1 \cdots n_d)} \right)^\top \right. \\ & \quad \left. \cdot \left(\mathbf{X}_u^{(n_1 \cdots n_d)^\top} \varepsilon_v^{(n_1 \cdots n_d)} \right) \right)_{u,v=1}^{p,p} \xrightarrow{\mathcal{P}} \mathbf{O}_{p \times p} \quad (49) \\ & \frac{1}{n_1 \cdots n_d} \left(\left(\mathbf{X}_u^{(n_1 \cdots n_d)^\top} \varepsilon_u^{(n_1 \cdots n_d)} \right)^\top \right. \\ & \quad \left. \cdot \left(\mathbf{X}_v^{(n_1 \cdots n_d)^\top} \varepsilon_v^{(n_1 \cdots n_d)} \right) \right)_{u,v=1}^{p,p} \xrightarrow{\mathcal{P}} \mathbf{O}_{p \times p}, \end{aligned}$$

where $\mathbf{O}_{p \times p}$ is the $p \times p$ -zero matrix. $\square$

Remark 6. Since $\widehat{\Sigma}_{n_1 \cdots n_d}^{-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d}) = \widehat{\Sigma}_{n_1 \cdots n_d}^{-1/2} \Sigma^{1/2} \Sigma^{-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d})$, without altering the convergence result presented in Theorem 1, the population

variance-covariance matrix Σ can be directly replaced by the consistence estimator $\widehat{\Sigma}_{n_1 \cdots n_d}$.

5. Calibration of the Tests

The limits of the test statistics are not distribution-free and we need therefore calibration for the distribution of the statistical tests. For the calibration we adapted the idea of residual based bootstrap for multivariate regression studied in Shao and Tu [32] for approximating the distributions of $\text{KS}_{n_1 \cdots n_d, \mathcal{A}}$ and $\text{CvM}_{n_1 \cdots n_d, \mathcal{A}}$.

For fixed $n_1, \dots, n_d$, let $F_{n_1 \cdots n_d}$ be the empirical distribution function of the vectors of least squares residuals $\{\mathbf{r}_{j_1 \cdots j_d} - \bar{\mathbf{R}}_{n_1 \cdots n_d} : 1 \leq j_k \leq n_k, k = 1, \dots, d\}$ centered at zero vector, where $\bar{\mathbf{R}}_{n_1 \cdots n_d} := (1/(n_1 \cdots n_d)) \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} \mathbf{r}_{j_1 \cdots j_d}$. Let $\mathcal{E}_{n_1 \cdots n_d}^* := (\mathcal{E}_{j_1 \cdots j_d}^*)_{j_1=1, \dots, n_d}^{n_1, \dots, n_d}$ be an array of independent and identically distributed random vectors sampled from $F_{n_1 \cdots n_d}$ and let $\widehat{\mathbf{g}}(\Xi_{n_1 \cdots n_d})$ be the ordinary LSE of $\mathbf{g}(\Xi_{n_1 \cdots n_d})$, where $\widehat{\mathbf{g}}(\Xi_{n_1 \cdots n_d}) = \text{pr}_{\prod_{i=1}^p \mathbf{w}_i} \mathbf{Z}_{n_1 \cdots n_d}$. Then we generate the array of p -dimensional bootstrap observations which is denoted in this paper by $\mathbf{Z}_{n_1 \cdots n_d}^* := (\mathbf{Z}_{j_1 \cdots j_d}^*)_{j_1=1, \dots, n_d}^{n_1, \dots, n_d}$ through the model:

$$\mathbf{Z}_{n_1 \cdots n_d}^* = \text{pr}_{\prod_{i=1}^p \mathbf{w}_i} \mathbf{Z}_{n_1 \cdots n_d} + \mathcal{E}_{n_1 \cdots n_d}^*. \quad (50)$$

Based on this model we get the array of p -dimensional bootstrap least squares residuals which is given by the component-wise projection of the bootstrap observations:

$$\begin{aligned} \mathbf{R}_{n_1 \cdots n_d}^* &:= (\mathbf{r}_{j_1 \cdots j_d}^*)_{j_1=1, \dots, n_d}^{n_1, \dots, n_d} \\ &= \mathbf{Z}_{n_1 \cdots n_d}^* - \text{pr}_{\prod_{i=1}^p \mathbf{w}_i} \mathbf{Z}_{n_1 \cdots n_d}^*. \end{aligned} \quad (51)$$

Hence, the bootstrap analog of $\text{KS}_{n_1 \cdots n_d, \mathcal{A}}$ and $\text{CvM}_{n_1 \cdots n_d, \mathcal{A}}$ is

$$\begin{aligned} \text{KS}_{n_1 \cdots n_d, \mathcal{A}}^* &:= \sup_{A \in \mathcal{A}} \left\| \widehat{\Sigma}_{n_1 \cdots n_d}^{*-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d}^*)(A) \right\|_{\mathbb{R}^p} \\ \text{CvM}_{n_1 \cdots n_d, \mathcal{A}}^* &:= \frac{1}{n_1 \cdots n_d} \sum_{A \in \mathcal{A}} \left\| \widehat{\Sigma}_{n_1 \cdots n_d}^{*-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d}^*)(A) \right\|_{\mathbb{R}^p}^2, \end{aligned} \quad (52)$$

where

$$\begin{aligned} \widehat{\Sigma}_{n_1 \cdots n_d}^* &:= \frac{1}{n_1 \cdots n_d} \left(\text{pr}_{\prod_{i=1}^p \mathcal{C}(\mathbf{X}_i^{(n_1 \cdots n_d)})^\perp} \mathbf{Z}^{*(n_1 \cdots n_d)} \right)^\top \\ & \quad \cdot \left(\text{pr}_{\prod_{i=1}^p \mathcal{C}(\mathbf{X}_i^{(n_1 \cdots n_d)})^\perp} \mathbf{Z}^{*(n_1 \cdots n_d)} \right). \end{aligned} \quad (53)$$

The question regarding the consistency of the bootstrap approximation of the p -dimensional processes $\widehat{\Sigma}_{n_1 \cdots n_d}^{*-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d}^*)(A)$ for $\widehat{\Sigma}_{n_1 \cdots n_d}^{-1/2} \mathbf{V}_{n_1 \cdots n_d}(\mathbf{R}_{n_1 \cdots n_d})(A)$ is summarized in the following theorem.

Theorem 7. Let $\{f_{i1}, \dots, f_{id}\}$ be an ONB of $\mathbf{W}_i$, for $i = 1, \dots, p$. Suppose the conditions of Theorem 1 are fulfilled. Then under H_0 it holds that

$$\Sigma^{*-1/2} \mathbf{V}_{n_1 \dots n_d} (\mathbf{R}_{n_1 \dots n_d}^*) \xrightarrow{\mathcal{D}} \mathbf{B}_p - \text{pr}_{\prod_{i=1}^p \mathbf{W}_i \mathcal{R}_B}^* \mathbf{B}_p, \quad (54)$$

where $\text{pr}_{\prod_{i=1}^p \mathbf{W}_i \mathcal{R}_B}^* \mathbf{B}_p$ is defined in Theorem 1.

Proof. We notice that $\{\mathcal{E}_{j_1 \dots j_d}^* : 1 \leq j_k \leq n_k, k = 1, \dots, d\}$ are independent and identically distributed with $E_*(\mathcal{E}_{1 \dots 1}^*) = \mathbf{0}$ and $\text{Cov}_*(\mathcal{E}_{1 \dots 1}^*) = \hat{\Sigma}_{n_1 \dots n_d} - \bar{\mathbf{R}}_{n_1 \dots n_d} \bar{\mathbf{R}}_{n_1 \dots n_d}^\top$. Hence, the invariance principle implies that $(\hat{\Sigma}_{n_1 \dots n_d} - \bar{\mathbf{R}}_{n_1 \dots n_d} \bar{\mathbf{R}}_{n_1 \dots n_d}^\top)^{-1/2} \mathbf{V}_{n_1 \dots n_d} (\mathcal{E}_{n_1 \dots n_d}^*)$ converges in distribution to $\mathbf{B}_p$. Hence under H_0 we have $\mathbf{R}_{n_1 \dots n_d}^* = \mathcal{E}_{n_1 \dots n_d}^* - \text{pr}_{\prod_{i=1}^p \mathbf{W}_i} \mathcal{E}_{n_1 \dots n_d}^*$ and $\Sigma_{n_1 \dots n_d}^{*-1/2}$ can be written as

$$\Sigma_{n_1 \dots n_d}^{*-1/2} = \Sigma_{n_1 \dots n_d}^{*-1/2} \left(\hat{\Sigma}_{n_1 \dots n_d} - \bar{\mathbf{R}}_{n_1 \dots n_d} \bar{\mathbf{R}}_{n_1 \dots n_d}^\top \right)^{1/2} \cdot \left(\hat{\Sigma}_{n_1 \dots n_d} - \bar{\mathbf{R}}_{n_1 \dots n_d} \bar{\mathbf{R}}_{n_1 \dots n_d}^\top \right)^{-1/2}, \quad (55)$$

where it can be shown easily that

$$\begin{aligned} \bar{\mathbf{R}}_{n_1 \dots n_d} \bar{\mathbf{R}}_{n_1 \dots n_d}^\top &= o_{\mathcal{D}}(1), \\ E_*(\hat{\Sigma}_{n_1 \dots n_d}^*) &= \hat{\Sigma}_{n_1 \dots n_d} + o_{\mathcal{D}}(1), \end{aligned} \quad (56)$$

with $o_{\mathcal{D}}(1)$ being the collection of the terms converging in probability to $\mathbf{0}_{p \times p}$. Then by recalling Theorem 5 and the linearity of $\mathbf{V}_{n_1 \dots n_d}$ we only need to show that

$$\left(\hat{\Sigma}_{n_1 \dots n_d} - \bar{\mathbf{R}}_{n_1 \dots n_d} \bar{\mathbf{R}}_{n_1 \dots n_d}^\top \right)^{-1/2} \mathbf{V}_{n_1 \dots n_d} (\text{pr}_{\prod_{i=1}^p \mathbf{W}_i} \mathcal{E}_{n_1 \dots n_d}^*) \xrightarrow{\mathcal{D}} \text{pr}_{\prod_{i=1}^p \mathbf{W}_i \mathcal{R}_B}^* \mathbf{B}_p. \quad (57)$$

The proof is established by imitating the steps of proving convergence result of Theorem 1. We are done. $\square$

6. Simulation Study

In this section, we report on a simulation study designed to investigate the finite sample size behavior of the KS and CvM tests. We simulate a multivariate model with four components defined on the unit rectangle $\mathbf{I}^2$. The hypothesis under study is $H_0 : \mathbf{g} \in \prod_{i=1}^4 \mathbf{W}_i$ against $H_1 : \mathbf{g} \notin \prod_{i=1}^4 \mathbf{W}_i$, where $\mathbf{W}_1 := [f_1]$, $\mathbf{W}_2 := [f_1, f_2, f_3]$, $\mathbf{W}_3 := [f_1, f_2, f_3, f_4, f_5, f_6]$, and $\mathbf{W}_4 := [f_1, f_2, f_3, f_4, f_5, f_6, f_7]$. Thereby we define $f_1(t, s) := 1$, $f_2(t, s) = t$, $f_3(t, s) = s$, $f_4(t, s) = t^2$, $f_5(t, s) = s^2$, $f_6(t, s) = ts$, and $f_7(t, s) = t^3$, for $(t, s) \in \mathbf{I}^2$. The samples are generated from a localized model $\mathbf{Y}(\ell/n_1, k/n_2) = (1/\sqrt{n_1 n_2}) \mathbf{g}(\ell/n_1, k/n_2) + \mathcal{E}(\ell/n_1, k/n_2)$ under the experimental design given by $\Xi_{n_1 \times n_2} = \{(\ell/n_1, k/n_2) \in \mathbf{I}^2 : 1 \leq \ell \leq n_1, 1 \leq k \leq n_2\}$, where $\mathbf{g}(\ell/n_1, k/n_2)$ is defined as

$$\mathbf{g}\left(\frac{\ell}{n_1}, \frac{k}{n_2}\right) := \begin{pmatrix} 5 + \rho \exp\left\{\frac{\ell}{n_1} \frac{k}{n_2}\right\} \\ 10 - 5 \frac{\ell}{n_1} + 10 \frac{k}{n_2} + \gamma \exp\left\{\frac{\ell}{n_1} \frac{k}{n_2}\right\} \\ 10 + 20 \frac{\ell}{n_1} - 25 \frac{k}{n_2} + 10 \frac{\ell^2}{n_1^2} - 5 \frac{k^2}{n_2^2} + 20 \frac{\ell}{n_1} \frac{k}{n_2} + \delta \exp\left\{\frac{\ell}{n_1} \frac{k}{n_2}\right\} \\ 30 - 30 \frac{\ell}{n_1} - 5 \frac{k}{n_2} + 20 \frac{\ell^2}{n_1^2} - 15 \frac{k^2}{n_2^2} + 10 \frac{\ell}{n_1} \frac{k}{n_2} + 10 \frac{\ell^3}{n_1^3} + \kappa \exp\left\{\frac{\ell}{n_1} \frac{k}{n_2}\right\} \end{pmatrix}, \quad (58)$$

for constants ρ, γ, δ , and κ determined prior to the generation of the samples. For fixed n_1 and n_2 and $1 \leq \ell \leq n_1$ and $1 \leq k \leq n_2$ the vector of random errors $\mathcal{E}(\ell/n_1, k/n_2)$ is generated independently from the 4-variate normal distribution with mean zero and variance-covariance matrix given by

$$\Sigma_{4 \times 4} := \begin{pmatrix} 9 & 3 & -6 & 12 \\ 3 & 26 & -7 & -11 \\ -6 & -7 & 9 & 7 \\ 12 & -11 & 7 & 65 \end{pmatrix}; \quad (59)$$

however, we assume in the computation that $\Sigma_{4 \times 4}$ is unknown. It is therefore estimated using $\hat{\Sigma}_{n_1 n_2}$ defined in Section 4. It is important to note that for computational reason we restricted the index set to the Vapnick Chervonenkis

Classes (VCC) of subsets of $\mathbf{I}^2$ which is given by the family of closed rectangle with the point $(0, 0)$ as the essential point. That is, the family $\{[0, t] \times [0, s] : 0 \leq t, s \leq 1\}$.

When ρ, γ, δ , and κ are set simultaneously to zero then we get the samples which coincide to the model specified under H_0 . Conversely, when at least one of them takes a nonzero value then we obtain the samples which can be regarded as from the alternative whose corresponding samples are generated by assigning nonzero values to either one of the constants, ρ, γ, δ , and κ , or the combinations of them.

Table 1 presents the empirical probabilities of rejection of H_0 for $\alpha = 0.05$ and some selected values of ρ, γ, δ , and κ . The empirical powers of the KS and CvM tests are denoted by $\hat{\alpha}_{\text{KS}}$ and $\hat{\alpha}_{\text{CvM}}$, respectively. The notations $\hat{\sigma}_{\text{KS}}$ and $\hat{\sigma}_{\text{CvM}}$ stand, respectively, for the standard deviation of the samples. The critical values of the statistics $\text{KS}_{n_1, n_2; \mathcal{A}}$ and $\text{CvM}_{n_1, n_2; \mathcal{A}}$ are

TABLE 1: The empirical probabilities of rejection of the KS and CvM tests for several selected values of ρ , γ , δ , and κ . The sample sizes are selected to 35×40 and 50×60 . The simulation results are based on 10000 runs.

Sample size	ρ	γ	δ	κ	$\hat{\alpha}_{KS}$	$\hat{\sigma}_{KS}$	$\hat{\alpha}_{CvM}$	$\hat{\sigma}_{CvM}$
35×40	0	0	0	0	4.90%	1.31299	4.60%	2.52127
	10	0	0	0	10.60%	1.52902	11.28%	4.11538
	20	0	0	0	32.80%	1.71455	33.40%	6.26419
	40	0	0	0	88.64%	1.86015	87.22%	10.37546
	0	50	0	0	5.03%	1.28540	5.33%	2.63926
	0	200	0	0	4.43%	1.22630	5.90%	2.57881
	0	300	0	0	4.60%	1.18783	7.03%	2.52180
	0	500	0	0	4.63%	0.90516	11.00%	2.32439
	0	0	50	0	4.83%	1.27888	5.03%	2.59347
	0	0	100	0	5.17%	1.29476	5.20%	2.68005
	0	0	200	0	6.03%	1.28627	5.23%	2.53931
	0	0	500	0	8.23%	1.27694	7.73%	2.71275
	0	0	0	50	5.00%	1.27241	5.03%	2.69462
	0	0	0	100	4.77%	1.23645	4.63%	2.56912
	0	0	0	300	4.77%	1.27318	5.07%	2.53470
	0	0	0	500	4.97%	1.26322	4.97%	2.55223
	10	50	0	0	11.34%	1.55144	12.22%	3.82300
	10	100	0	0	11.12%	1.49017	12.70%	3.81009
	10	200	0	0	11.20%	1.52118	12.06%	3.71303
	10	500	0	0	12.46%	1.10222	22.44%	2.87591
	20	50	50	50	32.78%	1.76263	33.62%	5.75683
	20	50	100	50	31.80%	1.71323	32.52%	5.98971
	20	50	200	50	28.16%	1.68594	29.16%	5.79669
	20	50	500	50	27.08%	1.52512	28.86%	4.92925
	50	50	50	20	98.60%	1.86506	97.60%	13.43552
	50	50	50	40	98.60%	1.83424	97.95%	13.10968
	50	50	50	60	98.55%	1.91241	98.25%	13.49794
50×60	0	0	0	0	4.75%	1.26388	4.55%	2.52770
	10	0	0	0	11.35%	1.48925	11.40%	3.70839
	20	0	0	0	33.65%	1.75109	33.40%	5.82381
	30	0	0	0	65.70%	1.82802	64.90%	8.31756
	0	50	0	0	4.70%	1.26907	4.95%	2.60854
	0	200	0	0	4.75%	1.23471	5.90%	2.53725
	0	300	0	0	5.35%	1.19808	7.35%	2.62291
	0	0	50	0	0.0485	1.27867	5.15%	2.57206
	0	0	100	0	4.35%	1.23893	4.45%	2.40576
	0	0	200	0	5.80%	1.27524	5.50%	2.62313
	0	0	500	0	8.45%	1.28708	6.50%	2.64556
	0	0	0	50	5.00%	1.28525	4.85%	2.71581
	0	0	0	100	4.50%	1.25700	4.70%	2.57227
	0	0	0	500	4.70%	1.25102	5.10%	2.51076
	20	50	0	0	32.10%	1.70264	32.30%	5.72887
	20	100	0	0	33.55%	1.74878	32.50%	5.76105
	20	200	0	0	33.30%	1.72065	34.35%	5.71531
	20	500	0	0	33.90%	1.61848	46.30%	5.77714
	30	50	100	0	62.55%	1.78912	61.80%	8.08037
	30	50	200	0	60.50%	1.79929	58.60%	7.76765
	30	50	500	0	56.35%	1.78203	54.00%	7.76498
	20	20	20	20	32.55%	1.70930	33.05%	5.60510
	30	50	50	50	63.10%	1.86549	61.65%	8.63149
	50	50	50	50	97.75%	1.88146	97.10%	13.27029
	50	100	50	50	97.60%	1.89668	96.90%	13.30456

FIGURE 1: The empirical power functions of the KS test (smooth line) and CvM test (dotted line) for the 4-variate model using 50×60 -point regular lattice. The simulation is under 10000 runs.

6.0742 and 7.9910 which are approximated by simulation. For $\rho = \gamma = \delta = \kappa = 0$ the values of $\hat{\alpha}_{KS}$ and $\hat{\alpha}_{CvM}$ fluctuate around 0.05 as it should be. This means that, independent of the selected number of the lattice points, both tests attain the specified level of significance.

Furthermore, Figure 1 exhibits the graphs of the empirical power function of the KS and CvM tests for $\alpha = 0.05$ associated with hypothesis H_0 specified above against $H_1 : g_i \notin \mathbf{W}_i$, for $i = 1, 2, 3, 4$. For the four cases we generate the error vectors independently from the same 4-variate normal distribution mentioned above. In the clockwise direction the left-top panel presents the graphs of the power function for testing H_0 against $H_1 : g_1 \notin \mathbf{W}_1$, the right-top panel is for H_0 against $H_1 : g_2 \notin \mathbf{W}_2$, the right-bottom is for H_0 against

$H_1 : g_3 \notin \mathbf{W}_3$, and left-bottom is for H_0 against $H_1 : g_4 \notin \mathbf{W}_4$. The common characteristic of the tests is that the power gets larger as the the model moves away from H_0 . The KS tests represented by smooth line tend to have slightly larger power. However, somewhat unexpectedly, in the second case, the CvM test has much larger power.

7. Example of Application

In this example, the proposed method is applied to a mining data studied in Tahir [20]. As introduced in Section 1, the data consist of a simultaneous measurement of the percentages of Nickel (Ni), Cobalt (Co), Ferum (Fe), and other substances like Calcium-Monoxide (CaO), Silicon-Dioxide (SiO_2), and

TABLE 2: Pearson's correlation matrix of the percentages of CaO, log SiO₂, log MgO, Ni, log Fe, and Co observed over a regular lattice of size 7 × 14. Source of data: Tahir [20].

	CaO	log SiO ₂	log MgO	Ni	log Fe	Co
CaO	1.0000	0.3949	0.4045	-0.1285	-0.1167	-0.0665
log SiO ₂	0.3949	1.0000	0.8459	-0.0003	-0.5414	-0.4556
log MgO	0.4045	0.8459	1.0000	-0.1331	-0.4968	-0.3134
Ni	-0.1285	-0.0003	-0.1331	1.0000	0.1652	0.1068
log Fe	-0.1166	-0.5414	-0.4968	0.1652	1.0000	0.5937
Co	-0.0665	-0.4556	-0.3134	0.1068	0.5937	1.0000

Magnesium-Monoxide (MgO). The sample was obtained by drilling bores set according to a three-dimensional lattice of size 7 × 14 × 10 with 7 equidistance rows running west to east, 14 equidistance columns running south to north, and 10 equidistance depths from the surface of the earth to the bottom. To simplify the computation of the test statistics we consider the experimental design as a two-dimensional lattice of size 7 × 14 by taking the average value of the samples measured in the same position. We further assume that the exploration region is given by a closed rectangle so that by suitable rescaling it can be transformed into a closed unit rectangle $\mathbf{I}^2$. Table 2 exhibits, respectively, the pairs scatter plot and Pearson's correlation coefficient among the percentages of Ni, CaO, Co, the logarithm of the percentages of SiO₂ (log SiO₂), MgO (log MgO), and Fe (log Fe). By this reason a multivariate analysis must be conducted in the statistical modelling taking into account the unknown covariance matrix of the vector of the variables. Furthermore, based on the individual scatter plot of the samples which are not presented in this work, it can be inferred that polynomials of lower order seem to be adequate to approximate the population model. More precisely, let $\mathbf{Y} := (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)^T$ be the vector of observations representing the observed percentages of CaO, log SiO₂, log MgO, Co, Ni, and log Fe, respectively. We aim to test the hypothesis

$$H_0 : \mathbf{E}(\mathbf{Y}) = \begin{pmatrix} \beta_{11} \\ \beta_{21} + \beta_{22}t + \beta_{23}s \\ \beta_{31} + \beta_{32}t + \beta_{33}s \\ \beta_{41} + \beta_{42}t + \beta_{43}s \\ \beta_{51} + \beta_{52}t + \beta_{53}s + \beta_{54}t^2 + \beta_{55}ts + \beta_{56}s^2 \\ \beta_{61} + \beta_{62}t + \beta_{63}s + \beta_{64}t^2 + \beta_{65}ts + \beta_{66}s^2 \end{pmatrix}, \quad (60)$$

$$0 \leq t, s \leq 1,$$

for some unknown constants β_{ij} , $i = 1, 2, 3, 4, 5, 6$ and $j = 1, \dots, d_i$, with $d_1 = 1$, $d_2 = d_3 = d_4 = 3$ and $d_5 = d_6 = 6$. For this case we have $\mathbf{W}_1 := [f_1]$, $\mathbf{W}_2 = \mathbf{W}_3 = \mathbf{W}_4 := [f_1, f_2, f_3]$, and $\mathbf{W}_5 = \mathbf{W}_6 := [f_1, f_2, f_3, f_4, f_5, f_6]$, with $f_1(t, s) := 1$, $f_2(t, s) = t$, $f_3(t, s) = s$, $f_4(t, s) = t^2$, $f_5(t, s) = ts$, and $f_6(t, s) = s^2$.

We obtained the values $\text{KS}_{7 \times 14; \mathcal{A}} = 1.44109$ and $\text{CvM}_{7 \times 14; \mathcal{A}} = 0.936687$ with the associated simulated p values of 0.98350 and 0.93670, respectively. We notice that in the computation we consider the VCC $\{[0, t] \times [0, s] : 0 \leq t, s \leq 1\}$ as the index sets instead of $\mathcal{A}$. Hence when using the KS test as well as CvM test the hypothesis will be not rejected for almost all commonly used values of α . There exists a significant evidence that the assumed model is appropriate for describing the functional relationship between the experimental conditional and the percentages of those elements.

In the practice some computational difficulties appear for testing using our proposed method. First, to the knowledge of the authors, the analytical formula for computing the critical and p values of the tests have been not yet available in the literatures; therefore we need to approximate them by simulation using computer. Second, although the test procedures are established for a much larger family of sets $\mathcal{A}$, in the application the computation is always restricted to the VCC of subsets of $\mathbf{I}^d$ like that of $\{\prod_{i=1}^d [0, t_i] \subset \mathbf{I}^d : 0 < t_i \leq 1, i = 1, \dots, d\}$ or $\{\prod_{i=1}^d [s_i, t_i] \subset \mathbf{I}^d : 0 \leq s_i < t_i \leq 1, i = 1, \dots, d\}$.

8. Concluding Remark

In this article we have developed an asymptotic method for checking the validity of a general multivariate spatial regression model by considering the multidimensional set-indexed partial sums of the residuals. For the calibration of the distribution of the test statistics we propose the residual based bootstrap for multivariate regression. It is shown by applying imitation technique that the residual bootstrap resampling technique is consistent. In a simulation study the finite sample size behavior of the KS and CvM statistics is investigated in greater detail. For the first-order model CvM test has much larger power, whereas for constant, second-order, and third-order models the powers of the two tests are almost the same.

Other possibilities of tests for multidimensional case can be obtained by incorporating a sampling technique according to an arbitrary experimental design. Sometimes because of technical, economic, or ecological reason, practitioners will not or cannot sample the observations equidistantly. One possible approach is to sample according to a continuous probability measure; see, for example, the sampling method

proposed in Bischoff [11]. Under this concern we get the so-called weighted KS and CvM tests which can be viewed as generalization of the KS and CvM tests studied in this paper.

Instead of considering the least squares residuals of the observations we can also define a test by directly investigating the partial sums of the observations. The limit process will be given by a type of signal plus a noise which is given by the multidimensional set-indexed Brownian sheet. Observing the limit process we can formulate likelihood ratio test based on the Cameron-Martin-Girsanov density formula of the limit process. Establishing such type of test will be of our concern in our future research project.

Appendix

A. Function of Bounded Variation on $\mathbf{I}^d$

Definition A.1. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a real-valued function with d variables. For $\alpha_k, \beta_k \in \mathbb{R}$, let $\Delta_{\alpha_k}^{\beta_k} f$ be a real-value function defined on $\mathbb{R}^d$, given by

$$\Delta_{\alpha_k}^{\beta_k} f := f(x_1, \dots, x_{k-1}, \beta_k, x_{k+1}, \dots, x_d) - f(x_1, \dots, x_{k-1}, \alpha_k, x_{k+1}, \dots, x_d), \quad (\text{A.1})$$

for $k = 1, \dots, d$. Furthermore, for $\alpha := (\alpha_k)_{k=1}^d$ and $\beta := (\beta_k)_{k=1}^d \in \mathbb{R}^d$, $\Delta_{\alpha}^{\beta} f$ is defined on $\mathbb{R}^d$ recursively starting from the last components of α and β . More precisely,

$$\Delta_{\alpha}^{\beta} f := \Delta_{\alpha_1}^{\beta_1} (\dots (\Delta_{\alpha_{d-1}}^{\beta_{d-1}} (\Delta_{\alpha_d}^{\beta_d} f)) \dots). \quad (\text{A.2})$$

Let $\{j_1, \dots, j_d\}$ be permutation of $\{1, 2, \dots, d\}$; then it holds that

$$\begin{aligned} \Delta_{\alpha}^{\beta} f &= \Delta_{\alpha_{j_1}}^{\beta_{j_1}} (\dots (\Delta_{\alpha_{j_{d-1}}}^{\beta_{j_{d-1}}} (\Delta_{\alpha_{j_d}}^{\beta_{j_d}} f)) \dots) \\ &= \Delta_{\alpha_1}^{\beta_1} (\dots (\Delta_{\alpha_{d-1}}^{\beta_{d-1}} (\Delta_{\alpha_d}^{\beta_d} f)) \dots). \end{aligned} \quad (\text{A.3})$$

This means that the operation of $\Delta_{\alpha}^{\beta} f$ does not depend on the order. By this reason we write $\Delta_{\alpha}^{\beta} f$ by $\Delta_{\alpha_1}^{\beta_1} \dots \Delta_{\alpha_{d-1}}^{\beta_{d-1}} \Delta_{\alpha_d}^{\beta_d} f$ ignoring the brackets. The reader is referred to Yeh [33] and Elstrodt [34], pp. 44-45.

Definition A.2 (see Yeh [33]). Let $\Gamma_k := \{[x_{k_0}, x_{k_1}], [x_{k_1}, x_{k_2}], \dots, [x_{k_{M_k-1}}, x_{k_{M_k}}]\}$ be a collection of M_k rectangles on the unit interval $[0, 1]$ with $0 = x_{k_0} \leq x_{k_1} \leq \dots \leq x_{k_{M_k}} = 1$, for $k = 1, \dots, d$. The Cartesian product $\mathcal{K} := \prod_{k=1}^d \Gamma_k$ which consists of $M_1 \times M_2 \times \dots \times M_d$ rectangles is called a nonoverlapping finite exact cover of $\mathbf{I}^d$. The family of all nonoverlapping finite exact cover of $\mathbf{I}^d$ is denoted by $\mathcal{F}(\mathcal{K})$.

Definition A.3 (see Yeh [33]). For $1 \leq w_k \leq M_k$, with $k = 1, \dots, d$, let $\mathbf{J}_{w_1, \dots, w_d}$ be the element of $\mathcal{K}$ defined by $\mathbf{J}_{w_1, \dots, w_d} := \prod_{k=1}^d [x_{k_{w_k-1}}, x_{k_{w_k}}]$. Let $\psi: \mathbf{I}^d \rightarrow \mathbb{R}$ be a real-valued function on $\mathbf{I}^d$. Operator $\Delta_{\mathbf{J}_{w_1, \dots, w_d}}$ acting on a function ψ is defined by

$$\Delta_{\mathbf{J}_{w_1, \dots, w_d}} \psi := \Delta_{x_{1_{w_1-1}}^{x_{1_{w_1}}}} \Delta_{x_{2_{w_2-1}}^{x_{2_{w_2}}}} \dots \Delta_{x_{d_{w_d-1}}^{x_{d_{w_d}}}} \psi. \quad (\text{A.4})$$

The variation of ψ over the finite exact cover $\mathcal{K}$ is defined by

$$v(\psi; \mathcal{K}) := \sum_{w_1=1}^{M_1} \dots \sum_{w_d=1}^{M_d} |\Delta_{\mathbf{J}_{w_1, \dots, w_d}} \psi|. \quad (\text{A.5})$$

Accordingly, the total variation of ψ over $\mathbf{I}^d$ is defined by

$$V(\psi; \mathbf{I}^d) := \sup_{\mathcal{K} \in \mathcal{F}(\mathcal{K})} v(\psi; \mathcal{K}). \quad (\text{A.6})$$

Furthermore, function ψ is said to have bounded variation in the sense of Vitaly on $\mathbf{I}^d$ if there exists a real number $M > 0$ such that $V(\psi; \mathbf{I}^d) \leq M$ for some real number $M > 0$. The class of such functions is denoted by $\text{BV}(\mathbf{I}^d)$.

Definition A.4 (see Yeh [33]). Let $(x_k)_{k=1}^d$ be a variable in $\mathbf{I}^d$. For fixed k , let $\mathbf{I}^k := [0, 1]^k$ be a k -dimensional unit closed rectangle constructed in the following way. We choose $d - k$ components of the variable $(x_k)_{k=1}^d$. For each choice from all possible elements of the set C_{d-k}^d , we set each x_i with 0 or 1 and let the remaining k variables satisfy $0 \leq x_i \leq 1$. Then for each k we get $2^{d-k} |C_{d-k}^d|$ unit closed rectangles $\mathbf{I}^k$. For convention we denote the collection of all $2^{d-k} |C_{d-k}^d|$ of closed rectangles $\mathbf{I}^k$ by $\mathcal{B}^k$ and the j th element of $\mathcal{B}^k$ will be denoted by $\mathbf{I}_j^k$. Function ψ is said to have bounded variation in the sense of Hardy on $\mathbf{I}^d$, if and only if for each $k = 1, \dots, d$ and $j = 1, \dots, 2^{d-k} |C_{d-k}^d|$ there exists a real number $M_{jk} > 0$ such that $V(\psi_{\mathbf{I}_j^k}(\cdot); \mathbf{I}_j^k) \leq M_{jk}$, where, for $k = 1, \dots, d$ and $j = 1, \dots, 2^{d-k} |C_{d-k}^d|$, $\psi_{\mathbf{I}_j^k}(\cdot)$ is a function with k variables obtained from the function $\psi(x_1, x_2, \dots, x_d)$ by setting the $d - k$ selected variables with 0 or 1, whereas the remaining k variables lie in the interval $[0, 1]$. The class of such functions will be denoted by $\text{BV}_H(\mathbf{I}^d)$.

B. Integration by Parts on $\mathbf{I}^d$

For family $\mathcal{B}^k$ defined in Definition A.4, let $\mathcal{F} := \bigcup_{k=0}^d \mathcal{B}^k$, where, for $k = 0$, the family $\mathcal{B}^0$ is a collection of 2^d different points in $\mathbf{I}^d$. As an example, for $d = 2$, we have $\mathcal{B}^0 = \{\mathbf{I}_1^0 = (0, 0), \mathbf{I}_2^0 = (0, 1), \mathbf{I}_3^0 = (1, 0), \mathbf{I}_4^0 = (1, 1)\}$. For each $k = 1, \dots, d$ and $j = 1, \dots, 2^{d-k} |C_{d-k}^d|$, let $\#(\mathbf{I}_j^k)$ be the number of 1's appearing in $\mathbf{I}_j^k$. Next, let φ and ψ be defined on $\mathbf{I}^d$. If φ is Riemann-Stieltjes integrable with respect to ψ on $\mathbf{I}_j^k \in \mathcal{B}^k$, we denote the integral by $\int_{\mathbf{I}_j^k}^{(R)} \varphi d^k \psi$. For $k = 0$, it is understood that $\int_{\mathbf{I}_j^0}^{(R)} \varphi d^0 \psi$ is defined as the product of φ and ψ at that point of $\mathbf{I}_j^0$ (see Yeh [33]).

Theorem B.1 (integration by parts (see Yeh [33])). *Let φ be Riemann-Stieltjes integrable with respect to ψ on each member*

of $\mathcal{F}$. Then ψ is Riemann-Stieltjes integrable with respect to φ on $\mathbf{I}^d$, and we have the formula

$$\begin{aligned} & \int_{\mathbf{I}^d}^{(R)} \psi(x_1, \dots, x_d) d\varphi(x_1, \dots, x_d) \\ &= \sum_{k=0}^d 2^{d-k} \sum_{j=1}^{|C_{d-k}^d|} (-1)^{(d-\#(\mathbf{I}_j^k))} \int_{\mathbf{I}_j^k}^{(R)} \varphi d^k \psi. \end{aligned} \quad (\text{B.1})$$

Moreover, if ψ have bounded variation in the sense of Hardy on $\mathbf{I}^d$ and φ is continuous on $\mathbf{I}^d$, then we have the inequality

$$\begin{aligned} & \left| \int_{\mathbf{I}^d}^{(R)} \psi(x_1, \dots, x_d) d\varphi(x_1, \dots, x_d) \right| \\ & \leq \|\varphi\|_{\infty} \left(2^d \|\psi\|_{\infty} + \sum_{k=1}^d 2^{d-k} \sum_{j=1}^{|C_{d-k}^d|} V(\psi_{\mathbf{I}_j^k}(\cdot); \mathbf{I}_j^k) \right). \end{aligned} \quad (\text{B.2})$$

C. Some Property of the Partial Sums Operator

Lemma C.1 (see Bischoff and Somayasa [9]). *For every one-dimensional pyramidal array $A_{n_1 \dots n_d} := (a_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$, it holds that $\mathbf{V}_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}) \in \mathcal{H}_B$, where $\mathcal{H}_B$ is the subspace defined in (15). Furthermore, for any arrays $A_{n_1 \dots n_d} := (a_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$ and $B_{n_1 \dots n_d} := (b_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$, we have*

$$\begin{aligned} & \langle \mathbf{V}_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}), \mathbf{V}_{n_1 \dots n_d}^{(i)}(B_{n_1 \dots n_d}) \rangle_{\mathcal{H}_B} \\ &= \langle A_{n_1 \dots n_d}, B_{n_1 \dots n_d} \rangle_{\mathbb{R}^{n_1 \times \dots \times n_d}}, \end{aligned} \quad (\text{C.1})$$

where $\mathbf{V}_{n_1 \dots n_d}^{(i)}$ is the one-dimensional component of the partial sums operator $\mathbf{V}_{n_1 \dots n_d}$.

Proof. Associated with $A_{n_1 \dots n_d}$ we can construct a step function $s_{A_{n_1 \dots n_d}} : \mathbf{I}^d \rightarrow \mathbb{R}$ defined by

$$s_{A_{n_1 \dots n_d}}(\mathbf{t}) := \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} a_{j_1 \dots j_d} \mathbf{1}_{C_{j_1 \dots j_d}}(\mathbf{t}), \quad \mathbf{t} \in \mathbf{I}^d, \quad (\text{C.2})$$

where $C_{j_1 \dots j_d} = \prod_{k=1}^d ((j_k - 1)/n_k, j_k/n_k]$, for $1 \leq j_k \leq n_k$. For any $B \in \mathcal{A}$, it holds that

$$\begin{aligned} h_{s_{A_{n_1 \dots n_d}}}(B) &:= \int_B s_{A_{n_1 \dots n_d}}(\mathbf{t}) \lambda_1^d(d\mathbf{t}) \\ &= \frac{1}{\sqrt{n_1 \dots n_d}} \mathbf{V}_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d})(B). \end{aligned} \quad (\text{C.3})$$

Hence, $\mathbf{V}_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}) \in \mathcal{H}_B$ having $\sqrt{n_1 \dots n_d} s_{A_{n_1 \dots n_d}}$ as the $L_2(\lambda_1^d)$ density. By the definition of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_B}$, we further get

$$\begin{aligned} & \langle \mathbf{V}_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}), \mathbf{V}_{n_1 \dots n_d}^{(i)}(B_{n_1 \dots n_d}) \rangle_{\mathcal{H}_B} \\ &= \int_{\mathbf{I}^d} \sqrt{n_1 \dots n_d} s_{A_{n_1 \dots n_d}}(\mathbf{t}) \sqrt{n_1 \dots n_d} s_{B_{n_1 \dots n_d}}(\mathbf{t}) \\ & \cdot \lambda_1^d(d\mathbf{t}) = \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} a_{j_1 \dots j_d} b_{j_1 \dots j_d} \\ &= \langle A_{n_1 \dots n_d}, B_{n_1 \dots n_d} \rangle_{\mathbb{R}^{n_1 \times \dots \times n_d}}. \end{aligned} \quad (\text{C.4})$$

□

Lemma C.2 (see Bischoff and Somayasa [9]). *For any $A_{n_1 \dots n_d} := (a_{j_1 \dots j_d})_{j_1=1, \dots, j_d=1}^{n_1, \dots, n_d}$ in $\mathbb{R}^{n_1 \times \dots \times n_d}$ it holds that, for $i = 1, \dots, p$,*

$$\begin{aligned} & \mathbf{V}_{n_1 \dots n_d}^{(i)}(\text{pr}_{\mathbf{W}_{i, n_1 \dots n_d}} A_{n_1 \dots n_d}) \\ &= \text{pr}_{\mathbf{W}_{i, n_1 \dots n_d}} \mathbf{V}_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}), \end{aligned} \quad (\text{C.5})$$

where

$$\begin{aligned} \mathbf{W}_{i, n_1 \dots n_d} &:= \{ \mathbf{V}_{n_1 \dots n_d}^{(i)}(B_{n_1 \dots n_d}) \mid B_{n_1 \dots n_d} \in \mathbf{W}_{i, n_1 \dots n_d} \} \\ &\subset \mathcal{H}_B. \end{aligned} \quad (\text{C.6})$$

Furthermore, by the definition of the component-wise projection, we finally get

$$\begin{aligned} & \mathbf{V}_{n_1 \dots n_d}(\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d}} A_{n_1 \dots n_d}) \\ &= \left(\text{pr}_{\mathbf{W}_{i, n_1 \dots n_d}} \mathbf{V}_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}) \right)_{i=1}^p, \end{aligned} \quad (\text{C.7})$$

for every p -dimensional array $\mathbf{A}_{n_1 \dots n_d} \in \prod_{i=1}^p \mathbb{R}^{n_1 \times \dots \times n_d}$.

Proof. For fixed i , let $\{f_{i1}(\Xi_{n_1 \dots n_d}), \dots, f_{id_i}(\Xi_{n_1 \dots n_d})\}$ be an ONB of $\mathbf{W}_{i, n_1 \dots n_d}$. Then by Lemma C.1 the corresponding ONB of $\mathbf{W}_{i, n_1 \dots n_d}$ is given by the set

$$\{ \mathbf{V}_{n_1 \dots n_d}^{(i)}(f_{i1}(\Xi_{n_1 \dots n_d})), \dots, \mathbf{V}_{n_1 \dots n_d}^{(i)}(f_{id_i}(\Xi_{n_1 \dots n_d})) \}. \quad (\text{C.8})$$

Hence, by the linearity of $\mathbf{V}_{n_1 \dots n_d}^{(i)}$ and by Lemma C.1, we get

$$\begin{aligned} & \mathbf{V}_{n_1 \dots n_d}^{(i)}(\text{pr}_{\mathbf{W}_{i, n_1 \dots n_d}} A_{n_1 \dots n_d}) \\ &= \mathbf{V}_{n_1 \dots n_d}^{(i)} \left(\sum_{j=1}^{d_i} \langle f_{ij}(\Xi_{n_1 \dots n_d}), A_{n_1 \dots n_d} \rangle_{\mathbb{R}^{n_1 \times \dots \times n_d}} \right. \\ & \cdot f_{ij}(\Xi_{n_1 \dots n_d}) \Big) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{d_i} \langle f_{ij}(\Xi_{n_1 \dots n_d}), A_{n_1 \dots n_d} \rangle_{\mathbb{R}^{n_1 \times \dots \times n_d}} \\
&\cdot V_{n_1 \dots n_d}^{(i)}(f_{ij}(\Xi_{n_1 \dots n_d})) \\
&= \sum_{j=1}^{d_i} \langle V_{n_1 \dots n_d}^{(i)}(f_{ij}(\Xi_{n_1 \dots n_d})), V_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}) \rangle_{\mathcal{H}_B} \\
&\cdot V_{n_1 \dots n_d}^{(i)}(f_{ij}(\Xi_{n_1 \dots n_d})) \\
&= \text{pr}_{\mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} V_{n_1 \dots n_d}^{(i)}(A_{n_1 \dots n_d}).
\end{aligned} \tag{C.9}$$

□

Lemma C.3 (see Bischoff and Somayasa [9]). Let $\{\tilde{s}_{i1}^{(n_1 \dots n_d)}, \dots, \tilde{s}_{id_i}^{(n_1 \dots n_d)}\}$ be an orthonormal set in $L_2(\lambda_1^d)$ obtained by the Gram-Schmidt procedure from the step functions:

$$\begin{aligned}
&\tilde{s}_{ij}^{(n_1 \dots n_d)}(\mathbf{t}) \\
&:= \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} f_{ij} \left(\frac{j_1}{n_1}, \dots, \frac{j_d}{n_d} \right) \mathbf{1}_{C_{j_1 \dots j_d}}(\mathbf{t}), \tag{C.10} \\
&\mathbf{t} \in \mathbf{I}^d,
\end{aligned}$$

for $i = 1, \dots, p$, and $j = 1, \dots, d_i$. Then $\{h_{\tilde{s}_{i1}^{(n_1 \dots n_d)}}, \dots, h_{\tilde{s}_{id_i}^{(n_1 \dots n_d)}}\}$ is an ONB of $\mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}$. The projection of any function $u_n \in \mathcal{H}_B$ into $\mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}$ with respect to this basis is given by

$$\begin{aligned}
\text{pr}_{\mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} u_n &= \sum_{j=1}^{d_i} \langle h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}, u_n \rangle_{\mathcal{H}_B} h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}} \\
&= \sum_{j=1}^{d_i} \int_{\mathbf{I}^d}^{(R)} \tilde{s}_{ij}^{(n_1 \dots n_d)}(\mathbf{t}) du(\mathbf{t}) h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}.
\end{aligned} \tag{C.11}$$

Moreover, if, for $i = 1, \dots, p$ and $j = 1, \dots, d_i$, f_{ij} is continuous on $\mathbf{I}^d$ and $\{f_{i1}, \dots, f_{id_i}\}$ is an ONB of $\mathbf{W}_i$, then $\|\tilde{s}_{ij}^{(n_1 \dots n_d)} - f_{ij}\|_\infty \rightarrow 0$ as $n_1, \dots, n_d \rightarrow \infty$. Consequently, it also holds that $\|h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}} - h_{f_{ij}}\|_{\mathcal{H}} \rightarrow 0$, for $n_1, \dots, n_d \rightarrow \infty$.

Proof. Since $h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}} = (1/\sqrt{n_1 \dots n_d}) V_{n_1 \dots n_d}^{(i)}(f_{ij}(\Xi_{n_1 \dots n_d}))$, by the linearity of $V_{n_1 \dots n_d}^{(i)}$ it follows that $\{h_{\tilde{s}_{i1}^{(n_1 \dots n_d)}}, \dots, h_{\tilde{s}_{id_i}^{(n_1 \dots n_d)}}\}$ builds a basis for $\mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}$ whenever the set $\{f_{i1}(\Xi_{n_1 \dots n_d}), \dots, f_{id_i}(\Xi_{n_1 \dots n_d})\}$ is a basis of $\mathbf{W}_{i, n_1 \dots n_d}$. Furthermore, if f_{ij} is continuous on $\mathbf{I}^d$, it can be shown that $\|\tilde{s}_{ij}^{(n_1 \dots n_d)} - f_{ij}\|_\infty \rightarrow 0$ as $n_1, \dots, n_d \rightarrow \infty$. Hence, by the definition of the Gram-Schmidt process and also by the continuity of $\|\cdot\|_{L_2(\lambda_1^d)}$, we can further show that $\|h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}} - h_{f_{ij}}\|_{\mathcal{H}} \rightarrow 0$ as $n_1, \dots, n_d \rightarrow \infty$. The last assertion is immediately obtained from the definition of $h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}$ and $h_{f_{ij}}$. □

Proposition C.4. The p -dimensional projection $\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}}$ is continuous uniformly on the space of continuous function $\mathcal{C}^p(\mathcal{A})$ for all $n_1 \geq 1, \dots, n_d \geq 1$.

Proof. Let $\mathbf{w}_1 := (w_{1i})_{i=1}^p$ and $\mathbf{w}_2 := (w_{2i})_{i=1}^p$ be any functions in $\mathcal{C}^p(\mathcal{A})$. Then, by the definition and the inequality presented in Theorem B.1, for any $A \in \mathcal{A}$ we have

$$\begin{aligned}
&\left| \left(\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} \mathbf{w}_1 \right)(A) - \left(\text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} \mathbf{w}_2 \right) \right. \\
&\cdot (A) \left. \right| \leq \sum_{i=1}^p \sum_{j=1}^{d_i} \left| \left\langle h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}, w_{1i} \right\rangle h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}(A) \right. \\
&- \left. \left\langle h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}, w_{2i} \right\rangle h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}(A) \right| \\
&= \sum_{i=1}^p \sum_{j=1}^{d_i} \left| \left\langle h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}, w_{1i} - w_{2i} \right\rangle \right| \left| h_{\tilde{s}_{ij}^{(n_1 \dots n_d)}}(A) \right| \tag{C.12} \\
&\leq \sum_{i=1}^p \sum_{j=1}^{d_i} \|w_{1i} - w_{2i}\|_{\mathcal{H}} \|f_{ij}\|_\infty \left(2^d \|f_{ij}\|_\infty \right. \\
&+ \left. \sum_{k=1}^d \sum_{\ell=1}^{2^{d-k} |C_{d-k}^d|} V(f_{ij_{\ell}^k}(\cdot); \mathbf{I}_\ell^k) \right) \leq \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{H}} K,
\end{aligned}$$

for some constant K defined by

$$\begin{aligned}
K &:= \sum_{i=1}^p \sum_{j=1}^{d_i} \|f_{ij}\|_\infty \\
&\cdot \left(2^d \|f_{ij}\|_\infty + \sum_{k=1}^d \sum_{\ell=1}^{2^{d-k} |C_{d-k}^d|} V(f_{ij_{\ell}^k}(\cdot); \mathbf{I}_\ell^k) \right). \tag{C.13}
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\left\| \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} \mathbf{w}_1 - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} \mathbf{w}_2 \right\|_{\mathcal{H}} \\
&\leq \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{H}} K.
\end{aligned} \tag{C.14}$$

Given any positive small number ϵ , there exists a small number $\delta := \epsilon/K$, such that, for any $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{C}^p(\mathcal{A})$, if $\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{H}} \leq \delta$, then

$$\left\| \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} \mathbf{w}_1 - \text{pr}_{\prod_{i=1}^p \mathbf{W}_{i, n_1 \dots n_d, \mathcal{H}_B}} \mathbf{w}_2 \right\|_{\mathcal{H}} \leq \epsilon. \tag{C.15}$$

□

Competing Interests

The authors declare that they have no competing interests.

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