Research Article

Bessel Equation in the Semiunbounded Interval $x \in [x_0, \infty]$: Solving in the Neighbourhood of an Irregular Singular Point

Qing-Hua Zhang,¹ Jian Ma,² and Yuanyuan Qu¹

¹The First Institute of Oceanography, State Oceanic Administration, Qingdao 266061, China

²College of Marine Sciences, Shanghai Ocean University, Shanghai 201306, China

Correspondence should be addressed to Jian Ma; majian@shou.edu.cn

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This study expresses the solution of the Bessel equation in the neighbourhood of $x = \infty$ as the product of a known-form singular divisor and a specific nonsingular function, which satisfies the corresponding derived equation. Considering the failure of the traditional irregular solution constructed with the power series, we adopt the corrected Fourier series with only limited smooth degree to approximate the nonsingular function in the interval $[x_0, \infty]$. In order to guarantee the series' uniform convergence and uniform approximation to the derived equation, we introduce constraint and compatibility conditions and hence completely determine all undetermined coefficients of the corrected Fourier series. Thus, what we found is not an asymptotic solution at $x \rightarrow \infty$ (not to mention a so-called formal solution), but a solution in the interval $[x_0, \infty]$ with certain regularities of distribution. During the solution procedure, there is no limitation on the coefficient property of the equation; that is, the coefficients of the equation can be any complex constant, so that the solution method presented here is universal.

1. Introduction

The general form of the Bessel equation is

$$x^2 y''(x) + \lambda_1 x y'(x) + (\lambda_2 x^2 + \lambda_3) y(x) = 0, \quad (1)$$

where λ_j ($j = 1, 2, 3$) are arbitrary complex constants, with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, and $x \in [0, \infty]$.

The Bessel equation only has two singular points on the real axis: $x = 0$ and $x = \infty$. For convenience, set division point at $x_0 > 0$, to separate the entire solution interval into two parts, $[0, x_0]$ and $[x_0, \infty]$, the latter of which is the semiunbounded interval that this study will solve the equation in. It is easy to verify that $x = 0$ is a regular singular point, but $x = \infty$ is irregular. Previous studies showed that the solution in the neighbourhood of the isolated singular point could be written as a combination of a known-form singular divisor and a specific nonsingular function. To find the solution in the neighbourhood of the regular singular point, one often uses the power series extension to reconstruct the nonsingular function. In the neighbourhood of the irregular singular point, attempts have been made with

power series traditionally, but already existing efforts cannot find a uniformly convergent solution, which can only give an asymptotic solution at $x \rightarrow \infty$. This so-called regular formal solution can only indicate the variation tendency of the solution [1–4].

However, the failure to obtain a solution with certain regularities of distribution using power series method does not necessarily mean inexistence of this kind of solution. As is known, the power series with uniform convergence in the complex number space converges to a complex analytic function, so that it is full differentiable. Yet, the Bessel equation is only a second-order differential equation, which demands limited smoothness (e.g., second-order derivative) only, but not full smoothness. Overdemanded smoothness could be the reason for the inability to find the solution with the uniformly convergent power series.

This study adopts the corrected Fourier series [5–10] with limited smoothness to construct the nonsingular function and insert it into the derived equations from the Bessel equation. By introducing the constraint and compatibility conditions at the same time, all coefficients of the corrected

Fourier series can be determined completely. Thus, one can obtain the solution with certain regularities of distribution in the interval $x \in [x_0, \infty]$, but not an asymptotic or formal solution. There is no limitation on the equation parameter $\{\lambda_j\}$ in the whole solution procedure, which could be any arbitrary complex constant, so that the solution method given in this study is universal.

2. The Solution of the Bessel Equation in the Semiunbounded Interval $[x_0, \infty]$

2.1. Obtaining the Derived Equation. For convenience, we introduce coordinate transformation

$$y(x) = y(x_0 \xi^{-1}) = y_e(\xi), \quad \xi = \frac{x_0}{x}, \quad (2)$$

which transforms the semiunbounded interval $x \in [x_0, \infty]$ to a bounded one $\xi \in [0, 1]$. Equation (1) turns into the following form with the new coordinate (Appendix A):

$$\xi^4 y_e''(\xi) + (2 - \lambda_1) \xi^3 y_e'(\xi) + (\lambda_{20} + \lambda_3 \xi^2) y_e(\xi) = 0, \quad (3)$$

$$\xi \in [0, 1], \quad \lambda_{20} = \lambda_2 x_0^2.$$

It is easy to find that $\xi = 0$ is an irregular singular point of the equation, and then the irregular solution can be written as

$$y_e(\xi) = R(\xi) V(\xi), \quad (4)$$

$$R(\xi) = \xi^\rho e^{\sigma \xi^{-1}} = \xi^\rho \exp(\sigma \xi^{-1}),$$

where ρ, σ in $R(\xi)$ are constants to be determined.

$V(\xi)$ is a nonsingular function with $V(0) \neq 0$, which satisfies the following equation:

$$\xi^4 V''(\xi) + P_1(\xi) V'(\xi) + P_2(\xi) V(\xi) = 0, \quad (5)$$

where

$$P_1(\xi) = 2\xi^4 \frac{R'}{R} + (2 - \lambda_1) \xi^3, \quad (6)$$

$$P_2(\xi) = \xi^4 \frac{R''}{R} + (2 - \lambda_1) \xi^3 \frac{R'}{R} + \lambda_{20} + \lambda_3 \xi^2.$$

Using the differential relationship,

$$\frac{R'}{R} = \rho \xi^{-1} - \sigma \xi^{-2}, \quad (7)$$

$$\frac{R''}{R} = \rho(\rho - 1) \xi^{-2} + (2 - 2\rho) \sigma \xi^{-3} + \sigma^2 \xi^{-4},$$

one can obtain

$$P_1(\xi) = (q_{11} + q_{12} \xi) \xi^2, \quad (8)$$

$$P_2(\xi) = q_{21} + q_{22} \xi + q_{23} \xi^2.$$

Hence, (5) can be written as the following form:

$$L\langle V \rangle \equiv \xi^4 V'' + (q_{11} + q_{12} \xi) \xi^2 V' + (q_{21} + q_{22} \xi + q_{23} \xi^2) V = 0, \quad (9)$$

where

$$q_{11} = -2\sigma,$$

$$q_{12} = 2 + 2\rho - \lambda_1,$$

$$q_{21} = \sigma^2 + \lambda_{20}, \quad (10)$$

$$q_{22} = (\lambda_1 - 2\rho) \sigma,$$

$$q_{23} = \rho(\rho + 1 - \lambda_1) + \lambda_3.$$

Let $\xi \rightarrow 0$, and, from

$$L\{V\} = 0, \quad (11)$$

$$[L\{V\}]' = 0,$$

one can obtain

$$q_{21} = q_{22} = 0; \quad (12)$$

that is,

$$\sigma^2 + \lambda_{20} = 0, \quad (13)$$

$$\lambda_1 - 2\rho = 0,$$

which determines that

$$\sigma = \begin{cases} \sigma_1 = \sigma_0, & \sigma_0 = \sqrt{-\lambda_{20}} = \sqrt{-\lambda_2} x_0, \\ \sigma_2 = -\sigma_0, & \sigma_0 = \sqrt{-\lambda_{20}} = \sqrt{-\lambda_2} x_0, \end{cases} \quad (14a)$$

$$\rho = \frac{\lambda_1}{2}, \quad (14b)$$

where

$$q_{11} = -2\sigma_1,$$

$$q_{12} = 2, \quad (15)$$

$$q_{23} = \frac{\lambda_1}{2} \left[1 - \frac{\lambda_1}{2} \right] + \lambda_3.$$

So, the derived equation is obtained:

$$L_1\{V, \sigma\} \equiv \xi^2 V'' + q_1(\xi, \sigma) V' + q_0 V = 0, \quad (16)$$

where

$$q_1(\xi, \sigma) = 2(\xi - \sigma), \quad (17)$$

$$q_0 = q_{23}.$$

2.2. Nonsingular Classic Solution of the Derived Solution. The form of the derived equation (see (16)) still seems to be a singular equation, but it can be guaranteed to have nonsingular classic solution after introducing certain constraint conditions (Appendix B). That is,

$$\frac{\partial}{\partial \xi} [L_1\{V, \sigma\}]_{\xi=0} = 0; \quad (18a)$$

that is,

$$-2\sigma V''(0, \sigma) + (2 + q_0) V'(0, \sigma) = 0. \quad (18b)$$

Because $\|V''(0, \sigma)\| < \infty$, $V(\xi)$ can be expanded to the following corrected Fourier series:

$$V(\xi) = \sum_{|n| \leq N} A_n \exp(i\alpha_n \xi) + \sum_{l=1}^3 a_l \frac{\xi^l}{l!},$$

$$V^{(k)}(\xi) = \sum_{|n| \leq N} A_n (i\alpha_n)^k \exp(i\alpha_n \xi) + \sum_{l=k}^3 a_l \frac{\xi^{l-k}}{(l-k)!}, \quad (19)$$

$$k = 1, 2,$$

which is uniformly convergent and where $\alpha_n = 2n\pi$. At the same time, we introduce the Fourier projection:

$$F^{-1} \langle \rangle_n = \int_0^1 \langle \rangle \exp(-i\alpha_n \xi) d\xi. \quad (20)$$

Inserting expansion equation (19) to the derived equation (see (16)), one can obtain (Appendix C)

$$\sum_n A_n GR(n, \xi, \sigma) \exp(i\alpha_n \xi) + \sum_{l=1}^3 a_l GS(l, \xi, \sigma) = 0. \quad (21)$$

After the Fourier projection,

$$\sum_{|n| \leq N} A_n R(n, n_0, \sigma) = \sum_{l=1}^3 a_l S(l, n_0, \sigma), \quad |n_0| \leq N, \quad (22)$$

where

$$R(n, n_0, \sigma) = F^{-1} \langle GR(n, \xi, \sigma) \exp(i\alpha_n \xi) \rangle_{n_0}$$

$$= F^{-1} \langle GR(n, \xi, \sigma) \rangle_{n_0 - n}, \quad (23)$$

$$S(l, n_0, \sigma) = -F^{-1} \langle GS(l, \xi, \sigma) \rangle_{n_0}.$$

It can be derived that

$$A_n = \sum_{l=1}^3 a_l A_e(l, n, \sigma), \quad (24)$$

where $A_e(l, n, \sigma)$ is the solution of the following equation:

$$\sum_n A_e(l, n, \sigma) R(n, n_0, \sigma) = S(l, n_0, \sigma). \quad (25)$$

Inserting solution (24) into extension (19), one can obtain

$$V(\xi, \sigma) = \sum_{l=1}^3 a_l Z_0(l, \xi, \sigma),$$

$$V^{(k)}(\xi, \sigma) = \sum_{l=1}^3 a_l Z_k(l, \xi, \sigma), \quad k = 1, 2, \quad (26)$$

where

$$Z_0(l, \xi, \sigma) = \sum_n A_e(l, n, \sigma) \exp(i\alpha_n \xi) + \frac{\xi^l}{l!}, \quad (27)$$

$$Z_k(l, \xi, \sigma) = Z_0^{(k)}(l, \xi, \sigma), \quad k = 1, 2.$$

The compatibility condition is required [5, 6] in Fourier projection of (16) (Appendix B):

$$L_1 \{V(\xi, \sigma)\}_0^1 = 0; \quad (28a)$$

that is,

$$V''(1, \sigma) + q_1(1, \sigma) V'(1, \sigma) - q_1(0, \sigma) V'(0, \sigma) + q_0[V(1, \sigma) - V(0, \sigma)] = 0. \quad (28b)$$

Insert solution (26) into constraint (18a) and (18b) and consistency (28a) and (28b) conditions to obtain the equation determining $\{a_l\}$ (Appendix D):

$$\sum_{l=1}^3 a_l \beta_{1l}(\sigma) = 0,$$

$$\sum_{l=1}^3 a_l \beta_{2l}(\sigma) = 0. \quad (29)$$

It can be solved that

$$a_l = \gamma(l, \sigma); \quad (30)$$

hence,

$$V(\xi, \sigma) = \sum_{l=1}^3 \gamma(l, \sigma) Z_0(l, \xi, \sigma). \quad (31)$$

Note that

$$V_1(\xi) = V(\xi, \sigma_1) = V(\xi, \sigma_0),$$

$$V_2(\xi) = V(\xi, \sigma_2) = V(\xi, -\sigma_0). \quad (32)$$

Finally, the solution of (3) is obtained:

$$\gamma_{e1}(\xi) = \xi^{\lambda_1/2} \exp(\sigma_0 \xi^{-1}) V_1(\xi),$$

$$\gamma_{e2}(\xi) = \xi^{\lambda_1/2} \exp(-\sigma_0 \xi^{-1}) V_2(\xi). \quad (33)$$

3. Examples and Discussion

As mentioned before, the solution method given here satisfies the equation with each parameter being any arbitrary complex constant, so that the first set of examples we will show are to provide the results of the functions $V_1(\xi)$ and $V_2(\xi)$ according to several sets of different complex constants. Their accuracy of approximation to satisfy the derived equation will also be estimated. The second set of examples aim to solve $V_1(\xi)$, $V_2(\xi)$ and, in turn, the solution of the Bessel equation, $\gamma_{e1}(\xi)$ and $\gamma_{e2}(\xi)$, with several sets of certain common real constants. Noting their linear combination as $y_1(x)$ and $y_2(x)$, we will compare them with already known Bessel functions and modified Bessel functions.

TABLE 1: Solution and relative error for Example 1.

ξ	$V_1(\xi)$	$V_2(\xi)$	Error($\xi, 1$)	Error($\xi, 2$)
0	(-1.3512, -2.7890)	(2.6727, 1.6044)	5.76×10^{-7}	1.70×10^{-7}
1/5	(-1.3573, -2.7902)	(2.6668, 1.6066)	1.47×10^{-4}	1.47×10^{-4}
2/5	(-1.3635, -2.7913)	(2.6608, 1.6086)	8.72×10^{-5}	8.75×10^{-5}
3/5	(-1.3698, -2.7923)	(2.6549, 1.6106)	1.19×10^{-4}	1.19×10^{-4}
4/5	(-1.3760, -2.7932)	(2.6488, 1.6124)	1.74×10^{-4}	1.75×10^{-4}
1	(-1.3822, -2.7939)	(2.6428, 1.6141)	6.43×10^{-7}	3.80×10^{-7}

TABLE 2: Solution and relative error for Example 2.

ξ	$V_1(\xi)$	$V_2(\xi)$	Error($\xi, 1$)	Error($\xi, 2$)
0	(2.6352, -0.3074)	(-1.4742, 1.3684)	1.65×10^{-7}	9.26×10^{-8}
1/5	(2.6327, -0.3118)	(-1.4777, 1.3668)	1.92×10^{-4}	2.83×10^{-4}
2/5	(2.6301, -0.3160)	(-1.4812, 1.3650)	9.90×10^{-5}	1.37×10^{-4}
3/5	(2.6275, -0.3202)	(-1.4846, 1.3632)	2.05×10^{-4}	1.99×10^{-4}
4/5	(2.6248, -0.3242)	(-1.4880, 1.3612)	2.82×10^{-4}	3.28×10^{-4}
1	(2.6220, -0.3281)	(-1.4914, 1.3592)	9.37×10^{-7}	1.90×10^{-7}

3.1. First Set of Examples

Example 1. Let $\lambda_1 = (1 + i)/2$, $\lambda_2 = 1$, $\lambda_3 = -\nu^2$ ($\nu = 0.3$), $x_0 = 10.0$, and $N = 50$, and calculate $V_1(\xi)$ and $V_2(\xi)$. To examine the accuracy of the solution to approximate the derived equation, note

$$\begin{aligned} g_1(\xi, j) &= \xi^2 V_j''(\xi), \\ g_2(\xi, j) &= q_1(\xi, \sigma_j) V_j'(\xi), \\ g_3(\xi, j) &= q_0 V_j(\xi), \\ g_0(\xi, j) &= g_1(\xi, j) + g_2(\xi, j) + g_3(\xi, j), \end{aligned} \quad (34)$$

with complex module

$$E_l(\xi, j) = \|g_l(\xi, j)\|, \quad l = 0, 1, 2, 3. \quad (35)$$

And then, we can define the relative accuracy of the solution $V_j(\xi)$ to satisfy the equation

$$\text{Error}(\xi, j) = \frac{E_0(\xi, j)}{\text{Max}\{E_1(\xi, j), E_2(\xi, j), E_3(\xi, j)\}}. \quad (36)$$

Values of the solution and relative error are shown in Table 1.

Example 2. Let $\lambda_1 = 1$, $\lambda_2 = (1 + i)/2$, $\lambda_3 = -\nu^2$ ($\nu = 0.3$), $x_0 = 10.0$, and $N = 50$, and the results are presented in Table 2.

Example 3. Let $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = (-\nu^2 + i)/2$ ($\nu = 0.3$), $x_0 = 10.0$, and $N = 50$, and the results are in Table 3.

From the above three examples, it is easy to see that functions $V_1(\xi)$ and $V_2(\xi)$ vary very slowly with ξ ; thus,

TABLE 3: Solution and relative error for Example 3.

ξ	$V_1(\xi)$	$V_2(\xi)$	Error($\xi, 1$)	Error($\xi, 2$)
0	(-1.0161, -0.4045)	(1.0966, -0.2249)	7.67×10^{-7}	1.11×10^{-6}
1/5	(-1.0220, -0.4044)	(1.0916, -0.2216)	1.57×10^{-4}	1.56×10^{-4}
2/5	(-1.0280, -0.4041)	(1.0865, -0.2185)	9.41×10^{-5}	9.44×10^{-5}
3/5	(-1.0339, -0.4037)	(1.0814, -0.2154)	1.27×10^{-4}	1.26×10^{-4}
4/5	(-1.0399, -0.4032)	(1.0762, -0.2125)	1.83×10^{-4}	1.85×10^{-4}
1	(-1.0458, -0.4025)	(1.0711, -0.2097)	7.30×10^{-7}	1.87×10^{-6}

it is hard to imagine to approximate them with uniform convergence by using the power series. On the other hand, when letting $x_0 = 10.0$ and $N = 50$, they satisfy the derived equation at relative errors smaller than 4×10^{-4} , meaning accuracy is significant enough.

3.2. Second Set of Examples

Example 1. To compare with Bessel function of the first kind, let $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -\nu^2$ ($\nu = 0.3$, $2\nu \neq \text{integer}$), $x_0 = 20.0$, and $N = 200$. For these parameters, the Bessel equation has solutions of the Bessel function of the first kind, $J_\nu(x)$ and $J_{-\nu}(x)$, in the neighbourhood of $x = 0$, which has infinite points of zero on the real axis. And when $x \gg 1$, they have the following Asymptotic Formula to determine the distribution of the zero points:

$$\begin{aligned} x_1(m) &= \left(m + \frac{\nu}{2} - \frac{1}{4}\right)\pi, \\ x_2(m) &= \left(m - \frac{\nu}{2} - \frac{1}{4}\right)\pi. \end{aligned} \quad (37)$$

The solutions obtained in this study $y_{e1}(\xi)$ and $y_{e2}(\xi)$ are conjugate complex ($\because \sigma_0 = i\sqrt{\lambda_2}x_0$; Appendix E); however, real solution compatible with $J_\nu(x)$ and $J_{-\nu}(x)$ can be obtained by linear combination of certain form (Appendix F):

$$\begin{aligned} y_1(x) &= b_{11}y_{e1}(\xi) + b_{12}y_{e2}(\xi), \\ y_2(x) &= b_{21}y_{e1}(\xi) + b_{22}y_{e2}(\xi), \end{aligned} \quad (38)$$

with the zero points noted as $\tilde{x}_1(m)$ and $\tilde{x}_2(m)$.

Table 4 compares the zero points $x_1(m)$ of $J_\nu(x)$ and $\tilde{x}_1(m)$ of $y_1(x)$ and the zero points $x_2(m)$ of $J_{-\nu}(x)$ and $\tilde{x}_2(m)$ of $y_2(x)$. It is easy to find that the solution obtained here (after rearranging) has consistent distribution of zero points with Bessel function of the first kind, indicating the credibility of the solution method provided by the present study [1, 2].

Example 2. Let $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 0$, $x_0 = 3.0$, and $N = 200$, which represent the Bessel equation with modified Bessel functions $I_0(x)$ and $K_0(x)$ in the neighbourhood of $x = 0$. With this set of parameters, the solutions obtained by this paper are $y_{e1}(\xi)$ and $y_{e2}(\xi)$ in the interval $x \in [x_0, \infty]$ or

TABLE 4: Comparisons of the zero points of the Bessel function in Example 1 between the asymptotic formula and the corrected Fourier series in this study.

m	$x_1(m)$	$\tilde{x}_1(m)$	$x_2(m)$	$\tilde{x}_2(m)$
7	21.6770	21.68	20.7345	20.73
8	24.8186	24.82	23.8761	23.87
9	27.9602	27.96	27.0177	27.01
10	31.1018	31.10	30.1593	30.16
11	34.2433	34.24	33.3009	33.30
12	37.3849	37.38	36.4425	36.44
13	40.5265	40.52	39.5841	39.58
14	43.6681	43.67	42.7257	42.72
15	46.8091	46.81	45.8672	45.86
16	49.9513	49.95	49.0088	49.00
17	53.0929	53.09	52.1504	52.15
18	56.2345	56.23	55.2920	55.29
19	59.3761	59.37	58.4336	58.43
20	62.5177	62.51	61.5752	61.57

Relative errors:

$$\text{Error}_1(m) = |(\tilde{x}_1(m) - x_1(m))/x_1(m)| < 1.40 \times 10^{-4},$$

$$\text{Error}_2(m) = |(\tilde{x}_2(m) - x_2(m))/x_2(m)| < 2.55 \times 10^{-4}.$$

$\xi \in [0, 1]$, with their linear combination also being solution of the equation

$$\begin{aligned} y_1(x) &= b_{11}y_{e1}(\xi) + b_{12}y_{e2}(\xi), \\ y_2(x) &= b_{21}y_{e1}(\xi) + b_{22}y_{e2}(\xi). \end{aligned} \quad (39)$$

From the table of Bessel functions, one can look up the values of functions $I_0(x)$ and $K_0(x)$ in the interval $x \in [0, 10]$. At the two arbitrarily selected points in the interval $x \in [3, 10]$, we linearly fit $y_1(x)$ and $I_0(x)$ and $y_2(x)$ and $K_0(x)$, to determine constant b_{ij} . The obtained distributions of functions $y_1(x)$ and $y_2(x)$ are compared with those of functions $I_0(x)$ and $K_0(x)$ in Table 5, which shows apparent consistency between the solution obtained with known modified Bessel functions and that calculated in this study.

Example 3. Let $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -1$, $x_0 = 3.0$, and $N = 200$, and this set of parameters corresponds to Bessel equation with modified Bessel functions $I_1(x)$ and $K_1(x)$ as the solution in the neighbourhood of $x = 0$. The corresponding solution obtained here is $y_{e1}(\xi)$ and $y_{e2}(\xi)$, whose linear combination can construct new solution functions, after linear fit with functions $I_1(x)$ and $K_1(x)$:

$$\begin{aligned} y_1(x) &= b_{11}y_{e1}(\xi) + b_{12}y_{e2}(\xi), \\ y_2(x) &= b_{21}y_{e1}(\xi) + b_{22}y_{e2}(\xi). \end{aligned} \quad (40)$$

Comparisons between their distributions and those of $I_1(x)$ and $K_1(x)$ are illustrated in Table 6. It can be easily seen that the solution obtained here is completely consistent with the known solution with modified Bessel functions [1, 2].

TABLE 5: Comparisons between the solution of Example 2 obtained with known modified Bessel functions and that calculated in this study.

x	$y_1(x)$	$I_0(x)$	$y_2(x)$	$K_0(x)$
3.0	4.881	4.881	3.474×10^{-2}	3.474×10^{-2}
3.5	7.384	7.378	1.960×10^{-2}	1.960×10^{-2}
4.0	11.31	11.30	1.116×10^{-2}	1.116×10^{-2}
4.5	17.49	17.48	6.400×10^{-3}	6.400×10^{-3}
5.0	27.26	27.24	3.691×10^{-3}	3.691×10^{-3}
5.5	42.71	42.69	2.139×10^{-3}	2.139×10^{-3}
6.0	67.26	67.23	1.244×10^{-3}	1.244×10^{-3}
6.5	106.3	106.3	7.259×10^{-4}	7.259×10^{-4}
7.0	168.6	168.6	4.248×10^{-4}	4.248×10^{-4}
7.5	268.3	268.2	2.492×10^{-4}	2.492×10^{-4}
8.0	427.7	427.6	1.465×10^{-4}	1.465×10^{-4}
8.5	683.4	683.2	8.626×10^{-5}	8.626×10^{-5}
9.0	1094	1094	5.088×10^{-5}	5.088×10^{-5}
9.5	1754	1753	3.006×10^{-5}	3.006×10^{-5}
9.9	2562	2561	1.974×10^{-5}	1.975×10^{-5}

Relative errors:

$$\text{Error}_1(x) = |(y_1(x) - I_0(x))/I_0(x)| < 0.88 \times 10^{-3},$$

$$\text{Error}_2(x) = |(y_2(x) - K_0(x))/K_0(x)| < 0.50 \times 10^{-3}.$$

TABLE 6: Comparisons between the solution of Example 3 obtained with known modified Bessel functions and that calculated in this study.

x	$y_1(x)$	$I_1(x)$	$y_2(x)$	$K_1(x)$
3.0	3.953	3.953	4.016×10^{-2}	4.016×10^{-2}
3.5	6.203	6.206	2.224×10^{-2}	2.224×10^{-2}
4.0	9.755	9.759	1.248×10^{-2}	1.248×10^{-2}
4.5	15.39	15.39	7.079×10^{-3}	7.078×10^{-3}
5.0	24.33	24.34	4.045×10^{-3}	4.045×10^{-3}
5.5	38.59	38.59	2.326×10^{-3}	2.326×10^{-3}
6.0	61.35	61.34	1.344×10^{-3}	1.344×10^{-3}
6.5	97.75	97.74	7.799×10^{-4}	7.799×10^{-4}
7.0	156.1	156.0	4.542×10^{-4}	4.542×10^{-4}
7.5	249.6	249.6	2.653×10^{-4}	2.653×10^{-4}
8.0	399.9	399.9	1.554×10^{-4}	1.554×10^{-4}
8.5	641.7	641.6	9.120×10^{-5}	9.120×10^{-5}
9.0	1031	1031	5.364×10^{-5}	5.364×10^{-5}
9.5	1659	1658	3.160×10^{-5}	3.160×10^{-5}
9.9	2428	2428	2.072×10^{-5}	2.072×10^{-5}

Relative errors:

$$\text{Error}_1(x) = |(y_1(x) - I_1(x))/I_1(x)| < 0.5 \times 10^{-3},$$

$$\text{Error}_2(x) = |(y_2(x) - K_1(x))/K_1(x)| < 1.5 \times 10^{-4}.$$

4. Conclusions

The present study proves that the solution of the Bessel equation in the interval with irregular singular point can be expressed as the product of a singular divisor and a non-singular function, the latter of which can be approximated with uniform convergence by corrected Fourier series with limited smoothness in the interval. Thus, we are able to obtain

the solution with certain regularities of distribution of the Bessel equation in the semiunbounded interval with irregular singular point, but no more an asymptotic solution at $x \rightarrow \infty$, or a regular formal solution. This indicates that it is incorrect to conclude existence of only asymptotic or regular formal solution in the neighbourhood of an irregular singular point only because of the inadequacy of certain solution method.

Appendix

A. Coordinate Transformation

There is a relationship between the old and new coordinates as follows:

$$\begin{aligned}\xi' &= \left(\frac{x_0}{x}\right)' = \frac{-1}{x_0} \xi^2, \\ \xi'' &= -\frac{2}{x_0} \xi \xi' = \frac{2}{x_0^2} \xi^3, \\ y'(x) &= y_e'(\xi) \xi' = -\frac{1}{x_0} \xi^2 y_e'(\xi), \\ y''(x) &= y_e''(\xi) \xi'^2 + y_e'(\xi) \xi'' \\ &= \frac{1}{x_0^2} \xi^4 y_e''(\xi) + \frac{2}{x_0^2} \xi^3 y_e'(\xi).\end{aligned}\quad (\text{A.1})$$

Inserting these relations into (1), one can derive (3) using the new coordinate after rearrangements.

B. Constraint and Compatibility

Based on Appendix B of [4], although the differential equation $L_1\{V(\xi)\} = f(\xi)$ still has a form of singular equation, its nonsingular solution can exist with certain constraint condition; that is,

$$\frac{\partial}{\partial \xi} [L_1\{V(\xi)\}]_{\xi=0} = f'(\xi)|_{\xi=0}. \quad (\text{B.1})$$

On the other hand, the classic solution of the equation not only demands the existence of its Fourier projected equation

$$F^{-1} \langle L_1\{V\} \rangle_n = F^{-1} \langle f(\xi) \rangle_n, \quad (\text{B.2})$$

but also requires the following compatibility condition:

$$L_1\{V\}|_0^1 = f(\xi)|_0^1. \quad (\text{B.3})$$

C. Parameters for (21) and (22)

Consider

$$GR(n, \xi, \sigma) = 2(\alpha_n)^2 \frac{\xi^2}{2!} + 2i\alpha_n \xi + (q_0 - 2i\alpha_n \sigma),$$

$$GS(l, \xi, \sigma) = (l^2 + l + q_0) \frac{\xi^l}{l!} - 2\sigma \frac{\xi^{l-1}}{(l-1)!},$$

$$\begin{aligned}R(n, n_0, \sigma) &= 2(\alpha_n)^2 I_{e0}(2, n_0 - n) + 2i\alpha_n I_{e0}(1, n_0 - n) \\ &\quad + (q_0 - 2i\alpha_n \sigma) \delta(n_0 - n), \\ S(l, n_0, \sigma) &= -\left\{ (l^2 + l + q_0) I_{e0}(l, n_0) - 2\sigma I_{e0}(l-1, n_0) \right\},\end{aligned}\quad (\text{C.1})$$

where $I_{e0}(l, n) = F^{-1} \langle \xi^l / l! \rangle_n$.

D. Determining Coefficients of (26)

In (29),

$$\begin{aligned}\beta_{1l}(\sigma) &= -2\sigma Z_2(l, 0, \sigma) + (2 + q_0) Z_1(l, 0, \sigma), \\ \beta_{2l}(\sigma) &= [\xi^2 Z_2(l, \xi, \sigma) + q_1(\xi, \sigma) Z_1(l, \xi, \sigma) \\ &\quad + q_0 Z_0(l, \xi, \sigma)]|_0^1.\end{aligned}\quad (\text{D.1})$$

Let $a_3 = 1$, which can derive

$$\begin{aligned}a_1 \beta_{11}(\sigma) + a_2 \beta_{12}(\sigma) &= -\beta_{13}(\sigma), \\ a_1 \beta_{21}(\sigma) + a_2 \beta_{22}(\sigma) &= -\beta_{23}(\sigma).\end{aligned}\quad (\text{D.2})$$

It can be solved that $a_l = \gamma(l, \sigma)$, where

$$\begin{aligned}\gamma(1, \sigma) &= \begin{vmatrix} -\beta_{13}(\sigma) & \beta_{12}(\sigma) \\ -\beta_{23}(\sigma) & \beta_{22}(\sigma) \end{vmatrix} \div \Delta, \\ \gamma(2, \sigma) &= \begin{vmatrix} \beta_{11}(\sigma) & -\beta_{13}(\sigma) \\ \beta_{21}(\sigma) & -\beta_{23}(\sigma) \end{vmatrix} \div \Delta, \\ \Delta &= \begin{vmatrix} \beta_{11}(\sigma) & \beta_{12}(\sigma) \\ \beta_{21}(\sigma) & \beta_{22}(\sigma) \end{vmatrix};\end{aligned}\quad (\text{D.3})$$

at the same time, let $\gamma(3, \sigma) = 1$.

E. Proof of Conjugate Relation

Because parameters $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -\nu^2$ are all real numbers,

$$\begin{aligned}\sigma_0 &= \sqrt{-\lambda_2} x_0 = ix_0, \\ \sigma_0^* &= -ix_0 = -\sigma_0,\end{aligned}\quad (\text{E.1})$$

and the parameters in (16) satisfy the following relation:

$$\begin{aligned}[q_1(\xi, \sigma)]^* &= q_1(\xi, -\sigma), \\ [q_0]^* &= q_0.\end{aligned}\quad (\text{E.2})$$

It is known that $V_1(\xi) = V(\xi, \sigma_0)$ and $V_2(\xi) = V(\xi, -\sigma_0)$ satisfy (16), noted as

$$\xi^2 V_1'' + q_1(\xi, \sigma_0) V_1' + q_0 V_1 = 0, \quad (\text{E.3})$$

$$\xi^2 V_2'' + q_1(\xi, -\sigma_0) V_2' + q_0 V_2 = 0. \quad (\text{E.4})$$

Solving the conjugate complex for (E.4),

$$\begin{aligned} 0 &= \left[\xi^2 V_2'' + q_1(\xi, -\sigma_0) V_2' + q_0 V_2 \right]^* \\ &= \xi^2 \left[V_2'' \right]^* + q_1^*(\xi, -\sigma_0) \left[V_2' \right]^* + q_0^* \left[V_2 \right]^* \\ &= \xi^2 (V_2^*)'' + q_1(\xi, \sigma_0) (V_2^*)' + q_0 V_2^*. \end{aligned} \quad (\text{E.5})$$

Comparing (E.5) and (E.3), a conjugate relation is obtained:

$$V_2^*(\xi) = V_1(\xi). \quad (\text{E.6})$$

It is known that

$$\begin{aligned} y_{e1}(\xi) &= \xi^{\lambda_1/2} \exp(\sigma_0 \xi^{-1}) V_1(\xi), \\ y_{e2}(\xi) &= \xi^{\lambda_1/2} \exp(\sigma_0 \xi^{-1}) V_2(\xi), \end{aligned} \quad (\text{E.7})$$

so the conjugate relation is obtained:

$$\begin{aligned} [y_{e2}(\xi)]^* &= \xi^{\lambda_1/2} \exp(-\sigma_0^* \xi^{-1}) V_2^*(\xi) \\ &= \xi^{\lambda_1/2} \exp(\sigma_0 \xi^{-1}) V_1(\xi) = y_{e1}(\xi). \end{aligned} \quad (\text{E.8})$$

F. Linear Combination for Real Solution

Taking x_1, x_2 large enough, $x_1 = 20$, $x_2 = 60$, for example, the corresponding $\xi_1 = 1/x_1$, $\xi_2 = 1/x_2$. Demanding

$$\begin{aligned} y_1(x_1) &= J_\nu(x_1), \\ y_1(x_2) &= J_\nu(x_2), \\ y_2(x_1) &= J_{-\nu}(x_1), \\ y_2(x_2) &= J_{-\nu}(x_2) \end{aligned} \quad (\text{F.1})$$

or

$$\begin{aligned} b_{11} y_{e1}(\xi_1) + b_{12} y_{e2}(\xi_1) &= J_\nu(x_1), \\ b_{11} y_{e1}(\xi_2) + b_{12} y_{e2}(\xi_2) &= J_\nu(x_2), \\ b_{21} y_{e1}(\xi_1) + b_{22} y_{e2}(\xi_1) &= J_{-\nu}(x_1), \\ b_{21} y_{e1}(\xi_2) + b_{22} y_{e2}(\xi_2) &= J_{-\nu}(x_2), \end{aligned} \quad (\text{F.2})$$

(b_{11}, b_{12}) and (b_{21}, b_{22}) can be determined.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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