Research Article

Homotopic Chain Maps Have Equal s-Homology and d-Homology

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The homotopy of chain maps on preabelian categories is investigated and the equality of standard homologies and $d$-homologies of homotopic chain maps is established. As a special case, if $X$ and $Y$ are the same homotopy type, then their $n$th $d$-homology $R$-modules are isomorphic, and if $X$ is a contractible space, then its $n$th $d$-homology $R$-modules for $n \neq 0$ are trivial.

1. Introduction and Preliminaries

It is known that the homotopic chain maps of abelian groups or more generally of $R$-modules have the same homologies; see [1, 2]. In this paper, the homotopy of chain maps on preabelian categories is investigated and it is proved that homotopic chain maps have the same $s$-homologies and the same $d$-homologies.

To this end, for a pointed category $\mathcal{C}$, following the notation of [3], we recall the following.

(i) For $f : A \to B$, the maps $k_f, \pi_f, p_f : A \to A, k_f, \pi_f, p_f : A \to B, c_f, \pi_f, p_f : B \to C_f$, and $P_f, \pi_f, p_f : C_f \to A$ are, respectively, the kernel, the cokernel, and the kernel pair of $f$; see [4, 5].

(ii) The image $I_f$ of $f$ is the coequalizer of the kernel pair of $f$. In a homological category, $I_f \cong \text{Cer}(k_f)$; see [4].

(iii) For a pair of maps $A \xrightarrow{f,g} B$, the maps $	ext{Equ}(f,g) \xrightarrow{\text{eq}} A$ and $B \xrightarrow{\text{cof}(f,g)} \text{Cer}(f,g)$ are, respectively, the equalizer and the coequalizer of $(f,g)$.

(iv) Given the diagram below in which the squares are commutative and the rows are coequalizers, $i$ is the unique map making the right square commute. Furthermore, $i$ is a regular epi.

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{r} & & \downarrow{1_B} \\
A' & \xrightarrow{g'} & B
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A' & \xrightarrow{f'} & B'
\end{array}
\end{equation}

(v) For a category $\mathcal{C}$ with a zero object, kernels, kernel pairs, and coequalizers of kernel pairs, the arrow category $\mathcal{C}$ of $\mathcal{C}$ has as objects the morphisms of $\mathcal{C}$ and as morphisms from $f : A \to B$ to $f' : A' \to B'$ the pairs $(\alpha, \beta)$ of morphisms of $\mathcal{C}$, making the following square commutative:
And the pair-chain category $\hat{\mathcal{C}}$ of $\mathcal{C}$ has as objects the pair-chains, that is, the composable pairs, $(f, g)$, of morphisms of $\mathcal{C}$, such that $gf = 0$, and as morphisms from $(f, g)$ to $(f', g')$ the triple $(\alpha, \beta, \gamma)$ of morphisms of $\mathcal{C}$, making the following squares commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{a} & & \downarrow{\beta} \\
A' & \xrightarrow{f'} & B'
\end{array} \quad \text{and} \quad 
\begin{array}{ccc}
C & \xrightarrow{g} & \ast \\
\downarrow{\gamma} & & \downarrow{} \\
C' & \xrightarrow{g'} & \ast
\end{array}
$$

Being functor of the following items are investigated and established in [3].

(iii) The kernel functor $K : \hat{\mathcal{C}} \to \mathcal{C}$ takes $(\alpha, \beta) : f \to f'$ to the left vertical arrow in the following commutative diagram:

$$
\begin{array}{ccc}
K_f & \xrightarrow{k_f} & A \\
\downarrow{K(\alpha, \beta)} & & \downarrow{a} \\
K_{f'} & \xrightarrow{k_{f'}} & A'
\end{array}
$$

(vii) The image functor $I : \hat{\mathcal{C}} \to \mathcal{C}$ takes $(\alpha, \beta) : f \to f'$ to the left vertical arrow in the following commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{e_f} & I_f \\
\downarrow{\alpha} & & \downarrow{I(\alpha, \beta)} \\
A' & \xrightarrow{e_{f'}} & I_{f'}
\end{array}
$$

(xii) Let $\mathcal{C} = R\text{mod}$ and $d = rpr_1 + spr_2 = + (r \times s)$ with $r, s \in R$. Let $(f, g)$ be a pair-chain. Then, $R_{fg} = \{[a, b] \in K^2 | ra + sb \in I_f, j^* \text{ is the inclusion},$ and $H_{fg} = K_g(I_{fg} - j_f(R_{fg})) = \{[a] : a \in K_g\}$, where $[a] = [b] | r(a - b) \in (r + s)K_g + I_f$ is the equivalence class under the equivalence relation $a \sim b$ if and only if $\exists m, n \in K_g$ such that $a - b = m - n$ and $rm + sn \in I_f$.

(xiii) As a special case of above example, for $d = +(r \times 1)$ or $d = +(1 \times r)$ with $r \in R$, we have $H_{fg} = K_g((1 + r)K_g + I_f)$.

We call the homology which is defined in [2, 4] the standard homology.

(xiv) The standard homology or $s$-homology functor $H^s$ takes $(f, g) \in \mathcal{C}$ to $\text{Coker}(j_{fg})$, and for a pair-chain map $(\alpha, \beta, \gamma) : (f, g) \to (f', g')$, we have the following commutative diagram:

$$
\begin{array}{ccc}
K_g & \xrightarrow{q} & H_{fg} \\
\downarrow{K(\alpha, \beta)} & & \downarrow{K(\alpha, \beta, \gamma)} \\
K_g' & \xrightarrow{q'} & H_{f'g'}
\end{array}
$$

where $q = \text{coker}(j_{fg})$ and $q' = \text{coker}(j_{f'g'})$.

2. Homotopy

**Definition 1.** Let $\mathcal{C}$ be an additive category. Two morphisms

$$
(f, g) \xrightarrow{(\alpha, \beta, \gamma)} (f', g')
$$

are homotopic if there exists a commutative diagram:}

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{a} & & \downarrow{\beta} \\
A' & \xrightarrow{f'} & B'
\end{array} \quad \text{and} \quad 
\begin{array}{ccc}
C & \xrightarrow{g} & \ast \\
\downarrow{\gamma} & & \downarrow{} \\
C' & \xrightarrow{g'} & \ast
\end{array}
$$

\end{align*}

In a pointed regular (homological, semiabelian, or abelian) category, $f_{fg}$ is monic; see [3, 4].
in $C$ are said to be homotopic whenever there is a pair of morphisms $(h, h')$ in $C$, as in the diagram below, such that $f' h + h' g = \beta - \beta'$; see [2],

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\alpha} & & \downarrow{g} \\
A' & \xrightarrow{f'} & B'
\end{array}
$$

Theorem 2. Let $C$ be a preabelian category. If the maps $(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$ in $C$ are homotopic, then $H^d(\alpha, \beta, \gamma) = H^d(\alpha', \beta', \gamma')$.

Proof. Consider

$$
(k_g j_{f', g'} e_f h + h' g) k_g = (\beta - \beta') k_g,
$$

$$
(k'_g j_{f', g'} e_f h k_g) k_g = (k(\beta, \gamma) - k(\beta', \gamma'))
$$

since $q' j_{f', g'} = 0$ we have $q'(k(\beta, \gamma) - k(\beta', \gamma')) = 0$.

$$
(H^d(\alpha, \beta, \gamma) - H^d(\alpha', \beta', \gamma'))q = 0, H^d(\alpha, \beta, \gamma) = H^d(\alpha', \beta', \gamma')
$$

Lemma 3. Let $C$ be a pointed category with pullbacks and pushouts and let $d$ be a kernel transformation in $C$. There is a natural transformation $p : H^d \to H^d : \check{C} \rightarrow C$. Furthermore, $p$ is pointwise regular epic.

Proof. Let $d : S \circ K \to K$ be a natural transformation. For $(f, g) \in \check{C}$ and $h : I_f \to 0$, $I_f = K_1$. Factoring $f = m e_f$ and using the morphism $(m, 0) : h \to g$, naturality of $d$ yields, $jd_h = d_g j$, and so $d_g(j, 0) = d_g(1, 0) = jd_h(1, 0)$. Setting $r = d_g(1, 0)$, we get $d_g(j, 0) = jr$. So, there exists a unique map $\psi$, such that the triangles in the following diagram commute:

$$
\begin{array}{ccc}
I_f & \xrightarrow{\psi} & K_g \\
\downarrow{\alpha} & & \downarrow{d_g} \\
R_{f g} & \xrightarrow{d_g} & I_f
\end{array}
$$

Therefore, $j = pr_1(j, 0) = pr_1 j^* \psi = j_1 \psi$ and $0 = pr_2(j, 0) = pr_2 j^* \psi = j_2 \psi$ so that $q j = q pr_1 j^* \psi = q pr_2 j^* \psi = q 0 = 0$. Then, there is a unique morphism $p_{fg} : H^g f \to H^g f'$ such that $p_{fg} c_j = q$; that is, the following diagram commutes:

$$
\begin{array}{ccc}
I_f & \xrightarrow{\psi} & K_g \\
\downarrow{\alpha} & & \downarrow{d_g} \\
R_{f g} & \xrightarrow{d_g} & I_f
\end{array}
$$

We have $p_{f', g} H^d(\alpha, \delta, \zeta) c_j = p_{f', g} c_j K(\delta, \zeta) = q_{f'} \psi K(\delta, \zeta)$ and $p_{f', g} H^d(\alpha, \delta, \zeta) p_{fg} c_j = H^d(\alpha, \delta, \zeta) p_{fg} c_j$. Since $c_j$ is epic, $p_{f', g} H^d(\alpha, \delta, \zeta) q_{fg} = H^d(\alpha, \delta, \zeta) p_{fg} c_j$ and the right parallelogram commutes. Then, the following diagram commutes:

$$
\begin{array}{ccc}
K_g & \xrightarrow{q_{fg}} & H^d \\
\downarrow{c_j} & & \downarrow{p_{fg}} \\
H^d(\alpha, \delta, \zeta) & \xrightarrow{\psi} & H^d(\alpha, \delta, \zeta)
\end{array}
$$

and so $p : H^d \to H^d : \check{C} \rightarrow C$ is a natural transformation.

Theorem 4. Let $C$ be a preabelian category. If $(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$ are homotopic, then $H^d(\alpha, \beta, \gamma) = H^d(\alpha', \beta', \gamma')$.
Proof. The equalities $p_{f+g} = q$, $p_{f'} = q'$, $H^d(\alpha, \beta, \gamma)q = q'K(\alpha, \beta)$, and $H^d'(\alpha', \beta', \gamma')q = q'K(\alpha', \beta')$ and the facts that $p$ is regular epic and standard homologies are equal imply $H^d(\alpha, \beta, \gamma) = H^d'(\alpha', \beta', \gamma')$ as desired.

**Definition 5.** Let $\mathcal{C}$ be a pointed category. A chain complex in $\mathcal{C}$ is a differential graded object of $\mathcal{C}$ of degree $-1$ as

$$\cdots \rightarrow C_{n+1} \xrightarrow{\nu_{n+1}} C_n \xrightarrow{\nu_n} C_{n-1} \rightarrow \cdots \tag{16}$$

in which $\nu_n \nu_{n+1} = 0$ for all $n \in \mathbb{Z}$ and a chain map $f_* : C_* \rightarrow D_*$ is a graded map $\{f_n : C_n \rightarrow D_n : n \in \mathbb{Z}\}$ as in the following diagram, in which all the squares commute:

$$\cdots \rightarrow C_{n+1} \xrightarrow{\nu_{n+1}} C_n \xrightarrow{\nu_n} C_{n-1} \xrightarrow{\nu_{n-1}} \cdots \tag{17}$$

$$\cdots \rightarrow D_{n+1} \xrightarrow{\mu_{n+1}} D_n \xrightarrow{\mu_n} D_{n-1} \xrightarrow{\mu_{n-1}} \cdots$$

These chain complexes and chain maps form a category that is denoted by $C_*(\mathcal{C})$.

**Definition 6.** Let $\mathcal{C}$ be an additive category and $C_* \xrightarrow{f_*} D_*$ two maps in $C_*(\mathcal{C})$; one says $f_*$ is chain homotopic to $g_*$ if there is a morphism $h_* : C_* \rightarrow D_*$ of degree $1$ as

$$\cdots \rightarrow C_{n+1} \xrightarrow{\nu_{n+1}} C_n \xrightarrow{\nu_n} C_{n-1} \xrightarrow{\nu_{n-1}} \cdots \tag{18}$$

$$\cdots \rightarrow D_{n+1} \xrightarrow{\mu_{n+1}} D_n \xrightarrow{\mu_n} D_{n-1} \xrightarrow{\mu_{n-1}} \cdots$$

such that $\mu_n h_n + h_{n-1} \nu_n = f_n - g_n$.

**Corollary 7.** Let $\mathcal{C}$ be a preabelian category and let $f_*$ and $g_*$ be homotopic in $C_*(\mathcal{C})$. Then, $H^d(f_*) = H^d(g_*)$ and $H^d'(f_*) = H^d'(g_*)$.

By the above theorems, $H^d(f_*) = H^d(g_*)$ and $H^d'(f_*) = H^d'(g_*)$, where $H_n(f) = H(f_{n+1}, f_n, f_{n-1})$.

Let $\text{Top}$, $\text{Set}^{\Delta^\text{op}}$, $\text{Rmod}^{\Delta^\text{op}}$, $\text{C}_* \text{Rmod}$, and $g\text{Rmod}$ be, respectively, categories of topological spaces, simplicial sets, simplicial $R$-modules, chain complexes of $R$-modules, and graded $R$-modules, and let $S_{\Delta^\text{op}} : \text{Top} \rightarrow \text{Set}^{\Delta^\text{op}}$, $\Delta^{\text{op}} : \text{Set}^{\Delta^\text{op}} \rightarrow \text{Rmod}^{\Delta^\text{op}}$, $\text{C}_* \text{Rmod} \rightarrow g\text{Rmod}$ be, respectively, singular functor, induced functor by free generator functor $F : \text{Set} \rightarrow \text{Rmod}$, chain complexes generator functor, and $d$-homology functor. If $X \in \text{Top}$, then the singular $d$-homology of $X$ is $H^d_*(X) = H^d_* o C_* o F^{\Delta^\text{op}} o S_{\Delta^\text{op}}(X)$. $R$ is PID.

**Example 8.** If $X \xrightarrow{f} Y$ is homotopic continuous maps, then $H^d(f) = H^d_*(g) : H^d_*(X) \rightarrow H^d_*(Y)$.

**Example 9.** Let $1$ be a terminal object in $\text{Top}$. Since $H^d_*(1) = 0$ for $n \neq 0$ and the natural transformation $p : H^d \rightarrow H^d : \overline{\mathcal{C}} \rightarrow \mathcal{C}$ is pointwise regular epic, $H^d_*(1) = 0$ for $n \neq 0$ and $H^d_0(1) = R/(j_1 - j_2)(R_{00})$ in which $R_{00} = Ker(d_0)$.

Let $\text{SimC}_x$ be simplicial complexes and let $S_* : \text{SimC}_x \rightarrow \text{Set}^{\Delta^\text{op}}$ be the simplicial functor. If $(K, S) \in \text{SimC}_x$, then the simplicial $d$-homology of $(K, S)$ is $H^d_*(X) = H^d_* o C_* o F^{\Delta^\text{op}} o S_*(K, S)$. 


Example 10. Let $(1, 1)$ be a terminal object in $\text{SimCx}$. Then, $H_n^1(1, 1) = 0$ for $n \neq 0$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References
