Research Article

Characterizations of Regular Ordered Semirings by Ordered Quasi-Ideals

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We introduce the notion of an ordered quasi-ideal of an ordered semiring and show that ordered quasi-ideals and ordered bi-ideals coincide in regular ordered semirings. Then we give characterizations of regular ordered semirings, regular ordered duo-semirings, and left (right) regular ordered semirings by their ordered quasi-ideals.

1. Introduction

The concept of a quasi-ideal was defined first by Steinfeld for semigroups and for rings [1–3] as a generalization of a right ideal and a left ideal. Then Iséki [4] introduced the notion of a quasi-ideal in a semiring without zero and investigated some of its properties. In 1994, Dönges [5] studied quasi-ideals of a semiring with zero, investigated connections between left (right) ideals, bi-ideals, and quasi-ideals and characterized regular semirings using their quasi-ideals. Later, Shabir et al. [6] have studied some properties of quasi-ideals, using quasi-ideals to characterize regular and intraregular semirings and regular duo-semirings. As a generalization of quasi-ideals of semirings the quasi-ideals of Γ -semirings were investigated by many authors; see, for example, [7–9].

In 2011, the notion of an ordered semiring was introduced by Gan and Jiang [10] as a semiring with a partially ordered relation on the semiring such that the relation is compatible to the operations of the semiring. In the paper, the concept of a left (right) ordered ideal, a minimal ordered ideal, and a maximal ordered ideal was defined. Then Mandal [11] studied fuzzy ideals in an ordered semiring with the least element zero and gave a characterization of regular ordered semirings by their fuzzy ideals.

In this paper, we introduce the notion of an ordered quasi-ideal of an ordered semiring and show that ordered

quasi-ideals and ordered bi-ideals coincide in regular ordered semirings. Then characterizations of regular ordered semirings, regular ordered duo-semirings, and left (right) regular ordered semirings by their ordered quasi-ideals have been investigated.

2. Preliminaries

An ordered semiring is a system $(S, +, \cdot, \leq)$ consisting of a nonempty set S such that $(S, +, \cdot)$ is a semiring, $(S, \leq)$ is a partially ordered set, and for any $a, b, x \in S$ the following conditions are satisfied:

- (i) if $a \leq b$ then $a + x \leq b + x$ and $x + a \leq x + b$;
- (ii) if $a \leq b$ then $ax \leq bx$ and $xa \leq xb$.

An ordered semiring S is said to be *additively commutative* if $a + b = b + a$ for all $a, b \in S$. An element $0 \in S$ is said to be an *absorbing zero* if $0a = 0 = a0$ and $a + 0 = a = 0 + a$ for all $a \in S$. In this paper we assume that S is an additively commutative ordered semiring with an absorbing zero 0 .

For any subsets A, B of S and $a \in S$, we denote

$$(A) = \{x \in S \mid x \leq a \text{ for some } a \in A\},$$

$$AB = \{ab \in S \mid a \in A, b \in B\},$$

$$\begin{aligned}
\Sigma A &= \left\{ \sum_{i \in I} a_i \in S \mid a_i \right. \\
&\quad \left. \in A \text{ and } I \text{ is a finite subset of } \mathbb{N} \right\}, \\
\Sigma AB &= \left\{ \sum_{i \in I} a_i b_i \in S \mid a_i \in A, b_i \right. \\
&\quad \left. \in B \text{ and } I \text{ is a finite subset of } \mathbb{N} \right\}, \\
\mathbb{N}a &= \Sigma \{a\}.
\end{aligned} \tag{1}$$

Now, we mention some properties of finite sums on an ordered semiring.

Remark 1. For any subsets A, B of S , the following statements hold:

- (i) $\Sigma(A) \subseteq (\Sigma A)$;
- (ii) $\Sigma(\Sigma A) = \Sigma A$;
- (iii) $A(\Sigma B) \subseteq \Sigma AB$ and $(\Sigma A)B \subseteq \Sigma AB$;
- (iv) $\Sigma(A\Sigma B) \subseteq \Sigma AB$ and $\Sigma(\Sigma A)B \subseteq \Sigma AB$;
- (v) $\Sigma(A + B) = \Sigma A + \Sigma B$.

We note that, for any $A \subseteq S$, $\Sigma A = A$ if and only if $A + A \subseteq A$ ($(A, +)$ is a subsemigroup of $(S, +)$).

Now, we give the basic properties of the operator (Σ) which are not difficult to verify.

Lemma 2. Let A, B, C be subsets of an ordered semiring S . Then the following statements hold:

- (i) $A \subseteq (A)$ and $((A)) = (A)$;
- (ii) If $A \subseteq B$ then $(A) \subseteq (B)$;
- (iii) $A(B) \subseteq (A)(B) \subseteq (AB)$ and $(A)B \subseteq (A)(B) \subseteq (AB)$;
- (iv) $A + (B) \subseteq (A) + (B) \subseteq (A + B)$ and $(A) + B \subseteq (A) + (B) \subseteq (A + B)$;
- (v) $A(B + C) \subseteq (AB + AC)$ and $(A + B)C \subseteq (AC + BC)$;
- (vi) $(A \cup B) = (A) \cup (B)$;
- (vii) $(A \cap B) \subseteq (A) \cap (B)$.

In (vii) of the above lemma, we have $(A \cap B) = (A) \cap (B)$ when $(A) = A$ and $(B) = B$.

Lemma 3. Let S be an ordered semiring and $\emptyset \neq A \subseteq S$. If $A \subseteq (\Sigma A^2 + \Sigma ASA)$ then $\Sigma A^2 \subseteq (\Sigma ASA)$.

Proof. Assume that $A \subseteq (\Sigma A^2 + \Sigma ASA)$. Then

$$\begin{aligned}
\Sigma A^2 &\subseteq \Sigma (\Sigma A^2 + \Sigma ASA) A \\
&\subseteq \Sigma ((\Sigma A^2) A + (\Sigma ASA) A) \subseteq \Sigma (\Sigma A^3 + \Sigma ASA) \\
&= (\Sigma (\Sigma A^3) + \Sigma (\Sigma ASA)) = (\Sigma A^3 + \Sigma ASA) \\
&= (\Sigma AAA + \Sigma ASA) \subseteq (\Sigma ASA + \Sigma ASA) \\
&= (\Sigma ASA).
\end{aligned} \tag{2}$$

□

Definition 4 (see [10]). Let S be an ordered semiring and $\emptyset \neq A \subseteq S$. Then A is said to be a *left ordered ideal* (*right ordered ideal*) if the following conditions are satisfied.

- (1) A is a left ideal (right ideal) of S .
- (2) If $x \leq a$ for some $a \in A$ then $x \in A$ (i.e., $A = (A)$).

We call A an *ordered ideal* if it is both left ordered ideal and right ordered ideal of S .

Example 5 (see [10]). Let $[0, 1]$ be the unit interval of real numbers. Define binary operations $\oplus$ and $\odot$ on $[0, 1]$ by letting $a, b \in [0, 1]$,

$$\begin{aligned}
a \oplus b &= \max \{a, b\}, \\
a \odot b &= \max \{a + b - 1, 0\},
\end{aligned} \tag{3}$$

and an ordered relation $\leq$ is the natural order on real numbers. It is easy to show that $L = ([0, 1], \oplus, \odot, \leq)$ is an ordered semiring. Let $I = [0, 1/2]$. Then we can prove that I is an ordered ideal of L .

Lemma 6. Let A be a nonempty subset of an ordered semiring S . Then

- (i) (ΣSA) is a left ordered ideal of S ;
- (ii) (ΣAS) is a right ordered ideal of S ;
- (iii) (ΣSAS) is an ordered ideal of S .

Proof. (i) Let $x, y \in (\Sigma SA)$. Then $x \leq x'$ and $y \leq y'$ for some $x', y' \in \Sigma SA$. It is clear that $x + y \leq x' + y' \in \Sigma SA$, and so $x + y \in (\Sigma SA)$. By Remark 1 and Lemma 2, we obtain $S(\Sigma SA) \subseteq (S\Sigma SA) \subseteq (\Sigma SSA) \subseteq (\Sigma SA)$. We have $((\Sigma SA)) = (\Sigma SA)$. Hence, (ΣSA) is a left ordered ideal of S .

(ii) and (iii) can be proved similar to (i). □

Corollary 7. Let S be an ordered semiring. Then, for any $a \in S$,

- (i) (Sa) is a left ordered ideal of S ;
- (ii) (aS) is a right ordered ideal of S ;
- (iii) (ΣSaS) is an ordered ideal of S .

Let A be a nonempty subset of an ordered semiring S . We denote $L(A)$, $R(A)$ and $I(A)$ as the smallest left ordered ideal, right ordered ideal, and ordered ideal of S containing

A , respectively. In particular, we can show that if A is a left ideal (right ideal, ideal) of S then $(A]$ is the smallest left ordered ideal (resp., right ordered ideal and ordered ideal) of S containing A .

Lemma 8. Let A be a nonempty subset of an ordered semiring S . Then

- (i) $L(A) = (\Sigma A + \Sigma SA]$;
- (ii) $R(A) = (\Sigma A + \Sigma AS]$;
- (iii) $I(A) = (\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]$.

Proof. (i) Since S has an absorbing zero, we have, for every $a \in A$, $a = a + 0 \in \Sigma A + \Sigma SA \subseteq (\Sigma A + \Sigma SA]$. Hence, $A \subseteq (\Sigma A + \Sigma SA]$. Let $x, y \in (\Sigma A + \Sigma SA]$. Then $x \leq x'$ and $y \leq y'$ for some $x', y' \in \Sigma A + \Sigma SA$. Thus $x' = a_1 + b_1$ and $y' = a_2 + b_2$ for some $a_1, a_2 \in \Sigma A$ and $b_1, b_2 \in \Sigma SA$. It is easy to show that $a_1 + a_2 \in \Sigma A$ and $b_1 + b_2 \in \Sigma SA$. It follows that $x + y \leq x' + y' \in \Sigma A + \Sigma SA$, and so $x + y \in (\Sigma A + \Sigma SA]$. By Remark 1 and Lemma 2, we obtain

$$\begin{aligned} S(\Sigma A + \Sigma SA) &\subseteq (S(\Sigma A + \Sigma SA)) \subseteq (\Sigma A + \Sigma SA) \\ &\subseteq (\Sigma SA + \Sigma SSA) \subseteq (\Sigma A + \Sigma SA) \\ &= (\Sigma SA) \subseteq (\Sigma A + \Sigma SA). \end{aligned} \quad (4)$$

Since $((\Sigma A + \Sigma SA)) = (\Sigma A + \Sigma SA)$, L is a left ordered ideal of S . Let K be any left ordered ideal of S containing A . It turns out $\Sigma A \subseteq K$ and $\Sigma SA \subseteq K$, so $\Sigma A + \Sigma SA \subseteq K$. It follows that $(\Sigma A + \Sigma SA) \subseteq (K) = K$. Therefore, $(\Sigma A + \Sigma SA)$ is the smallest left ordered ideal of S containing A .

(ii) and (iii) can be proved similar to (i). $\square$

As a special case of Lemma 8, if $A = \{a\}$ then we have the following corollary.

Corollary 9. Let S be an ordered semiring. Then, for any $a \in S$,

- (i) $L(a) = (\mathbb{N}a + Sa]$;
- (ii) $R(a) = (\mathbb{N}a + aS]$;
- (iii) $I(a) = (\mathbb{N}a + Sa + aS + \Sigma SaS]$.

An element e of an ordered semiring S is said to be an *identity* if $ea = a = ae$ for all $a \in S$. If S has an identity, then we denote 1 as the identity of S .

It is not difficult to show that if S has an identity, then $L(A) = (\Sigma SA]$, $R(A) = (\Sigma AS]$ and $I(A) = (\Sigma SAS]$ for any $A \subseteq S$. In particular case, we have $L(a) = (Sa]$, $R(a) = (aS]$ and $I(a) = (\Sigma SaS]$ for any $a \in S$.

3. Ordered Quasi-Ideals in Ordered Semirings

Here, we present a notion of an ordered quasi-ideal of an ordered semiring. Then, in ordered semiring with an identity, we show that every ordered quasi-ideal can be expressed as an intersection of an ordered left ideal and an ordered right ideal.

Definition 10. Let $(S, +, \cdot, \leq)$ be an ordered semiring and let $(Q, +)$ be a subsemiring of $(S, +)$. Then Q is said to be

an *ordered quasi-ideal* of S if the following conditions are satisfied:

- (1) $(\Sigma SQ) \cap (\Sigma QS) \subseteq Q$;
- (2) if $x \leq q$ for some $q \in Q$ then $x \in Q$ (i.e., $Q = (Q]$).

It is clear that every left ordered ideal (right ordered ideal and ordered ideal) of an ordered semiring S is an ordered quasi-ideal of S . Moreover, each ordered quasi-ideal of S is a subsemiring of S ; indeed, $QQ \subseteq (QQ) \subseteq (SQ) \cap (QS) \subseteq (\Sigma SQ) \cap (\Sigma QS) \subseteq Q$.

Example 11. Let $S = \{a, b, c, d\}$. Define binary operations $+$ and $\cdot$ on S by the following equations:

$$\begin{array}{c|cccc} + & a & b & c & d \\ \hline a & a & b & c & d \\ b & b & b & b & b \\ c & c & b & c & d \\ d & d & b & d & d \end{array}, \quad (5)$$

$$\begin{array}{c|cccc} \cdot & a & b & c & d \\ \hline a & a & a & a & a \\ b & a & b & b & b \\ c & a & c & c & c \\ d & a & b & b & b \end{array}$$

Then $(S, +, \cdot)$ is an additively commutative semiring with an absorbing zero a . Define a binary relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (b, d)\}. \quad (6)$$

We give the covering relation “ $<$ ” and the figure of S :

$$< := \{(b, d)\}. \quad (7)$$

Now, $(S, +, \cdot, \leq)$ is an ordered semiring. Let $Q = \{a, b\}$. We have $(\Sigma SQ) \cap (\Sigma QS) = \{a, b, c\} \cap \{a, b\} = Q$ and $(Q) = Q$. Hence, Q is an ordered quasi-ideal of S but is not a left ordered ideal of S , since $SQ = \{a, b, c\} \not\subseteq Q$.

Lemma 12. Let S be an ordered semiring and let $\{Q_i \mid i \in I\}$ be a family of ordered quasi-ideals of S . Then $\bigcap_{i \in I} Q_i$ is an ordered quasi-ideal of S .

Let A be a nonempty subset of an ordered semiring S . We denote $Q(A)$ the smallest ordered quasi-ideal of S containing A .

Theorem 13. Let S be an ordered semiring and let A be a nonempty subset of S . Then $Q(A) = (\Sigma A + ((\Sigma SA) \cap (\Sigma AS)))$.

Proof. Let $Q = (\Sigma A + ((\Sigma SA] \cap (\Sigma AS]))$. Since S has an absorbing zero, we have $a = a + 0 \in \Sigma A + ((\Sigma SA] \cap (\Sigma AS)) \subseteq Q$ for every $a \in A$. Hence, $A \subseteq Q$. Let $x, y \in Q$. Then $x \leq x'$ and $y \leq y'$ for some $x', y' \in \Sigma A + ((\Sigma SA] \cap (\Sigma AS))$. Thus $x' = a_1 + b_1$ and $y' = a_2 + b_2$ for some $a_1, a_2 \in \Sigma A$ and $b_1, b_2 \in (\Sigma SA] \cap (\Sigma AS]$. Clearly, $a_1 + a_2 \in \Sigma A$ and $b_1 + b_2 \in (\Sigma SA] \cap (\Sigma AS]$. It follows that $x + y \leq x' + y' \in \Sigma A + ((\Sigma SA] \cap (\Sigma AS))$, and so $x + y \in Q$. By Remark 1 and Lemma 2, we obtain

$$\begin{aligned}
 (\Sigma SQ] \cap (\Sigma QS] &\subseteq (\Sigma SQ] \\
 &= (\Sigma S(\Sigma A + ((\Sigma SA] \cap (\Sigma AS)))) \\
 &\subseteq (\Sigma S(\Sigma A + (\Sigma SA])) \\
 &\subseteq (\Sigma (\Sigma A + S(\Sigma SA))) \\
 &\subseteq (\Sigma (\Sigma SA + (\Sigma SA))) \\
 &\subseteq (\Sigma (\Sigma SA + \Sigma SA)) \\
 &\subseteq (\Sigma (\Sigma SA + \Sigma SA)) \subseteq (\Sigma (\Sigma SA)) \\
 &\subseteq ((\Sigma SA]) = (\Sigma SA).
 \end{aligned} \tag{8}$$

Similarly, we can show that $(\Sigma SQ] \cap (\Sigma QS] \subseteq (\Sigma AS]$. Thus $(\Sigma SQ] \cap (\Sigma QS] \subseteq (\Sigma SA] \cap (\Sigma AS] \subseteq \Sigma A + ((\Sigma SA] \cap (\Sigma AS)) \subseteq Q$. Since $(Q] = Q$, we obtain that Q is an ordered quasi-ideal of S containing A . Let K be any ordered quasi-ideal of S containing A . It follows that $(\Sigma SA] \cap (\Sigma AS] \subseteq (\Sigma SK] \cap (\Sigma KS] \subseteq K$. So $\Sigma A + ((\Sigma SA] \cap (\Sigma AS)) \subseteq K$. Hence, $Q = (\Sigma A + ((\Sigma SA] \cap (\Sigma AS))) \subseteq (K] = K$. Therefore, Q is the smallest ordered quasi-ideal of S containing A . $\square$

As a special case of Theorem 13, if $A = \{a\}$ then we have the following corollary.

Corollary 14. *Let S be an ordered semiring. Then $Q(a) = (\mathbb{N}a + ((Sa] \cap (aS]))$ for any $a \in S$.*

If S has an identity, then it is easy to check that $Q(A) = (\Sigma SA] \cap (\Sigma AS]$ for any $A \subseteq S$. In particular case, we have $Q(a) = (Sa] \cap (aS]$ for any $a \in S$.

Let $\mathcal{Q}(S)$ be the set of all ordered quasi-ideals of an ordered semiring S . Using Lemma 12, we define the operations $\wedge$ and $\vee$ on $\mathcal{Q}(S)$ by letting $P_1, P_2 \in \mathcal{Q}(S)$,

$$\begin{aligned}
 P_1 \wedge P_2 &= P_1 \cap P_2, \\
 P_1 \vee P_2 &= Q(P_1 \cup P_2).
 \end{aligned} \tag{9}$$

Then we obtain the following theorem.

Theorem 15. *Let S be an ordered semiring. Then $(\mathcal{Q}(S), \wedge, \vee)$ is a complete lattice.*

Theorem 16. *The intersection of a left ordered ideal L and a right ordered ideal R of an ordered semiring S is an ordered quasi-ideal of S .*

Proof. It is easy to show that $L \cap R$ is a subsemigroup of $(S, +)$. By Remark 1 and Lemma 2, we obtain

$$\begin{aligned}
 (\Sigma S(L \cap R]) \cap (\Sigma (L \cap R)S] &\subseteq (\Sigma S(L \cap R]) \\
 &= (\Sigma (SL \cap SR)) \subseteq (\Sigma SL] \subseteq L, \\
 (\Sigma S(L \cap R]) \cap (\Sigma (L \cap R)S] &\subseteq (\Sigma (L \cap R)S] \\
 &= (\Sigma (LS \cap RS)) \subseteq (\Sigma RS] \subseteq R.
 \end{aligned} \tag{10}$$

Hence, $(\Sigma S(L \cap R]) \cap (\Sigma (L \cap R)S] \subseteq L \cap R$. Let $s \in S$ such that $s \leq x$ for some $x \in L \cap R$. Then $s \in (L \cap R] \subseteq (L] \cap (R] = L \cap R$. $\square$

The converse of Theorem 16 is not true as Example 2.1 page 8 in [2] given by A. H. Clifford.

Corollary 17. *Let S be an ordered semiring. Then the following statements hold.*

- (i) $(\Sigma SA] \cap (\Sigma AS]$ is an ordered quasi-ideal of S , for any $A \subseteq S$.
- (ii) $(Sa] \cap (aS]$ is an ordered quasi-ideal of S , for any $a \in S$.

Proof. (i) By Lemma 6, we have $(\Sigma SA]$ and $(\Sigma AS]$ a left and a right ordered ideal of S , respectively. Then by Theorem 16, we have that $(\Sigma SA] \cap (\Sigma AS]$ is an ordered quasi-ideal of S .

- (ii) It is a particular case of (i). $\square$

Now, we will show that the converse of Theorem 16 is true if S contains an identity as the following theorem.

Theorem 18. *Let S be an ordered semiring with identity. Then every ordered quasi-ideal Q of S can be written in the form $Q = R \cap L$ for some right ordered ideal R and left ordered ideal L of S .*

Proof. Assume that S has an identity. Let Q be an ordered quasi-ideal of S . Then $R(Q) = (\Sigma QS]$ and $L(Q) = (\Sigma SQ]$. We obtain $Q \subseteq R(Q) \cap L(Q)$ and $R(Q) \cap L(Q) = (\Sigma QS] \cap (\Sigma SQ] \subseteq Q$. Hence, $Q = R(Q) \cap L(Q)$. $\square$

4. Regular Ordered Semirings

In this section, we show that in regular ordered semirings the converse of Theorem 16 is true and ordered quasi-ideals coincide with ordered bi-ideals. Then we give characterizations of regular ordered semirings, regular ordered duo-semirings, and left regular and right regular ordered semirings by their ordered quasi-ideals.

Definition 19 (see [11]). An element a of an ordered semiring S is said to be *regular* if $a \leq axa$ for some $x \in S$. An ordered semiring S is said to be *regular* if every element $a \in S$ is regular.

The following lemma is characterizations of regular ordered semiring which directly follows Definition 19.

Lemma 20. Let S be an ordered semiring. Then the following statements are equivalent:

- (i) S is regular;
- (ii) $A \subseteq (\Sigma ASA)$ for each $A \subseteq S$;
- (iii) $a \in (aSa)$ for any $a \in S$.

Now, we will show that the converse of Theorem 16 is true in regular ordered semirings.

Theorem 21. Every ordered quasi-ideal of a regular ordered semiring S can be written in the form $Q = R \cap L$ for some right ordered ideal R and left ordered ideal L of S .

Proof. Let Q be an ordered quasi-ideal of S . By Lemma 8, we have $R(Q) = (\Sigma Q + \Sigma QS)$ and $L(Q) = (\Sigma Q + \Sigma SQ)$. Now, $Q \subseteq R(Q) \cap L(Q)$. Let $q \in Q$. Since S is regular, there exists $x \in S$ such that $q \leq qxq \in \Sigma QS$. So $Q \subseteq (\Sigma QS)$. Since $Q + Q \subseteq Q$, $\Sigma Q = Q$. It follows that

$$\begin{aligned} (\Sigma QS) &\subseteq (\Sigma Q + \Sigma QS) = (Q + \Sigma QS) \subseteq ((\Sigma QS) + \Sigma QS) \\ &\subseteq (\Sigma QS). \end{aligned} \quad (11)$$

This implies that $R(Q) = (\Sigma QS)$. Similarly, we can show that $L(Q) = (\Sigma SQ)$. Hence, $R(Q) \cap L(Q) = (\Sigma QS) \cap (\Sigma SQ) \subseteq Q$. Therefore, $Q = R(Q) \cap L(Q)$. $\square$

Definition 22. Let $(S, +, \cdot, \leq)$ be an ordered semiring. A subsemigroup $(B, +)$ of $(S, +)$ is said to be an *ordered bi-ideal* of S if the following conditions hold:

- (1) $BSB \subseteq B$;
- (2) if $x \leq b$ for some $b \in B$, then $x \in B$ (i.e., $B = (B)$).

We note that condition (1) of Definition 22 is equivalent to $\Sigma BSB \subseteq B$.

Theorem 23. Every ordered quasi-ideal of an ordered semiring S is an ordered bi-ideal of S .

Proof. Let Q be an ordered quasi-ideal of S . Then $\Sigma QSQ \subseteq \Sigma QS \subseteq (\Sigma QS)$ and $QSQ \subseteq \Sigma SQ \subseteq (\Sigma SQ)$. So, $\Sigma QSQ \subseteq (\Sigma SQ) \cap (\Sigma QS) \subseteq Q$. Hence, Q is an ordered bi-ideal of S . $\square$

The converse of Theorem 23 is not generally true as the following example.

Example 24. Let $S = \{a, b, c, d, e\}$. Define binary operations $+$ and $\cdot$ by the following equations:

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	d
c	c	d	d	d	d
d	d	d	d	d	d
e	e	d	d	d	e

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	b	b	b
d	a	a	b	b	b
e	a	a	b	b	b

(12)

Then $(S, +, \cdot)$ is an additively commutative semiring with an absorbing zero a . Define a binary relation $\leq$ on S by

$$\begin{aligned} \leq := & \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, e), \\ & (a, d), (b, d), (c, d), (e, d)\}. \end{aligned} \quad (13)$$

We give the covering relation " $<$ " and the figure of S :

$$< := \{(a, b), (a, c), (a, e), (b, d), (c, d), (e, d)\}. \quad (14)$$

Then $(S, +, \cdot, \leq)$ is an ordered semiring but not regular, since $d \not\leq dxd$ for any $x \in S$. Let $B = \{a, e\}$. It is easy to show that B is an ordered bi-ideal but not an ordered quasi-ideal of S , since $(\Sigma SB) \cap (\Sigma BS) = \{a, b\} \not\subseteq B$.

Now, we show that in regular ordered semirings, ordered bi-ideals and ordered quasi-ideals coincide as the following theorem.

Theorem 25. Let S be a regular ordered semiring. Then ordered bi-ideals and ordered quasi-ideals coincide in S .

Proof. By Theorem 23, we have that every ordered quasi-ideal of S is an ordered bi-ideal of S . Now, we show that every ordered bi-ideal of S is an ordered quasi-ideal of S . Let B be an ordered bi-ideal of S . Let $a \in (\Sigma SB) \cap (\Sigma BS)$. By Lemma 20, Remark 1, and Lemma 2, we obtain $a \in (aSa) \subseteq ((\Sigma BS)S(\Sigma SB)) \subseteq ((\Sigma BSS)(\Sigma SB)) \subseteq ((\Sigma BS)(\Sigma SB)) \subseteq (\Sigma(BS(\Sigma SB))) \subseteq (\Sigma(\Sigma BSSB)) \subseteq (\Sigma BSB) \subseteq B$. Hence, B is an ordered quasi-ideal of S . $\square$

Theorem 26. Let S be an ordered semiring. Then the following statements are equivalent:

- (i) S is regular;
- (ii) $(\Sigma RL) = R \cap L$ for every right ordered ideal R and left ordered ideal L of S ;
- (iii) $B = (\Sigma BSB)$ for each ordered bi-ideal B of S ;
- (iv) $Q = (\Sigma QSQ)$ for each ordered quasi-ideal Q of S .

Proof. (i) $\Rightarrow$ (ii): assume that S is regular and let R and L be a right ordered ideal and a left ordered ideal of S , respectively. So, $(\Sigma RL) \subseteq (\Sigma R) = R$ and $(\Sigma RL) \subseteq (\Sigma L) = L$. Hence, $(\Sigma RL) \subseteq R \cap L$. Let $a \in R \cap L$. Since S is regular, $a \leq axa$ for some $x \in S$. Since $a \in R$, $xa \in L$. It follows that $a \leq a(xa) \in RL$. This means $a \in (RL) \subseteq (\Sigma RL)$. Therefore, $(\Sigma RL) = R \cap L$.

(ii) $\Rightarrow$ (iii): assume that (ii) holds. Let B be an ordered bi-ideal of S . It is clear that $(\Sigma BSB) \subseteq B$. By assumption, $B \subseteq R(B) \cap L(B) = (\Sigma R(B)L(B))$. By Lemma 8, Remark 1, and Lemmas 2 and 3, we have

$$\begin{aligned} B &\subseteq (\Sigma B + \Sigma BS) \cap (\Sigma B + \Sigma SB) \\ &= (\Sigma ((\Sigma B + \Sigma BS) (\Sigma B + \Sigma SB))) \\ &\subseteq (\Sigma ((\Sigma B + \Sigma BS) (\Sigma B + \Sigma SB))) \\ &\subseteq (\Sigma (\Sigma B (\Sigma B + \Sigma SB) + \Sigma BS (\Sigma B + \Sigma SB))) \\ &\subseteq (\Sigma (\Sigma B^2 + \Sigma BSB + \Sigma BSB + \Sigma BSSB)) \\ &\subseteq (\Sigma (\Sigma B^2 + \Sigma BSB)) = (\Sigma (\Sigma B^2) + \Sigma (\Sigma BSB)) \\ &= (\Sigma B^2 + \Sigma BSB) \subseteq ((\Sigma BSB) + \Sigma BSB) \subseteq (\Sigma BSB). \end{aligned} \quad (15)$$

(iii) $\Rightarrow$ (iv): it follows from Theorem 23.

(iv) $\Rightarrow$ (i): let $a \in S$. Then $Q(a) = (Q(a)SQ(a))$. By Corollary 14, Remark 1, and Lemma 2, we have

$$\begin{aligned} a &\in (\mathbb{N}a + ((Sa) \cap (aS))) \\ &= (\Sigma ((\mathbb{N}a + ((Sa) \cap (aS))) S (\mathbb{N}a + ((Sa) \cap (aS)))) \\ &\subseteq (\Sigma ((\mathbb{N}a + (aS)) S (\mathbb{N}a + (Sa)))) \\ &\subseteq (\Sigma ((\mathbb{N}a + aS) S (\mathbb{N}a + Sa))) \\ &\subseteq (\Sigma (((\mathbb{N}a + aS) S) (\mathbb{N}a + Sa))) \\ &\subseteq (\Sigma ((aS) (\mathbb{N}a + Sa))) \subseteq (\Sigma (aS (\mathbb{N}a + Sa))) \\ &\subseteq (\Sigma aSa) = (aS a). \end{aligned} \quad (16)$$

By Lemma 20, S is regular. $\square$

Theorem 27. Let S be a regular ordered semiring. Then the following statements hold:

(i) every ordered quasi-ideal Q of S can be written in the form $Q = R \cap L = (RL)$ for some right ordered ideal R and left ordered ideal L of S ;

(ii) $(Q^2) = (Q^3)$ for each ordered quasi-ideal Q of S .

Proof. (i) It is obvious by Theorems 21 and 26.

(ii) Let Q be an ordered quasi-ideal of S . Clearly, $((QQ)Q) \subseteq (QQ)$. Let $x \in (QQ)$. Then $x \leq q_1 q_2$ for some $q_1, q_2 \in Q$. Since S is regular, there exists $s \in S$ such that $x \leq q_1 q_2 \leq (q_1 q_2)s(q_1 q_2) \in QQSQQ$. Hence, $x \in (Q(QSQ)Q) \subseteq (QQQ)$. Therefore, $(Q^2) = (Q^3)$. $\square$

Theorem 28. Let S be an ordered semiring. Then S is regular if and only if $B \cap I \cap L \subseteq (BIL)$ for every ordered bi-ideal B , every ordered ideal I , and every left ordered ideal L of S .

Proof. Let B, I , and L be an ordered bi-ideal, an ordered ideal, and a left ordered ideal of S , respectively. Let $a \in B \cap I \cap L$. Since S is regular, $a \leq axa \leq axaxaxa \in BIL$. Hence, $B \cap I \cap L \subseteq (BIL)$.

Conversely, assume that $B \cap I \cap L \subseteq (BIL)$ for every ordered bi-ideal B , every ordered ideal I , and every left ordered ideal L of S . Then we obtain $R \cap L = R \cap S \cap L \subseteq (RSL) \subseteq (RL) \subseteq (\Sigma RL)$ for every right ordered ideal R and left ordered ideal L of S . On the other hand, we have $(\Sigma RL) \subseteq R \cap L$. Hence, $(\Sigma RL) = R \cap L$. By Theorem 26, S is regular. $\square$

Definition 29. An ordered semiring S is said to be an *ordered duo-semiring* if every one-sided (right or left) ordered ideal of S is an ordered ideal of S .

We note that every multiplicatively commutative ordered semiring is an ordered duo-semiring, but the converse is not generally true. Now, we give an example of a multiplicatively noncommutative ordered semiring which is an ordered duo-semiring.

Example 30. Let $S = \{a, b, c, d, e\}$. Define binary operations $+$ and $\cdot$ by the following equations:

$$\begin{array}{c|ccccc} + & a & b & c & d & e \\ \hline a & a & b & c & d & e \\ b & b & c & c & c & c \\ c & c & c & c & c & c \\ d & d & c & c & c & c \\ e & e & c & c & c & c \end{array}, \quad \begin{array}{c|ccccc} \cdot & a & b & c & d & e \\ \hline a & a & a & a & a & a \\ b & a & e & c & e & c \\ c & a & c & c & c & c \\ d & a & c & c & e & c \\ e & a & c & c & c & c \end{array} \quad (17)$$

Then $(S, +, \cdot)$ is an additively commutative semiring with an absorbing zero a . Define a binary relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (e, c)\}. \quad (18)$$

We give the covering relation " $<$ " and the figure of S :

$$< := \{(e, c)\}. \quad (19)$$

Then $(S, +, \cdot, \leq)$ is an ordered semiring which is not multiplicatively commutative, since $bd \neq db$. We have all one-sided ordered ideals of S which are as follows:

$$\{a\}, \{a, c\}, \{a, c, e\}, \{a, b, c, e\}, \{a, c, d, e\}, S. \quad (20)$$

It is not difficult to check that all of them are ordered ideals of S . This shows that S is an ordered duo-semiring.

Lemma 31. *Let S be an ordered semiring. Then the following conditions are equivalent:*

- (i) S is an ordered duo-semiring;
- (ii) $R(A) = L(A)$ for each $A \subseteq S$;
- (iii) $R(a) = L(a)$ for each $a \in S$.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i): let L be a left ordered ideal of S and let $x \in L, s \in S$. By assumption, we have $xs \in R(x)S \subseteq R(x) = L(x) \subseteq L(L) = L$. It follows that L is a right ordered ideal of S . Similarly, we have that every right ordered ideal of S is a left ordered ideal of S . Hence, S is an ordered duo-semiring. $\square$

Theorem 32. *Let S be an ordered duo-semiring. Then S is regular if and only if $(\Sigma Q_1 Q_2) = Q_1 \cap Q_2$ for each two ordered quasi-ideals Q_1 and Q_2 of S .*

Proof. Assume that S is a regular ordered semiring. Let Q_1 and Q_2 be ordered quasi-ideals of S . By Theorem 21, Q_1 and Q_2 can be written in the forms

$$\begin{aligned} Q_1 &= R_1 \cap L_1, \\ Q_2 &= R_2 \cap L_2 \end{aligned} \quad (21)$$

for some R_1, R_2 and L_1, L_2 which are right ordered ideals and left ordered ideals of S , respectively. Since S is an ordered duo-semiring, R_1, R_2, L_1 , and L_2 are ordered ideals of S . It follows that Q_1 and Q_2 are ordered ideals of S . By Theorem 26, we have $(\Sigma Q_1 Q_2) = Q_1 \cap Q_2$.

Conversely, assume that $(\Sigma Q_1 Q_2) = Q_1 \cap Q_2$ for each two ordered quasi-ideals Q_1 and Q_2 of S . Let $A \subseteq S$. By assumption, $A \subseteq Q(A) \cap Q(A) = (\Sigma Q(A)Q(A))$. By Theorem 13, Remark 1, and Lemmas 2 and 3, we have

$$\begin{aligned} A &\subseteq (\Sigma ((\Sigma A + ((\Sigma SA) \cap (\Sigma AS))) \\ &\quad \cdot (\Sigma A + ((\Sigma SA) \cap (\Sigma AS)))) \subseteq (\Sigma ((\Sigma A + (\Sigma AS)) \\ &\quad \cdot (\Sigma A + (\Sigma SA)))) \subseteq (\Sigma ((\Sigma A + \Sigma AS) (\Sigma A + \Sigma SA))) \\ &\subseteq (\Sigma ((\Sigma A + \Sigma AS) (\Sigma A + \Sigma SA))) \\ &\subseteq (\Sigma (\Sigma A (\Sigma A + \Sigma SA) + \Sigma AS (\Sigma A + \Sigma SA))) \\ &\subseteq (\Sigma (\Sigma A^2 + \Sigma ASA + \Sigma ASA + \Sigma ASSA)) \\ &\subseteq (\Sigma (\Sigma A^2 + \Sigma ASA)) = (\Sigma (\Sigma A^2) + \Sigma (\Sigma ASA)) \\ &= (\Sigma A^2 + \Sigma ASA) \subseteq ((\Sigma ASA) + \Sigma ASA) \subseteq (\Sigma ASA). \end{aligned} \quad (22)$$

By Lemma 20, S is a regular ordered semiring. $\square$

Theorem 33. *Let S be an ordered duo-semiring. Then the following conditions are equivalent:*

- (i) S is regular;

(ii) $(\Sigma L_1 L_2) = L_1 \cap L_2$ and $(\Sigma R_1 R_2) = R_1 \cap R_2$ for each two left ordered ideals L_1, L_2 and right ordered ideals R_1, R_2 of S ;

(iii) $(\Sigma RL) = R \cap L = (\Sigma LR)$, for each right ordered ideal R and left ordered ideal L of S .

Proof. It is obvious by Theorem 26. $\square$

Definition 34. Let S be an ordered semiring. Then an element $a \in S$ is said to be *left regular* (right regular) if $a \leq xa^2$ ($a \leq a^2x$) for some $x \in S$. An ordered semiring S is said to be *left regular* (right regular) if every element $a \in S$ is left regular (right regular).

Example 35. Let $S = \{a, b, c, d, e, f\}$. Define binary operations $+$ and $\cdot$ on S by the following equations:

$$\begin{array}{c|cccccc} + & a & b & c & d & e & f \\ \hline a & a & b & c & d & e & f \\ b & b & b & b & b & b & b \\ c & c & c & c & c & c & c \\ d & d & d & d & d & d & d \\ e & e & e & e & e & e & e \\ f & f & f & f & f & f & f \end{array}, \quad \begin{array}{c|cccccc} \cdot & a & b & c & d & e & f \\ \hline a & a & a & a & a & a & a \\ b & a & b & b & b & b & b \\ c & a & b & b & b & b & b \\ d & a & b & b & d & b & d \\ e & a & e & e & e & e & e \\ f & a & e & e & f & e & f \end{array} \quad (23)$$

Then $(S, +, \cdot)$ is a semiring with an absorbing zero a . Define a binary relation $\leq$ on S by

$$\begin{aligned} \leq &:= \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (b, c), \\ &\quad (b, d), (b, e), (b, f), (c, d), (c, e), (c, f), (d, f), (e, f)\}. \end{aligned} \quad (24)$$

We give the covering relation " $\prec$ " and the figure of S :

$$\prec := \{(b, c), (c, d), (c, e), (d, f), (e, f)\}. \quad (25)$$

Now, $(S, +, \cdot, \leq)$ is an ordered semiring. Clearly, a, b, d, e , and f are left regular. We consider $c \leq ec^2 = eb = e$. This implies

that S is left regular. Since there does not exist $x \in S$ such that $c \leq cxc$, S is not regular.

Example 36. Consider the ordered semiring $S = (\mathbb{N} \cup \{0\}, \max, \min, \leq)$ where $\leq$ is the natural order relation on numbers. Since $n \leq \min\{n, n, n\}$ for any $n \in \mathbb{N}$, we get S a regular, left regular, and right regular ordered semiring.

The following lemmas can be easily proved using Definition 34.

Lemma 37. Let S be an ordered semiring. Then the following statements are equivalent:

- (i) S is left regular;
- (ii) $A \subseteq (\Sigma SA^2]$ for each $A \subseteq S$;
- (iii) $a \in (Sa^2]$ for each $a \in S$.

Lemma 38. Let S be an ordered semiring. Then the following statements are equivalent:

- (i) S is right regular;
- (ii) $A \subseteq (\Sigma A^2 S]$ for each $A \subseteq S$;
- (iii) $a \in (a^2 S]$ for each $a \in S$.

Definition 39. Let T be a nonempty subset of an ordered semiring S . Then T is said to be *semiprime* if for any $a \in S$, if $a^2 \in T$, then $a \in T$.

We note that a nonempty subset T of S is semiprime if and only if, for any $\emptyset \neq A \subseteq S$, $A^2 \subseteq T$ implies $A \subseteq T$. Because, if T is semiprime, $\emptyset \neq A \subseteq T$, and $a \in A$ then $a^2 \in T$ and so $a \in T$; that is, $A \subseteq T$.

Theorem 40. Let S be an ordered semiring. Then S is left regular and right regular if and only if every ordered quasi-ideal of S is semiprime.

Proof. Let Q be an ordered quasi-ideal of S . Let A be a nonempty subset of S such that $A^2 \subseteq Q$. Since S is left regular and by Lemma 37, we have $A \subseteq (\Sigma SA^2]$. Since S is right regular and by Lemma 38, we have $A \subseteq (\Sigma A^2 S]$. Hence, $A \subseteq (\Sigma SA^2] \cap (\Sigma A^2 S] \subseteq (\Sigma SQ] \cap (\Sigma QS] \subseteq Q$. Therefore, Q is semiprime.

Conversely, assume that every ordered quasi-ideal of S is semiprime. Let $A \subseteq S$. By Theorem 13, we have $Q(A^2) = (\Sigma A^2 + ((\Sigma SA^2] \cap (\Sigma A^2 S]))$. Since $A^2 \subseteq Q(A^2)$ is semiprime, $A \subseteq Q(A^2) = (\Sigma A^2 + ((\Sigma SA^2] \cap (\Sigma A^2 S]))$. Then we obtain

$$\begin{aligned} A &\subseteq (\Sigma A^2 + ((\Sigma SA^2] \cap (\Sigma A^2 S])) \\ &\subseteq (\Sigma A^2 + (\Sigma SA^2]) \subseteq (\Sigma A^2 + \Sigma SA^2], \\ A &\subseteq (\Sigma A^2 + ((\Sigma SA^2] \cap (\Sigma A^2 S])) \\ &\subseteq (\Sigma A^2 + (\Sigma A^2 S]) \subseteq (\Sigma A^2 + \Sigma A^2 S]. \end{aligned} \quad (26)$$

In case $A \subseteq (\Sigma A^2 + \Sigma SA^2]$, we get

$$\begin{aligned} \Sigma A^2 &\subseteq \Sigma A (\Sigma A^2 + \Sigma SA^2] \subseteq \Sigma (A \Sigma A^2 + A \Sigma SA^2] \\ &\subseteq \Sigma (\Sigma A^3 + \Sigma ASA^2] \subseteq \Sigma (\Sigma A^3 + \Sigma SA^2] \\ &\subseteq (\Sigma (\Sigma A^3) + \Sigma (\Sigma SA^2]) \subseteq (\Sigma A^3 + \Sigma SA^2] \\ &\subseteq (\Sigma SA^2 + \Sigma SA^2] = (\Sigma SA^2] \end{aligned} \quad (27)$$

and so $A \subseteq (\Sigma A^2 + \Sigma SA^2] \subseteq ((\Sigma SA^2] + \Sigma SA^2] \subseteq (\Sigma SA^2]$. By Lemma 37, S is left regular. Similarly, in case $A \subseteq (\Sigma A^2 + \Sigma A^2 S]$, we get $A \subseteq (A^2 S]$. By Lemma 38, S is right regular. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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