Research Article

Univalent and Starlike Properties for Generalized Struve Function

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We derive conditions on the parameters p , b , and c so that the function $zw_{p,b,c}(z)$, where $w_{p,b,c}(z)$ is the normalized form of generalized Struve function, belongs to the class $S_1^*(\alpha)$. Also, some sufficient conditions for the function $zw_{p,b,c}(z)$, to be in the class $\mathcal{U}(\lambda)$, are obtained.

1. Introduction and Preliminaries

Let $\mathbb{U} := \{z : |z| < 1\}$ denote the unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{A}$ denote the class of functions which are analytic and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ denote the class of functions such that

$$\mathcal{S} = \{f : f \in \mathcal{A} \text{ and } f \text{ univalent in } \mathbb{U}\}. \quad (2)$$

Suppose that f and g are two analytic functions in $\mathbb{U}$, and g is univalent in $\mathbb{U}$. We say that f is subordinate to g , written $f(z) \prec g(z)$ or $f \prec g$, if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

A function $f \in \mathcal{A}$ belongs to the class of starlike functions of order α denoted by $S^*(\alpha)$ if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{U}. \quad (3)$$

A subclass of the class of starlike functions denoted by $S_1^*(\alpha)$ for $0 \leq \alpha < 1$ consists of functions for which

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad z \in \mathbb{U}. \quad (4)$$

We also note that $S_1^*(0) \subset S^*$. In [1–3] the authors have discussed the coefficient bounds and other extremal properties of the class $S_1^*(\alpha)$.

For $0 < \lambda \leq 1$, consider the class

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda \text{ for } z \in \mathbb{U} \right\}. \quad (5)$$

From [4], we have the strict inclusion $\mathcal{U}(1) := \mathcal{U} \subset \mathcal{S}$. Very recently, Obradović et al. [5] have discussed the geometric behaviour of functions in $\mathcal{U}(\lambda)$. Also the class has been widely studied by many authors in [4, 6–8].

Let us consider the following second-order linear nonhomogenous differential equation (for more details see [9, 10]):

$$\begin{aligned} z^2 u''(z) + bzu'(z) + [cz^2 - p^2 + (1-b)p]u(z) \\ = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+b/2)}, \end{aligned} \quad (6)$$

where $b, p, c \in \mathbb{C}$. The function $u_{p,b,c}(z)$ which is called the generalized Struve function of order p is defined as a particular solution of (6) and has the series representation as follows:

$$u_{p,b,c}(z) = \sum_{n \geq 0} \frac{(-1)^n (c)^n}{\Gamma(n+3/2) \Gamma(p+n+(b+2)/2)} \left(\frac{z}{2}\right)^{2n+p+1}, \quad (7)$$

$$z \in \mathbb{C},$$

where Γ stands for the Euler gamma function.

Now, we consider the function $w_{p,b,c}(z)$ defined in terms of the generalized Struve function $u_{p,b,c}(z)$ by the transformation:

$$w_{p,b,c}(z) = 2^p \sqrt{\pi} \Gamma\left(p + \frac{b+2}{2}\right) z^{(-p-1)/2} u_{p,b,c}(\sqrt{z}). \quad (8)$$

By using the well-known Pochhammer symbol (or the shifted factorial) $(\chi)_n$ defined for $\chi, n \in \mathbb{C}$ in terms of the Euler Γ -function, we have

$$(\chi)_n = \frac{\Gamma(\chi+n)}{\Gamma(\chi)}$$

$$= \begin{cases} 1, & \text{if } n = 0, \chi \in \mathbb{C}^*, \\ \chi(\chi+1) \cdots (\chi+n-1), & \text{if } n \in \mathbb{N}, \chi \in \mathbb{C}. \end{cases} \quad (9)$$

We obtain the following series representation for the function $w_{p,b,c}$ given by (8):

$$w_{p,b,c}(z) = \sum_{n \geq 0} \frac{(-c/4)^n}{(3/2)_n (k)_n} z^n, \quad (10)$$

where $k = p + (b+2)/2 \neq 0, -1, -2, \dots$. Also, this function is analytic on $\mathbb{C}$ and satisfies the following second-order inhomogeneous differential equation:

$$4z^2 w''(z) + 2(2p+b+3)zw'(z) + (cz+2p+b)w(z) = (2p+b), \quad (11)$$

where $p, b, c \in \mathbb{C}$.

Recently, the class $\mathcal{U}(\lambda)$ and its generalizations have been widely studied by many authors [4–8, 11]. By applying the admissible function method authors in [12] have obtained conditions on the triplet a, b , and c such that $zF_{a,b,c}(z)$ is in the class $\mathcal{U}$, where $F_{a,b,c}(z)$ is the confluent hypergeometric function. In [13], the authors have derived conditions on the parameters a, b , and c such that the function ${}_2F_1(a, b, c; z)$ is in $S_1^*(0)$, where ${}_2F_1(a, b, c; z)$ is the Gaussian hypergeometric function. Moreover, in [9, 10] Yagmur and Orhan have obtained sufficient conditions for the generalized Struve function to be convex, starlike, and univalent. Most of these results were motivated by the research on geometric properties of Gaussian and confluent hypergeometric function.

Motivated by the above-mentioned works, in this paper we use the method of differential subordination to show that

$zw_{p,b,c}(z)$ is in the class $S^*(\alpha)$ and also we provide sufficient conditions for the function $zw_{p,b,c}(z)$ to be in the class $\mathcal{U}$ and hence univalent.

To prove our main results, we will need the following lemmas.

Lemma 1 (see [14]). Let (nonconstant) function $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 0$, $p(z) \neq 0$ ($z \in \mathbb{U}$).

If $|p(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then

$$z_0 p'(z_0) = kp(z_0),$$

$$\Re\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq k, \quad (12)$$

where k is a real number and $k \geq 1$.

Lemma 2 (see [15]). If an analytic function f has the form $f(z) = z + a_2 z^2 + \dots$, $z \in \mathbb{U}$ and satisfies the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \quad (13)$$

then f is univalent in $\mathbb{U}$.

Lemma 3 (see [15]). Let β be a complex number, $\Re(\beta) > 0$, and let α be a complex number, $|\alpha| \leq 1$, $\alpha \neq -1$, and $h(z) = z + a_2 z^2 + \dots$ a regular function on $\mathbb{U}$. If

$$\left| \alpha |z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1 \quad (14)$$

for all $z \in \mathbb{U}$, then the function

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} h'(t) dt \right)^{1/\beta}$$

$$= z + \frac{2a_2}{(1+\beta)} z^2$$

$$+ \left(\frac{3a_3}{2+\beta} + \frac{2\beta(1-\beta)a_2^2}{(\beta+1)^3} \right) z^3 + \dots \quad (15)$$

is regular and univalent in $\mathbb{U}$.

Lemma 4 (see [16]). Let $\Omega \subset \mathbb{C}$ and let q be analytic and univalent on $\mathbb{U}$ except for those $\xi \in \partial\mathbb{U}$ for which $\lim_{z \rightarrow \xi} q(z) = \infty$. Suppose that $\psi : \mathbb{C}^3 \times \mathbb{U} \rightarrow \mathbb{C}$ satisfies the condition

$$\psi(q(\zeta), m\zeta q'(\zeta), \zeta^2 q''(\zeta); z) \notin \Omega, \quad (16)$$

where $q(z)$ is finite, $m \geq n \geq 1$, and $|\xi| = 1$. If p and q analytic in $\mathbb{U}$, $p(z) = p(0) + p_n z^n + \dots$, $p(0) = q(0)$, and further if

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega \quad (17)$$

then $p(z) \prec q(z)$ in $\mathbb{U}$.

Suppose that $p(z)$ is analytic in $\mathbb{U}$ with $p(z) = p_n z^n + p_{n+1} z^{n+1} + \dots$ and $q(z) = Mz$. Then the condition (16) reduces to a simple form:

$$\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega \quad (18)$$

whenever $K \geq nM$, $\operatorname{Re}(Le^{-i\theta}) \geq (n-1)K$, $z \in \mathbb{U}$, and $\theta \in \mathbb{R}$.

2. Main Results

Theorem 5. Let $\delta > 0$, $p, b \in \mathbb{R}$, and $c \in \mathbb{C}$ such that $|c|\delta < 3[(2p+b)+2] - |c|\delta$; then

$$|w_{p,b,c}(\delta z) - 1| < \frac{|c|\delta}{3[(2p+b)+2] - |c|\delta}, \quad z \in \mathbb{U}. \quad (19)$$

Proof. Let $p(z) = w_{p,b,c}(\delta z) - 1$, $\delta > 0$; then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 0$. Since the function $w_{p,b,c}(z)$ satisfies the differential equation (11) and $w_{p,b,c}(\delta z) = p(z) + 1$, $p(z)$ satisfies the following inhomogeneous differential equation:

$$4z^2 p''(z) + 2(2p+b+3)zp'(z) + (c\delta z + 2p+b)(p(z)+1) = (2p+b). \quad (20)$$

Using Lemma 4, we will show that $|p(z)| < M$, where $M = |c|\delta/3[(2p+b)+2] - |c|\delta$. For this, if we let

$$\psi(r, s, t; z) = 4t + 2(2p+b+3)s + (c\delta z + 2p+b)(1+r) - (2p+b) \quad (21)$$

and $\Omega = \{0\}$, then it is sufficient to prove that $\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega$ whenever $K \geq M$, $\operatorname{Re}(Le^{-i\theta}) \geq 0$, $z \in \mathbb{U}$, and θ is real, we have that

$$\begin{aligned} |\psi(Me^{i\theta}, Ke^{i\theta}, L; z)| &= |4L + 2(2p+b+3)Ke^{i\theta} \\ &\quad + (c\delta z + 2p+b)(1 + Me^{i\theta}) - (2p+b)| = |4Le^{-i\theta} \\ &\quad + 2(2p+b+3)K + (c\delta z + 2p+b)(M + e^{-i\theta}) \\ &\quad - (2p+b)e^{-i\theta}| \geq 4\operatorname{Re}(Le^{-i\theta}) + 2(2p+b+3)M \\ &\quad + \operatorname{Re}((c\delta z + 2p+b)M) \\ &\quad + \operatorname{Re}((c\delta z + 2p+b)e^{-i\theta}) - \operatorname{Re}((2p+b)e^{-i\theta}) \\ &\geq [2(2p+b+3) + (2p+b)]M - \operatorname{Re}(c\delta ze^{-i\theta}) \\ &\quad - \operatorname{Re}(c\delta zM) \geq 3[(2p+b)+2]M - |c|\delta M - |c|\delta \\ &= 0. \end{aligned} \quad (22)$$

In the last stage of the inequalities we have used the definition of M and shown that

$$\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega = \{0\} \quad (23)$$

whenever $K \geq M$, θ is real, and $z \in \mathbb{U}$. Hence, $|p(z)| < M$. $\square$

Choosing $\delta = 1$ in the above theorem we have the following Corollary.

Corollary 6. Let $p, b \in \mathbb{R}$, and $c \in \mathbb{C}$ be such that $3[(2p+b)+2] - 2|c| > 0$; then

$$|w_{p,b,c}(z) - 1| < \frac{|c|}{3[(2p+b)+2] - |c|}, \quad z \in \mathbb{U}. \quad (24)$$

The next results give sufficient conditions for the function $g_{p,b,c}(z) = zw_{p,b,c}(z)$ to be starlike and univalent in the open unit disc.

Theorem 7. If $p, b \in \mathbb{R}$, and $c \in \mathbb{C}$ such that $2|c| < 3[(2p+b)+2]$ and

$$\begin{aligned} &\frac{(2p+b)|c|}{[3[(2p+b)+2] - 2|c|]} + |c| \\ &\leq (1-\alpha)[2(2p+b+3) - 4(1-\alpha)], \end{aligned} \quad (25)$$

then the function $g_{p,b,c}(z)$ belongs to the class $S_1^*(\alpha)$.

Proof. For $g_{p,b,c}(z) = zw_{p,b,c}(z)$ to be in the class $S_1^*(\alpha)$ we need to prove that

$$\left| \frac{zg'_{p,b,c}(z)}{g_{p,b,c}(z)} - 1 \right| < (1-\alpha), \quad z \in \mathbb{U}. \quad (26)$$

Upon setting

$$\frac{[zw_{p,b,c}(z)]'}{w_{p,b,c}(z)} - 1 = \tilde{q}(z) \quad (27)$$

we observe that $\tilde{q}(z)$ is analytic in $\mathbb{U}$ and $\tilde{q}(0) = \tilde{q}'(0) - 1 = 0$. Also

$$\begin{aligned} zw'_{p,b,c}(z) &= \tilde{q}(z)w_{p,b,c}(z), \\ z^2 w''_{p,b,c}(z) &= [(\tilde{q}(z) - 1)\tilde{q}(z) + z\tilde{q}'(z)]w_{p,b,c}(z). \end{aligned} \quad (28)$$

Now, since $w_{p,b,c}(z)$ satisfies the inhomogeneous differential equation (11), in terms of $\tilde{q}(z)$ we see that $\tilde{q}(z)$ satisfies the following equation:

$$\psi(\tilde{q}(z), z\tilde{q}'(z); z) = 0, \quad (29)$$

where

$$\begin{aligned} \psi(r, s; z) &= 4[r(r-1) + s] + 2(2p+b+3)r \\ &\quad + (cz + 2p+b) - (2p+b) \frac{1}{g_{p,b,c}(z)}. \end{aligned} \quad (30)$$

Now, we claim that $|\tilde{q}(z)| < 1 - \alpha$, $0 \leq \alpha < 1$. By using Lemma 4 with $\Omega = \{0\}$, $n = 1$, and $q(z) = (1-\alpha)z$, we need to show that

$$\psi((1-\alpha)e^{i\theta}, Ke^{i\theta}; z) \notin \Omega, \quad (31)$$

where θ is real, $K \geq (1 - \alpha)$, and $z \in \mathbb{U}$:

$$\psi((1 - \alpha)e^{i\theta}, Ke^{i\theta}; z) = e^{i\theta}C - D, \quad (32)$$

where

$$C = 4[(1 - \alpha)[(1 - \alpha)e^{i\theta} - 1] + K] + 2(2p + b + 3)(1 - \alpha), \quad (33)$$

$$D = (2p + b) \left[\frac{1}{g_{p,b,c}(z)} - 1 \right] - cz.$$

Also, when $\delta = 1$ in Theorem 5 we have that if $2|c| < 3(2p + b) + 6$ then

$$|w_p(z) - 1| < \frac{|c|}{3(2p + b) + 6} - |c| \leq 1. \quad (34)$$

Also,

$$\begin{aligned} |C| &\geq \operatorname{Re} C > 4(1 - \alpha)[(1 - \alpha)e^{i\theta} - 1] + (1 - \alpha)4 \\ &+ 2(2p + b + 3)(1 - \alpha) \geq (1 - \alpha)[4(1 - \alpha)\cos\theta \\ &- 4(1 - \alpha) + 4(1 - \alpha) + 2(2p + b + 3)] \geq (1 - \alpha) \\ &\cdot [2(2p + b + 3) - 4(1 - \alpha)] = \tilde{\beta}, \text{ say.} \end{aligned} \quad (35)$$

Now, if we show that $|D| < \tilde{\beta}$, then we have

$$|\psi(r, s; z)| \geq |e^{i\theta}C - D| \geq |C| - |D| = 0. \quad (36)$$

In order to establish that we need the following results which state that for $r < 1$ and $|w - 1| < r$ if and only if

$$\left| \frac{1}{w} - \frac{1}{1 - r^2} \right| < \frac{r}{1 - r^2}. \quad (37)$$

In particular, $|w - 1| < r$ implies that

$$\left| \frac{1}{w} - 1 \right| < \frac{r}{1 - r}. \quad (38)$$

When $r = |c|/[3((2p + b) + 2) - |c|]$, we have

$$\left| \frac{1}{g_{p,b,c}(z)} - 1 \right| < \frac{r}{1 - r} = \frac{|c|}{3(2p + b) + 6 - 2|c|}. \quad (39)$$

Using the last inequality, we have

$$|D| \leq (2p + b) \left| \frac{1}{g_{p,b,c}(z)} - 1 \right| + |c||z| < \frac{(2p + b)|c|}{3[(2p + b) + 2] - 2|c|} + |c| \quad (40)$$

and so we have $|D| < \tilde{\beta}$ provided that

$$\frac{(2p + b)|c|}{3[(2p + b) + 2] - 2|c|} + |c| \leq \tilde{\beta} \quad (41)$$

holds, where $\tilde{\beta}$ is given by (35). And, we observe that the above condition is stated in the theorem. Thus, $|\tilde{q}(z)| < (1 - \alpha)$ in $\mathbb{U}$ and hence $g_{p,b,c}(z)$ belongs to $S_1^*(\alpha)$. $\square$

Theorem 8. Let $p, b \in \mathbb{R}$, and $c \in \mathbb{C}$. If $g_{p,b,c}(z)$ satisfies any one of the following inequalities:

$$\left| \frac{z^2 g'_{p,b,c}(z)}{[g_{p,b,c}(z)]^2} \left[\frac{(zg_{p,b,c}(z))''}{g'_{p,b,c}(z)} - \frac{2zg'_{p,b,c}(z)}{g_{p,b,c}(z)} \right] \right| < 1 \quad (z \in \mathbb{U}), \quad (42)$$

$$\left| \frac{[g_{p,b,c}(z)]^2}{z^2 g'_{p,b,c}(z)} \left[\frac{[zg_{p,b,c}(z)]''}{g'_{p,b,c}(z)} - \frac{2zg'_{p,b,c}(z)}{g_{p,b,c}(z)} \right] \right| < \frac{1}{4} \quad (z \in \mathbb{U}), \quad (43)$$

$$\left| \frac{[zg_{p,b,c}(z)]'' / g'_{p,b,c}(z) - 2zg'_{p,b,c}(z) / g_{p,b,c}(z)}{z^2 g'_{p,b,c}(z) / (g_{p,b,c}(z))^2 - 1} \right| < \frac{1}{2} \quad (z \in \mathbb{U}), \quad (44)$$

$$\operatorname{Re} \left[\frac{z^2 g_{p,b,c}(z)}{(g_{p,b,c}(z))^2} \left[\frac{[zg_{p,b,c}(z)]'' / g'_{p,b,c}(z) - 2zg'_{p,b,c}(z) / g_{p,b,c}(z)}{z^2 g'_{p,b,c}(z) / [g_{p,b,c}(z)]^2} \right] \right] < 1 \quad (z \in \mathbb{U}), \quad (45)$$

then $g_{p,b,c}$ is in $\mathcal{U}$ and hence $g_{p,b,c}(z)$ is univalent in $\mathbb{U}$.

Proof. Define a function $p(z)$ by

$$\frac{z^2 g'_{p,b,c}(z)}{[g_{p,b,c}(z)]^2} = 1 + p(z) \quad (z \in \mathbb{U}), \quad (46)$$

and then $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 0$. Differentiating (46) gives

$$\frac{[zg_{p,b,c}(z)]''}{g'_{p,b,c}(z)} - \frac{2zg'_{p,b,c}(z)}{g_{p,b,c}(z)} = \frac{zp'(z)}{1+p(z)}. \quad (47)$$

Hence, from (46) and (47), we have

$$\begin{aligned} E_1(z) &= \frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} \left(\frac{[zg_{p,b,c}(z)]''}{g'_{p,b,c}(z)} - \frac{2zg'_{p,b,c}(z)}{g_{p,b,c}(z)} \right) = zp'(z), \\ E_2(z) &= \frac{(g_{p,b,c}(z))^2}{z^2 g'_{p,b,c}(z)} \left(\frac{[zg_{p,b,c}(z)]''}{g'_{p,b,c}(z)} - \frac{2zg'_{p,b,c}(z)}{g_{p,b,c}(z)} \right) = \frac{zp'(z)}{(1+p(z))^2}, \\ E_3(z) &= \frac{[zg_{p,b,c}(z)]'' / g'_{p,b,c}(z) - 2zg'_{p,b,c}(z) / g_{p,b,c}(z)}{z^2 g'_{p,b,c}(z) / (g_{p,b,c}(z))^2 - 1} = \frac{zp'(z)}{p(z)} \frac{1}{(1+p(z))}, \\ E_4(z) &= \frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} \left(\frac{[zg_{p,b,c}(z)]'' / g'_{p,b,c}(z) - 2zg'_{p,b,c}(z) / g_{p,b,c}(z)}{z^2 g'_{p,b,c}(z) / (g_{p,b,c}(z))^2 - 1} \right) = \frac{zp'(z)}{p(z)}. \end{aligned} \quad (48)$$

Now suppose that there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z| < |z_0|} |p(z)| = |p(z_0)| = 1, \quad (49)$$

and then from Lemma 1, we have

$$z_0 p'(z_0) = k p(z_0). \quad (50)$$

Therefore, letting $p(z_0) = e^{i\theta}$ in each of (48), we obtain that

$$\begin{aligned} |E_1(z_0)| &= |z_0 p'(z_0)| = |k e^{i\theta}| \geq 1, \\ |E_2(z_0)| &= \left| \frac{z_0 p'(z_0)}{(1+p(z_0))^2} \right| = \frac{k}{|1+e^{i\theta}|^2} \geq \frac{1}{4}, \\ |E_3(z_0)| &= \left| \frac{z_0 p'(z_0)}{p(z_0)} \frac{1}{1+p(z_0)} \right| = \frac{k}{|1+e^{i\theta}|} \geq \frac{1}{2}, \end{aligned} \quad (51)$$

$$\operatorname{Re} \{E_4(z_0)\} = \operatorname{Re} \left\{ \frac{z_0 p'(z_0)}{p(z_0)} \right\} = k \geq 1,$$

which contradicts our assumption (42)–(45), respectively. Therefore, $|w(z)| < 1$ for all $z \in \mathbb{U}$; then from (46) we have

$$\left| \frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} - 1 \right| = |w(z)| < 1, \quad (52)$$

which implies $g_{p,b,c}$ is in the class $\mathcal{U}$ and hence univalent. $\square$

Theorem 9. Let $c > 0$, $d \geq 0$, such that $c + 2d \leq 1$. If $g_{p,b,c}(z)$ satisfies the inequality

$$\Re \left\{ \frac{[zg_{p,b,c}(z)]''}{g'_{p,b,c}(z)} - \frac{2zg'_{p,b,c}(z)}{g_{p,b,c}(z)} \right\} < \frac{c+d}{(1+c)(1-d)}, \quad (53)$$

then $g_{p,b,c}(z)$ is univalent in $\mathbb{U}$.

Proof. Define a function $q(z)$ as follows:

$$\frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} = \frac{1+aq(z)}{1-bq(z)} \quad (z \in \mathbb{U}). \quad (54)$$

We see that $q(z)$ is analytic in $\mathbb{U}$ and $q(0) = 0$. Differentiation of (54) gives

$$\begin{aligned} \frac{[zg_{p,b,c}(z)]''}{g'_{p,b,c}(z)} - \frac{2zg'_{p,b,c}(z)}{g_{p,b,c}(z)} &= \frac{(c+d)zq'(z)}{(1+cq(z))(1-dq(z))} = E_5(z), \text{ say.} \end{aligned} \quad (55)$$

Now suppose there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z| < |z_0|} |q(z)| = |q(z_0)| = 1. \quad (56)$$

Then from Lemma 1, we have

$$z_0 q'(z_0) = k q(z_0). \quad (57)$$

Now letting $q(z_0) = e^{i\theta}$, $(\theta \in [0, 2\pi])$ in (55), we have

$$\begin{aligned} \operatorname{Re}(E_5(z_0)) &= k(a+b) \\ &\cdot \operatorname{Re}\left(\frac{q(z_0)}{(1+cq(z_0))(1-dq(z_0))}\right) = k \\ &\cdot \operatorname{Re}\left(\frac{1}{(1-dq(z_0))} - \frac{1}{(1+cq(z_0))}\right) = k \\ &\cdot \operatorname{Re}\left(\frac{1-de^{i\theta}}{1+d^2-2d\cos\theta} - \frac{1+ce^{i\theta}}{1+c^2+2c\cos\theta}\right) \\ &= k\left(\frac{1}{2+(d^2-1)/(1-d\cos\theta)}\right. \\ &\quad \left.- \frac{1}{2+(c^2-1)/(1+c\cos\theta)}\right), \end{aligned} \quad (58)$$

where $\theta \neq \cos^{-1}(-1/c)$ and $\theta \neq \cos^{-1}(1/d)$. Since $k \geq 1$ we have

$$\operatorname{Re}(E(z_0)) > \frac{c+d}{(1+c)(1-d)}. \quad (59)$$

This contradicts the hypothesis and therefore $|q(z)| < 1$ for all $z \in \mathbb{U}$. Thus,

$$\left| \frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} - 1 \right| = \left| \frac{(c+d)q(z)}{1-dq(z)} \right| < \frac{c+d}{1-d} \leq 1 \quad (60)$$

$(z \in \mathbb{U}).$

In view of Lemma 2 it implies that $g_{p,b,c}$ is in $\mathcal{U}$ and hence univalent. $\square$

Theorem 10. Let $M \geq 1$, ν be a real number such that $\nu \geq 2M+1$ and let γ be a complex number which satisfies the inequality

$$|\gamma| \leq 1 - \frac{1}{\nu}(2M+1). \quad (61)$$

If $g_{p,b,c}$ is univalent in $\mathbb{U}$ and $|g_{p,b,c}(z)| \leq M$ for all $z \in \mathbb{U}$, then the function

$$E(z) = \left\{ \nu \int_0^z t^{\nu-1} \frac{g_{p,b,c}(t)}{t} dt \right\}^{1/\nu} \quad (62)$$

is univalent in $\mathbb{U}$, where the values of the complex powers are taken with their principal values.

Proof. Define a function

$$p(z) = \int_0^z \frac{g_{p,b,c}(t)}{t} dt. \quad (63)$$

Then we have $p(0) = p'(0) - 1 = 0$.

Also

$$p'(z) = \frac{g_{p,b,c}(z)}{z}, \quad (64)$$

$$\frac{zp''(z)}{p'(z)} = \frac{zg'_{p,b,c}(z)}{g_{p,b,c}(z)} - 1. \quad (65)$$

From (65), we have

$$\begin{aligned} \left| \frac{zp''(z)}{p'(z)} \right| &\leq \left| \frac{zg'_{p,b,c}(z)}{g_{p,b,c}(z)} \right| + 1 \\ &= \left| \frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} \right| \left| \frac{g_{p,b,c}(z)}{z} \right| + 1. \end{aligned} \quad (66)$$

From the hypothesis, we have $|g_{p,b,c}(z)| \leq M$ ($z \in \mathbb{U}$); then, by the Schwarz Lemma (cf [17], we obtain that

$$|g_{p,b,c}(z)| \leq M|z| \quad (z \in \mathbb{U}). \quad (67)$$

Now, since $g_{p,b,c}(z)$ is univalent in $\mathbb{U}$

$$\begin{aligned} \left| \frac{zp''(z)}{p'(z)} \right| &\leq \left| \frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} \right| M + 1 \\ &\leq \left| \frac{z^2 g'_{p,b,c}(z)}{(g_{p,b,c}(z))^2} - 1 \right| M + M + 1 \leq 2M + 1. \end{aligned} \quad (68)$$

Using (68), we have

$$\begin{aligned} \left| \gamma |z|^{2\nu} + (1 - |z|^{2\nu}) \frac{zp''(z)}{\nu p'(z)} \right| &\leq |\gamma| + \frac{1}{\nu} \left| \frac{zp''(z)}{\nu p'(z)} \right| \\ &\leq |\gamma| + \frac{1}{\beta} (2M + 1). \end{aligned} \quad (69)$$

So, from (61) we have

$$\left| \gamma |z|^{2\nu} + (1 - |z|^{2\nu}) \frac{zp''(z)}{\nu p'(z)} \right| \leq 1. \quad (70)$$

Applying Lemma 3, we obtain the function $E(z)$ defined by (62) which is univalent in $\mathbb{U}$. $\square$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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