Research Article

Null Controllability of a Nonlinear Dissipative System and Application to the Detection of the Incomplete Parameter for a Nonlinear Population Dynamics Model

Yacouba Simporé¹ and Oumar Traoré²

¹LAMI, Université de Ouagadougou, 01 BP 7021, Ouaga 01, Burkina Faso

²LAMI, Université de Ouagadougou, 12 BP 417, Ouaga 12, Burkina Faso

Correspondence should be addressed to Oumar Traoré; otraoret@gmail.com

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We first prove a null controllability result for a nonlinear system derived from a nonlinear population dynamics model. In order to tackle the controllability problem we use an adapted Carleman inequality. Next we consider the nonlinear population dynamics model with a source term called the pollution term. In order to obtain information on the pollution term we use the method of sentinel.

1. Introduction and Main Result

For a given real function G , in this paper we consider the following nonlinear system:

$$\begin{aligned} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q &= \beta q(t, 0, x) G\left(\int_0^A \beta q da\right) \\ &+ \chi_\Theta h + \chi_\omega v \\ &\text{in } Q = (0, T) \times (0, A) \times \Omega, \quad (1) \\ q(t, a, x) &= 0 \quad \text{on } \Sigma \\ q(t, A, x) &= 0 \quad \text{in } (0, T) \times (0, A) \\ q(T, a, x) &= 0 \quad \text{in } (0, A) \times \Omega, \end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$. Let T be a positive real and let ω, Θ be two open subsets of Ω such that $\bar{\omega} \subset \Omega$ and $\Theta \subset \Omega$. Here χ_ω, χ_Θ are the characteristic function of ω and Θ , Δ is the Laplacian with respect to the spatial variable, and $\beta(a)$ and $\mu(a)$ denote, respectively, the natural fertility and the natural death rate of individuals of age a .

The purpose of this paper is to prove a null controllability result for (1) at any time T . This means more precisely that we prove here the existence of a control $v \in L^2(Q_\omega)$ such that the associated solution of (1) verifies $q(0, a, x) = 0$ a.e. in $(0, A) \times \Omega$.

For the sequel we assume that the following assumptions hold:

$$\begin{aligned} \mu(a) &\geq 0 \\ \lim_{a \rightarrow A} \int_0^a \mu(s) ds &= +\infty, \quad \mu \in L^1_{\text{loc}}(0, A), \end{aligned} \quad (H_1)$$

$$\begin{aligned} \beta &\in C^2([0, A]) \\ \beta(a) &\geq 0 \quad \text{in } [0, A] \\ \exists 0 &< a_0 < a_1 < A \end{aligned} \quad (H_2)$$

$$\text{such that } \beta(a) = 0 \quad \text{in } (0, a_0) \cup (a_1, A)$$

$$G \in L^\infty(\mathbb{R}). \quad (H_3)$$

We set $Q_\omega = (0, T) \times (0, A) \times \omega$ and $Q_\Theta = (0, T) \times (0, A) \times \Theta$.

Let us now present the main results of this paper.

For this aim we introduce the weight

$$e_M(t, a) = \exp\left(\frac{M}{at(A-a)(T-t)}\right) \quad (2)$$

and define the Hilbert space

$$L^2(e_M) = \left\{ f \in L^2(Q) : \int_Q f^2 e_M dx da dt < \infty \right\}, \quad (3)$$

endowed with the natural norm. M is a real that will be defined in the following.

We have the following.

Theorem 1. Let $h \in L^2(e_M)$; then under the assumptions (H_1) , (H_2) , and (H_3) there exists a control $v \in L^2(Q_\omega)$ satisfying $q(0, a, x) = 0$ a.e. in $(0, A) \times \Omega$.

The rest of this paper is as follows: the proof of the main result is given in Section 2; in Section 3, we describe the method of sentinel; and Section 4 is devoted to the information provided by the sentinel.

2. Proof of the Null Controllability of the Nonlinear System

2.1. An Adapted Observability Inequality. We recall here that there exists a function $\Psi \in C^2(\bar{\Omega})$ such that $\Psi(x) = 0$, for all $x \in \partial\Omega$; $\Psi(x) > 0$ for all $x \in \Omega$ and $\nabla\Psi \neq 0$ for all $\bar{\Omega} - \tilde{\omega}$, where $\tilde{\omega}$ is an open set such that $\tilde{\omega} \subset \omega \subset \Omega$. See [1].

Now we consider the following system:

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} - \Delta w + \mu w &= f \quad \text{in } Q \\ w &= 0 \quad \text{on } \Sigma \end{aligned} \quad (4)$$

$$w(0, a, x) = w^0(a, x) \quad \text{in } (0, A) \times \Omega$$

$$w(t, 0, x) = g(t, x) \quad \text{in } (0, T) \times \Omega.$$

Proposition 2. There exist positive constants $\alpha_1 > 1$ and $\lambda_1 > 1$ and there exists a positive constant $C(A, T)$ which depends on A and T such that, for $\alpha \geq \alpha_1$, $\lambda \geq \lambda_1$, and for all solution of (4) the following inequality holds:

$$\begin{aligned} \alpha^3 \lambda^4 \int_Q e^{-2\alpha\eta} p^3 |w|^2 dx da dt \\ \leq C \int_Q e^{-2\alpha\eta} f^2 dx da dt \\ + C \alpha^3 \lambda^4 \int_{Q_\omega} e^{-2\alpha\eta} p^3 |w|^2 dx da dt \end{aligned} \quad (5)$$

with

$$\begin{aligned} \eta(t, a, x) &= \frac{e^{2\lambda\Psi\|_\infty} - e^{\lambda\Psi(x)}}{at(T-t)(A-a)}, \\ p(t, a, x) &= \frac{e^{\lambda\Psi(x)}}{at(T-t)(A-a)}. \end{aligned} \quad (6)$$

For the proof of this proposition see [2].

For $f = 0$ we obtain an Observability Inequality.

Lemma 3. There exist $M > 0$ and $C > 0$ such that the following inequality is true:

$$\begin{aligned} \int_Q \exp\left(\frac{-M}{at(T-t)(A-a)}\right) w^2 dx da dt \\ \leq C \int_{Q_\omega} w^2 dx da dt; \end{aligned} \quad (7)$$

herein w is the solution of system (4).

Proof. If $f = 0$, (5) become

$$\begin{aligned} \int_Q e^{-2\alpha\eta} p^3 |w|^2 dx da dt \\ \leq C \int_{Q_\omega} e^{-2\alpha\eta} p^3 |w|^2 dx da dt. \end{aligned} \quad (8)$$

Choosing $\lambda = \lambda_1$ and $\alpha = \alpha_0 \geq \alpha_1$ such that $e^{-2\alpha_0\eta} p^3 \leq 1$, we obtain the result with $M = 2\alpha_0 \exp(2\lambda_1 \|\Psi\|_\infty)$. $\square$

In the following we consider the linear system:

$$\begin{aligned} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q &= \beta q(t, 0, x) b(t, x) + \chi_\Theta h \\ &+ \chi_\omega v \quad \text{in } Q \\ q(t, a, x) &= 0 \quad \text{on } \Sigma \end{aligned} \quad (9)$$

$$q(t, A, x) = 0 \quad \text{in } (0, T) \times (0, A)$$

$$q(T, a, x) = 0 \quad \text{in } (0, A) \times \Omega,$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial a} - \Delta \varphi + \mu \varphi = 0 \quad \text{in } Q$$

$$\varphi = 0 \quad \text{on } \Sigma$$

$$\varphi(0, a, x) = \varphi^0 \quad \text{in } (0, A) \times \Omega \quad (10)$$

$$\varphi(t, 0, x) = \int_0^A \beta \varphi b(t, x) da$$

$$\text{in } (0, T) \times \Omega,$$

where $a(t, x)$ and $b(t, x)$ are in $L^\infty((0, T) \times \Omega)$. Each system (9)-(10) is well posed and admits a unique solution. See [3] for the proof of existence and uniqueness of the solution.

Remark 4. For the linear case, that is, $G(x) = x$ can use the Hahn-Banach theorem for proving the existence of a control v which verifies $\|q(0, \cdot, \cdot)\|_{L^2((0, A) \times \Omega)} \leq \epsilon$, one writes

$$R(0) = \{q(0, a, x); v \in L^2(Q_\omega) \text{ when } q \text{ solves (9)}\}. \quad (11)$$

It is easy to see that $R(0)$ is a nonempty subspace of $L^2((0, A) \times \Omega)$ and that

$$R^1(0) \text{ is dense in } L^2((0, A) \times \Omega). \quad (12)$$

It suffices to show that

$$R^\perp(0) \subset \{0\}. \quad (13)$$

Suppose that

$$\varphi^0 \in R^\perp\{0\}. \quad (14)$$

Multiplying (9) by φ and integrating by parts, we have

$$\begin{aligned} & \int_Q (\chi_\Theta h + \chi_\omega v) \varphi \, dx \, da \, dt \\ &= \int_0^A \int_\Omega q(0, a, x) \varphi^0(a, x) \, da \, dx. \end{aligned} \quad (15)$$

Since

$$\varphi^0 \in R^\perp\{0\}, \quad (16)$$

then

$$\int_0^A \int_\Omega q(0, a, x) \varphi^0 \, da \, dx = 0. \quad (17)$$

This implies that

$$\int_Q (\chi_\Theta h + \chi_\omega v) \varphi \, dx \, da \, dt = 0 \quad (18)$$

for every $v \in L^2(Q_\omega)$.

Then $\varphi = 0$ on $(\Theta \cap \omega) \times (0, T) \times (0, A)$. The unique continuation property of φ gives

$$\varphi = 0 \quad \text{on } (0, T) \times (0, A) \times \Omega. \quad (19)$$

Therefore $\varphi^0 = 0$. So $R^\perp\{0\} \subset \{0\}$.

However, this method cannot allow the study of the nonlinear case.

2.2. The Approached Controllability of the Linear System.
Now, for $\delta > 0$ we consider the functional J_δ defined by

$$\begin{aligned} J_\delta(v) &= \frac{1}{2} \int_{Q_\omega} |v|^2 \, dx \, da \, dt \\ &+ \frac{1}{\delta} \int_0^A \int_\Omega q^2(0, a, x) \, dx \, da, \end{aligned} \quad (20)$$

where q solves

$$\begin{aligned} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q &= \beta q(t, 0, x) b(t, x) + \chi_\Theta h \\ &+ \chi_\omega v \quad \text{in } Q \\ q(t, a, x) &= 0 \quad \text{on } \Sigma \\ q(t, A, x) &= 0 \quad \text{in } (0, T) \times \Omega \\ q(T, a, x) &= 0 \quad \text{in } (0, A) \times \Omega. \end{aligned} \quad (21)$$

Proposition 5. *The functional J_δ is continuous, strictly convex, and coercive. Therefore, J_δ reaches its minimum at a point $v_\delta \in L^2(Q_\omega)$. Moreover, setting q_δ the associated solution of (21) and φ_δ the solution of (10) with $\varphi_\delta(0, a, x) = -(1/\delta)q_\delta(0, a, x)$ has $\chi_\omega v_\delta = \chi_\omega \varphi_\delta$ and there exist $C > 0$, $K_0 > 0$ independent of δ , such that*

$$\int_{Q_\omega} |\varphi_\delta|^2 \, dx \, da \, dt \leq C \|h\|_{L^2(e_M)}^2, \quad (22)$$

$$\int_0^A \int_\Omega q_\delta^2(0, a, x) \, dx \, da \leq \delta K_0 \|h\|_{L^2(e_M)}^2.$$

Remark 6. As we will see Proposition 5 solves the problem of existence of the approached sentinel for the linear case.

Proof. It is easy to check that J_δ is coercive, continuous, and strictly convex. Then it admits a unique minimizer v_δ . The maximum principle gives that $\chi_\omega v_\delta = \chi_\omega \varphi_\delta$.

Multiplying (21) with $v = v_\delta$ by φ_δ and integrating by parts, we obtain

$$\begin{aligned} & -\frac{1}{\delta} \int_0^A \int_\Omega q_\delta^2(0, a, x) \, dx \, da \\ &= \int_{Q_\Theta} h \varphi_\delta \, dx \, da \, dt + \int_{Q_\omega} v_\delta \varphi_\delta \, dx \, da \, dt. \end{aligned} \quad (23)$$

So

$$\begin{aligned} & \frac{1}{\delta} \int_0^A \int_\Omega q_\delta^2(0, a, x) \, dx \, da + \int_{Q_\omega} \varphi_\delta^2 \, dt \, da \, dx \\ &\leq \int_{Q_\Theta} |h \varphi_\delta| \, dx \, da \, dt. \end{aligned} \quad (24)$$

Using the inequality of Young, we obtain for any $R > 0$

$$\begin{aligned} & \frac{1}{\delta} \int_0^A \int_\Omega q_\delta^2(0, a, x) \, dx \, da + \int_{Q_\omega} \varphi_\delta^2 \, dt \, da \, dx \\ &\leq 2R \|h\|_{L^2(e_M)}^2 \\ &+ \frac{1}{2R} \int_Q \exp\left(\frac{-M}{at(T-t)(A-a)}\right) \varphi_\delta^2 \, dx \, da \, dt. \end{aligned} \quad (25)$$

Now, inequalities (7) and (25) give

$$\begin{aligned} & \frac{1}{\delta} \int_0^A \int_\Omega q_\delta^2(0, a, x) \, dx \, da + \int_{Q_\omega} \varphi_\delta^2 \, dx \, da \, dt \\ &\leq 2R \|h\|_{L^2(e_M)}^2 + \frac{C}{2R} \int_{Q_\omega} \varphi_\delta^2 \, dx \, da \, dt. \end{aligned} \quad (26)$$

Choosing R such that $2R = C$ we have

$$\begin{aligned} & \frac{1}{\delta} \int_0^A \int_\Omega q_\delta^2(0, a, x) \, dx \, da + \frac{1}{2} \int_{Q_\omega} \varphi_\delta^2 \, dx \, da \, dt \\ &\leq 2R \|h\|_{L^2(e_M)}^2. \end{aligned} \quad (27)$$

We conclude that

$$\begin{aligned} \int_{Q_\omega} |\varphi_\delta|^2 dt da dx &\leq 2R \|h\|_{L^2(e_M)}^2, \\ \int_0^A \int_\Omega q_\delta^2(0, a, x) dx da &\leq 2R\delta \|h\|_{L^2(e_M)}^2. \end{aligned} \quad (28)$$

□

Proposition 7. *The function q_δ solution of system (21) verifies*

$$\|q_\delta\|_{L^2((0,T) \times (0,A); H^1(\Omega))} \leq C \left(\|h\|_{L^2(e_M)} + \|h\|_{L^2(Q)} \right). \quad (29)$$

Proof. Let λ_0 be a positive constant. Put $\hat{q}_\delta = e^{-\lambda_0 t} q_\delta$. Then, $\hat{q}_\delta$ solves

$$\begin{aligned} -\frac{\partial \hat{q}_\delta}{\partial t} - \frac{\partial \hat{q}_\delta}{\partial a} - \Delta \hat{q}_\delta + (\mu + \lambda_0) \hat{q}_\delta \\ = \beta \hat{q}_\delta(t, 0, x) b(t, x) + \chi_\Theta e^{-\lambda_0 t} h + \chi_\omega e^{-\lambda_0 t} \varphi_\delta \end{aligned} \quad \text{in } Q \quad (30)$$

$$\hat{q}_\delta(t, a, x) = 0 \quad \text{on } \Sigma$$

$$\hat{q}_\delta(t, A, x) = 0 \quad \text{in } (0, T) \times \Omega$$

$$\hat{q}_\delta(T, a, x) = 0 \quad \text{in } (0, A) \times \Omega.$$

Multiplying (30) by $\hat{q}_\delta$ and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2} \int_0^A \int_\Omega \hat{q}_\delta^2(0, a, x) dx da + \frac{1}{2} \int_0^T \int_\Omega \hat{q}_\delta^2(t, 0, x) dx dt \\ + \int_Q |\nabla \hat{q}_\delta|^2 dx da dt + \lambda_0 \|\hat{q}_\delta\|_{L^2(Q)}^2 \\ \leq \frac{1}{2} \int_0^T \int_\Omega \hat{q}_\delta^2(t, 0, x) dx dt \\ + A \|\beta\|_\infty^2 \|b\|_\infty^2 \|\hat{q}_\delta\|_{L^2(Q)}^2 + \|h\|_{L^2(Q)}^2 \\ + C \|\hat{q}_\delta\|_{L^2(Q)}^2 + 2R \|h\|_{L^2(e_M)}^2. \end{aligned} \quad (31)$$

Choosing $\lambda_0 > A\|\beta\|_\infty^2 \|b\|_\infty^2 + C + 1$, we obtain

$$\|\nabla \hat{q}_\delta\|_{L^2(Q)} + \|\hat{q}_\delta\|_{L^2(Q)} \leq C \left(\|h\|_{L^2(Q)} + \|h\|_{L^2(e_M)} \right). \quad (32)$$

This gives the desired estimation. □

2.3. Study of the Nonlinear Case. In the following we set

$$b(t, x) = G(\eta). \quad (33)$$

Let Λ be an operator defined on $L^2((0, T) \times \Omega)$ by

$$\Lambda(\eta) = \int_0^A \beta(a) q_\delta(\eta) da, \quad (34)$$

where $q_\delta(\eta)$ is the solution of the following system:

$$\begin{aligned} -\frac{\partial q_\delta}{\partial t} - \frac{\partial q_\delta}{\partial a} - \Delta q_\delta + \mu q_\delta \\ = \beta q_\delta(t, 0, x) G(\eta) + \chi_\Theta h + \chi_\omega \varphi_\delta \quad \text{in } Q \\ q_\delta(t, a, x) = 0 \quad \text{on } \Sigma \\ q_\delta(t, A, x) = 0 \quad \text{in } (0, T) \times (0, A) \\ q_\delta(T, a, x) = 0 \quad \text{in } (0, A) \times \Omega. \end{aligned} \quad (35)$$

Then we have the following.

Proposition 8. *The operator Λ is continuous, bounded, and compact of $L^2((0, T) \times \Omega)$. Then Λ admits a fixed point.*

Proof. The proof will be done in two steps as follows.

Step 1 (boundedness and compactness of Λ). Let

$$Y(t, a, x) = \int_0^A \beta(a) q_\delta da, \quad (36)$$

and denote $(0, T) \times \Omega = Q_T$.

Then Y is the solution of the following system:

$$\begin{aligned} \frac{\partial Y(\eta)}{\partial t} - \Delta Y(\eta) + \int_0^A \mu \beta(a) q_\delta da = Z(\eta) \quad \text{in } Q_T \\ Y(\eta)(t, x) = 0 \quad \text{on } (0, T) \times \partial\Omega \\ Y(\eta)(0, x) = 0 \quad \text{in } \Omega \end{aligned} \quad (37)$$

with

$$\begin{aligned} Z(\eta) \\ = \int_0^A (\beta'_a q_\delta + \chi_\omega \beta \varphi_\delta + \beta^2 q_\delta(t, 0, x) G(\eta) + \beta h) da. \end{aligned} \quad (38)$$

Under assumptions (H_1) , (H_2) , and (H_3) and the results and owing to the estimation on q_δ and φ_δ the function $Z(\eta)$ satisfies

$$\begin{aligned} \|Z(\eta)\|_{L^2((0,T) \times \Omega)} \\ \leq C \left(\|\beta'_a\|_\infty, \|\beta\|_\infty, \|G\|_\infty \right) \left(\|h\|_{L^2(e_M)} + \|h\|_{L^2(Q)} \right). \end{aligned} \quad (39)$$

System (37) is a heat equation where the source term and the initial condition are bounded, respectively, in $L^2(Q_T)$ and $L^2(\Omega)$.

Then $Y(\eta)$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\partial Y(\eta)/\partial t$ is bounded in $L^2(0, T; H_0^{-1}(\Omega))$. Hence, using the Lions-Aubin lemma we conclude that Λ is bounded and compact in $L^2(Q_T)$.

Step 2 (continuity of Λ). Let $\eta_n \rightarrow \eta$ strongly in $L^2(Q_T)$. Then we can show that the sequence $Y(\eta_n)$ converges to $Y(\eta)$ strongly. For all $\eta_n \in \mathbb{R}$, $q_\delta(\eta_n)$ and $\varphi_\delta(\eta_n)$ are bounded independently to η_n . Therefore $Z(\eta_n)$ is bounded in $L^2(Q_T)$. Then we can extract a subsequence $X(\eta_{n_k})$ such that

$$\begin{aligned} Y_{\eta_k} &\longrightarrow Y(\eta) \quad \text{on } Q_T \\ Z_{\eta_k} &\longrightarrow Z(\eta) \quad \text{weakly in } Q_T \\ \int_0^A \mu \beta q_\delta(\eta_n) da &\longrightarrow \int_0^A \mu \beta q_\delta(\eta) da \\ &\quad \text{weakly in } L^2(Q_T). \end{aligned} \quad (40)$$

Then $Y(\eta)$ is a solution of (37). We deduce that the sequence $(Y(\eta_n))$ converges to $Y(\eta)$.

Since the operator Λ is continuous, bounded, and compact on $L^2(Q_T)$ onto $L^2(Q_T)$, Schauder's fixed-point theorem (see [4, 5]) implies that Λ admits a fixed point.

There exists $\eta \in L^2(Q_T)$ such that

$$\Lambda(\eta) = \eta = \int_0^A \beta q_\delta da. \quad (41)$$

Then q_δ solves

$$\begin{aligned} -\frac{\partial q_\delta}{\partial t} - \frac{\partial q_\delta}{\partial a} - \Delta q_\delta + \mu q_\delta \\ = \beta q_\delta(t, 0, x) G\left(\int_0^A \beta q_\delta da\right) + \chi_\Theta h + \chi_\omega \varphi_\delta \\ \text{in } Q \end{aligned} \quad (42)$$

$$q_\delta(t, a, x) = 0 \quad \text{on } \Sigma$$

$$q_\delta(t, A, x) = 0 \quad \text{in } (0, T) \times (0, A)$$

$$q_\delta(T, a, x) = 0 \quad \text{in } (0, A) \times \Omega.$$

From the foregoing, we have that φ_δ and q_δ are bounded in $L^2(Q)$.

Then we can extract a subsequence of (q_δ) still denoted by (q_{δ_n}) , such that

$$\begin{aligned} q_\delta &\rightharpoonup q \quad \text{in } L^2(Q), \\ \int_0^A \beta q_\delta da &\longrightarrow \int_0^A \beta q da \quad \text{in } L^2(Q_T). \end{aligned} \quad (43)$$

By the same idea

$$\chi_\omega \varphi_\delta \rightharpoonup \chi_\omega \nu \quad \text{in } L^2(Q). \quad (44)$$

Since $\int_0^A \beta q_\delta da \rightarrow \int_0^A \beta q da$, then there exists a subsequence still denoted by $\int_0^A \beta q_\delta da$ such that $\int_0^A \beta q_\delta da \rightarrow \int_0^A \beta q da$ a.e in Q_T .

Now since G is continuous, then

$$G\left(\int_0^A \beta q_\delta da\right) \longrightarrow G\left(\int_0^A \beta q da\right) \quad \text{a.e in } Q_T. \quad (45)$$

Therefore, one derives that q solves the following system:

$$\begin{aligned} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q &= \beta q(t, 0, x) G\left(\int_0^A \beta q da\right) \\ &\quad + \chi_\Theta h + \chi_\omega \nu \quad \text{in } Q \\ q(t, a, x) &= 0 \quad \text{on } \Sigma \\ q(t, A, x) &= 0 \quad \text{in } (0, T) \times (0, A) \\ q(T, a, x) &= 0 \quad \text{in } (0, A) \times \Omega, \end{aligned} \quad (46)$$

and we have also, for $\delta \rightarrow 0$, $q(0, a, x) = 0$ a.e. in $(0, A) \times \Omega$. $\square$

3. Application to the Detection of the Incomplete Parameter for a Nonlinear Population Dynamics Model

For a given positive real function F , we consider the following nonlinear population dynamics model:

$$\begin{aligned} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \Delta y + \mu y &= \lambda \hat{\xi} \quad \text{in } Q \\ y(t, a, x) &= 0 \quad \text{on } \Sigma \\ y(t, 0, x) &= F\left(\int_0^A \beta(a) y(\lambda, \tau) da\right) \\ &\quad \text{in } (0, T) \times \Omega \\ y(0, a, x) &= y^0(a, x) + \tau \hat{y}^0(a, x) \\ &\quad \text{in } (0, A) \times \Omega, \end{aligned} \quad (47)$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$. Let T be a positive real and let ω be an open subset such that $\bar{\omega} \subset \Omega$. Here, $y(t, a, x)$ is the distribution of individuals of age a at time t and location $x \in \Omega$, χ_ω is the characteristic function of ω , A is the maximal live expectancy, Δ is the Laplacian with respect to the spatial variable, and $\beta(a)$ and $\mu(a)$ denote, respectively, the natural fertility and the natural death rate of individuals of age a . Thus, the formula $\int_0^A \beta(a) y(t, a, x) da$ denotes the distribution of newborn individuals at time t and location x . In an oviparus species it denotes the total eggs at time t and position x . Therefore, the quantity $F(\int_0^A \beta(a) y(t, a, x) da)$ is the distribution of eggs that hatch at time t and position x .

System (47) describes the evolution of internal controlled age and space structured population under inhospitable boundary conditions in the case that the flux of individuals has the form $-\nabla y(t, a, x)$.

In system (47) y^0 is given in $L^2((0, A) \times \Omega)$.

The data of system (47) are incomplete in the following sense:

$$(i) \quad \hat{y}^0 \in L^2((0, A) \times \Omega) \text{ is unknown and } \|\hat{y}^0\|_{L^2((0, A) \times \Omega)} = 1;$$

- (ii) $\hat{\xi} \in L^2(Q)$ is unknown and $\|\hat{\xi}\|_{L^2(Q)} = 1$;
- (iii) $\lambda \in \mathbb{R}$ is unknown and small enough;
- (iv) $\tau \in \mathbb{R}$ is unknown and small enough.

We denote by $y_{\text{obs}} = m_0$ the observation that is a function in $L^2(Q_\Theta)$, where $\Theta \subset \Omega$ is the observation set. We know that (47) has a unique solution that we denote by $y(\lambda, \tau) = y(t, s, x, \lambda, \tau)$ in some relevant space. The question is as follows:

How to calculate the incomplete term $\lambda \hat{\xi}$ independently from the variation $\tau \hat{y}^0$ around the initial data?

The question above is natural and leads to some developments. An answer is given by the least squares method. This method consists in considering the unknowns $\{\lambda \xi, \tau y_0\} = \{\nu, \vartheta\}$, as control variables; then the state $y(t, s, x, \nu, \vartheta)$ has to be driven as close as possible to m_0 . So one studies an optimal control problem. By this way one looks to the pair $\{\nu, \vartheta\}$ but unfortunately there is possibility to find ν or ϑ independently. The sentinel method of Lions [6] is a particular least squares method which is adapted to the identification of parameters in ecosystems with incomplete data. Many papers are devoted to this topic in the literature. The sentinel concept relies on the following three objects: some state equation (e.g., (47)), some observation function (y_{obs}), and some control function ν to be determined. Many papers use the definition of Lions in the theoretical aspect (see, e.g., Bodart [7], Bodart and Fabre [8, 9]), as well as in the numerical one (see Bodart and Demeestere [10], Demeestere [11], and Kernevez [12]).

In [13] the author studies the detection of incomplete parameter for a linear population dynamic model. In this paper we are concerned by the nonlinear case. Moreover our method allows us to easily address the numerical aspect.

We suppose that $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$.

Proposition 9. *The functions $\lambda \mapsto y(\lambda, \tau)$ and $\tau \mapsto y(\lambda, \tau)$ are differentiable at the point 0.*

Proof. Let

$$\hat{y}(t, a, x) = e^{-\lambda_0 t} (y(\lambda, \tau) - y_0), \quad (48)$$

where $y(\lambda, 0) = y_0$.

It is easy to prove that $\hat{y}$ is the solution of the following system:

$$\begin{aligned} \frac{\partial \hat{y}}{\partial t} + \frac{\partial \hat{y}}{\partial a} - \Delta \hat{y} + (\mu + \lambda_0) \hat{y} &= 0 \quad \text{in } Q \\ \hat{y}(t, a, x) &= 0 \quad \text{on } \Sigma \\ \hat{y}(t, 0, x) &= e^{-\lambda_0 t} \left(F \left(\int_0^A \beta y(\lambda, \tau) da \right) - F \left(\int_0^A \beta y_0 da \right) \right) \\ \hat{y}(0, a, x) &= \tau \hat{y}^0(a, x). \end{aligned} \quad (49)$$

Multiplying (49) by $\hat{y}$ and integrating by parts, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^A \int_\Omega \hat{y}^2(T, a, x) dx dt \\ & + \frac{1}{2} \int_0^T \int_\Omega \hat{y}^2(t, A, x) dx da + \int_Q |\nabla \hat{y}|^2 dx da dt \\ & + \int_Q (\mu + \lambda_0) \hat{y}^2 dx da dt \\ & = \frac{1}{2} \int_0^A \int_\Omega \tau^2 |\hat{y}^0|^2 dx da \\ & + \frac{1}{2} \int_0^T \int_\Omega \hat{y}^2(t, 0, x) dx dt. \end{aligned} \quad (50)$$

We have

$$\begin{aligned} & |\hat{y}(t, 0, x)| \\ & = e^{-\lambda_0 t} \left| F \left(\int_0^A \beta y(\lambda, \tau) da \right) - F \left(\int_0^A \beta y_0 da \right) \right|. \end{aligned} \quad (51)$$

This implies

$$|\hat{y}(t, 0, x)|^2 \leq K \left| \int_0^A \beta \hat{y} \right|^2; \quad (52)$$

then

$$\int_0^T \int_\Omega \hat{y}^2(t, 0, x) dx dt \leq K \|\beta\|_\infty^2 \|\hat{y}\|_{L^2(Q)}^2. \quad (53)$$

Choosing $\lambda_0 = 1 + K\|\beta\|_\infty^2$ and using equality (51) we obtain

$$\|\nabla \hat{y}\|_{L^2(Q)}^2 + \|\hat{y}\|_{L^2(Q)}^2 \leq C\tau^2. \quad (54)$$

This gives

$$y(\lambda, \tau) \rightarrow y_0 \quad \text{a.e on } Q \text{ uniformly} \quad (55)$$

and then

$$\begin{aligned} \int_0^A \beta y(\lambda, \tau) da &\rightarrow \int_0^A \beta y_0 da \\ &\text{a.e on } Q_T \text{ uniformly.} \end{aligned} \quad (56)$$

Now we consider

$$\begin{aligned} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - \Delta z + (\mu + \lambda_0) z &= 0 \quad \text{in } Q \\ z(t, a, \sigma) &= 0 \quad \text{on } \Sigma \\ z(t, 0, x) &= F' \left(\int_0^A \beta y_0 da \right) \int_0^A \beta e^{-\lambda_0 t} z da \\ &\text{in } (0, T) \times \Omega \\ z(0, a, x) &= \hat{y}^0(a, x) \quad \text{in } (0, A) \times \Omega; \end{aligned} \quad (57)$$

$z_\tau = \hat{y}/\tau$ and $p_\tau = z_\tau - z$ solving

$$\frac{\partial p_\tau}{\partial t} + \frac{\partial p_\tau}{\partial a} - \Delta p_\tau + (\mu + \lambda_0) p_\tau = 0 \quad \text{in } Q$$

$$p_\tau(t, a, \sigma) = 0 \quad \text{on } \Sigma$$

$$p_\tau(t, 0, x)$$

$$= \frac{1}{\tau} e^{-\lambda_0 t} \left(F \left(\int_0^A \beta y(\lambda, \tau) da \right) - F \left(\int_0^A \beta y_0 da \right) \right) \quad (59)$$

$$- F' \left(\int_0^A \beta y_0 da \right) \int_0^A e^{-\lambda_0 t} \beta z da$$

$$\text{in } (0, T) \times \Omega$$

$$p_\tau(0, a, x) = \hat{y}^0(a, x) \quad \text{in } (0, A) \times \Omega.$$

From the regularity of F and the uniform convergence of $\tau \mapsto y(\lambda, \tau)$, we have

$\forall \epsilon > 0, \exists \tau_0$ such that $\forall \tau < \tau_0$;

$$\left| \frac{e^{-\lambda_0 t}}{\tau} F \left(\int_0^A \beta y(\lambda, \tau) da \right) - \frac{e^{-\lambda_0 t}}{\tau} F \left(\int_0^A \beta y_0 da \right) \right.$$

$$\left. - F' \left(\int_0^A \beta y_0 da \right) \int_0^A \beta e^{-\lambda_0 t} z da \right| \quad (60)$$

$$\leq C \int_0^A e^{\lambda_0 t} |\beta p_\tau| da + \epsilon \int_0^A e^{-\lambda_0 t} \beta |z_\tau| da.$$

Multiplying (59) by p_τ and integrating by parts, we obtain

$$\frac{1}{2} \int_0^A \int_\Omega p_\tau^2(T, a, x) dx dt$$

$$+ \frac{1}{2} \int_0^T \int_\Omega p_\tau^2(t, A, x) dx da + \int_Q |\nabla p_\tau|^2 dx da dt \quad (61)$$

$$+ \int_Q (\mu + \lambda_0) p_\tau^2 dx da dt$$

$$= \frac{1}{2} \int_0^T \int_\Omega p_\tau^2(t, 0, x) dx dt.$$

Using (60), we obtain

$$\frac{1}{2} \int_0^A \int_\Omega p_\tau^2(T, a, x) dx dt$$

$$+ \frac{1}{2} \int_0^T \int_\Omega p_\tau^2(t, A, x) dx da + \int_Q |\nabla p_\tau|^2 dx da dt \quad (62)$$

$$+ \int_Q (\mu + \lambda_0) p_\tau^2 dx da dt$$

$$\leq C \|\beta\|_\infty^2 \epsilon + A \|\beta\|_\infty^2 \|p_\tau\|_{L^2(Q)}^2.$$

Choosing λ_0 such that $\lambda_0 > 1 + A\|\beta\|_\infty^2$ we obtain

$$\|p_\tau\|_{L^2((0,T) \times (0,A); H^1(\Omega))}^2 \leq C\epsilon. \quad (63)$$

This implies $p_\tau \rightarrow 0$ a.e. in $L^2((0, T) \times (0, A); H_0^1(\Omega))$ as $\tau \rightarrow 0$. Then $z_\tau \rightarrow z$ a.e. in $L^2((0, T) \times (0, A); H_0^1(\Omega))$.

Therefore $\tau \mapsto e^{-\lambda_0 t} y(\tau, \lambda)$ is differentiable at 0. We conclude that $\tau \mapsto y(\lambda, \tau)$ is differentiable at 0.

The proof of the differentiability of the function $\lambda \mapsto y(\lambda, \tau)$ goes similarly. $\square$

Now let h be some function in $L^2(Q_\Theta)$ and let Q_ω be an open and nonempty subset of Q . For a control function $v \in L^2(Q_\omega)$, we define the functional

$$s(\lambda, \tau) = \int_{Q_\Theta} h y(\lambda, \tau) dx da dt$$

$$+ \int_{Q_\omega} v y(\lambda, \tau) dx da dt. \quad (64)$$

Using Proposition 2, one has

$$\frac{\partial S}{\partial \tau}(0, 0) = \int_{Q_\Theta} h y_\tau dx da dt + \int_{Q_\omega} v y_\tau dx da dt, \quad (65)$$

where

$$y_\tau = \lim_{\tau \rightarrow 0} \frac{y(0, \tau) - y(0, 0)}{\tau} \quad (66)$$

is the solution of the system

$$\frac{\partial y_\tau}{\partial t} + \frac{\partial y_\tau}{\partial a} - \Delta y_\tau + \mu y_\tau = 0 \quad \text{in } Q$$

$$y_\tau(t, a, x) = 0 \quad \text{on } \Sigma$$

$$y_\tau(t, 0, x) = \int_0^A \left(\beta F' \left(\int_0^A \beta y(0, 0) da \right) y_\tau \right) da \quad (67)$$

$$\text{in } (0, T) \times \Omega$$

$$y_\tau(0, a, x) = \hat{y}^0(a, x) \quad \text{in } (0, A) \times \Omega.$$

We say that S defines a sentinel for problem (47) if there exists v such that S is insensitive (at first order) with respect to the missing terms $\tau \hat{y}^0$; this means

$$\frac{\partial S}{\partial \tau}(0, 0) = 0 \quad (68)$$

for any $\hat{y}^0$ and if the norm $\|v\|_{L^2(Q_\omega)}$ of v is minimal.

The functional S assumed to be nonzero is an approached sentinel if, for any $\delta > 0$, there exists $v \in L^2(Q_\omega)$ such that

$$\left| \frac{\partial S}{\partial \tau}(0, 0) \right|$$

$$= \left| \int_{Q_\Theta} h y_\tau(0, 0) dx da dt + \int_{Q_\omega} v y_\tau(0, 0) dx da dt \right| \quad (69)$$

$$\leq \delta^2.$$

The existence of sentinel is equivalent to a controllability problem.

Proposition 10. Let us consider the following system:

$$\begin{aligned} & -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q \\ & = \beta q(t, 0, x) F' \left(\int_0^A \beta y(0, 0) da \right) + \chi_\Theta h + \chi_\omega v \\ & \text{in } Q \end{aligned} \quad (70)$$

$$q(t, a, x) = 0 \quad \text{on } \Sigma$$

$$q(t, A, x) = 0 \quad \text{in } (0, T) \times (0, A)$$

$$q(T, a, x) = 0 \quad \text{in } (0, A) \times \Omega,$$

where y_0 is the solution of system (47) with $\lambda = \tau = 0$. The approached sentinel condition (69) is equivalent to find $v \in L^2(Q_\omega)$ such that

$$\|q(0, \cdot, \cdot)\|_{L^2((0, A) \times \Omega)} \leq \delta \quad (71)$$

for any $\delta > 0$. The exact sentinel condition is equivalent to finding $v \in L^2(Q_\omega)$ such that

$$q(0, a, x) = 0 \quad \text{a.e in } (0, A) \times \Omega. \quad (72)$$

Proof. Multiplying (70) by y_τ and integrating by parts, we obtain

$$\begin{aligned} & \int_0^A \int_\Omega q(0, a, x) \hat{y}^0 dx da dt \\ & = \int_{Q_\Theta} h y_\tau dx da dt + \int_{Q_\omega} v y_\tau dx da dt. \end{aligned} \quad (73)$$

And then the approached sentinel condition

$$\left| \int_Q (\chi_\Theta h + \chi_\omega v) y_\tau dt da dx \right| \leq \delta^2 \quad (74)$$

is equivalent to

$$\left| \int_0^A \int_\Omega q(0, a, x) \hat{y}^0 dx da \right| \leq \delta^2 \quad (75)$$

for every

$$\hat{y}^0 \in L^2((0, A) \times \Omega). \quad (76)$$

That is equivalent to

$$\|q(0, a, x)\|_{L^2((0, A) \times \Omega)} \leq \delta. \quad (77)$$

□

We have the following result.

Proposition 11. Let $h \in L^2(e_M)$, if $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$; then under assumptions (H_1) , (H_2) , and (H_3) there exists a unique control $v \in L^2(Q_\omega)$ satisfying (68).

Proof. Replacing $b(t, x)$ in (21) by $F'(\int_0^A \beta y(0, 0) da)$ we obtain the approached sentinel, and we obtain the sentinel when δ goes to zero. □

4. Information Provided by the Sentinel

The function

$$y_\lambda = \lim_{\lambda \rightarrow 0} \frac{y(\lambda, 0) - y(0, 0)}{\lambda} \quad (78)$$

is the solution of the system

$$\begin{aligned} & \frac{\partial y_\lambda}{\partial t} + \frac{\partial y_\lambda}{\partial a} - \Delta y_\lambda + \mu y_\lambda = \hat{\xi}(t, a, x) \\ & \text{in } (0, T) \times (0, A) \times \Omega \\ & y_\lambda(t, a, x) = 0 \quad \text{on } \Sigma \end{aligned} \quad (79)$$

$$y_\lambda(t, 0, x) = \int_0^A \left(\beta F' \left(\int_0^A \beta y(0, 0) da \right) y_\lambda \right) da \quad \text{in } (0, T) \times \Omega$$

$$y_\lambda(0, a, x) = 0 \quad \text{in } (0, A) \times \Omega.$$

We have

$$\begin{aligned} S_{\text{obs}}(\lambda, \tau) & \approx S(0, 0) + \lambda \frac{\partial S}{\partial \lambda}(0, 0) \\ & + \tau \frac{\partial S}{\partial \tau}(0, 0) \iff \end{aligned} \quad (80)$$

$$S_{\text{obs}}(\lambda, \tau) - S(0, 0) \approx \lambda \frac{\partial S}{\partial \lambda}(0, 0).$$

Multiplying system (22) by y_λ and integrating by parts, we obtain

$$\int_Q q \hat{\xi} dx da dt = \int_Q (\chi_\Theta h + \chi_\omega v) y_\lambda dx da dt \quad (81)$$

and then

$$\begin{aligned} & \int_Q \lambda q \hat{\xi} dx da dt \\ & = \int_Q (\chi_\Theta h + \chi_\omega v) (m_0 - y(0, 0)) dx da dt. \end{aligned} \quad (82)$$

Knowing h , v , m_0 , and $y(0, 0)$, q can be calculated. Thus we obtain an integral equation in $\lambda \hat{\xi}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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