A Note on Primitivity of Ideals in Skew Polynomial Rings of Automorphism Type

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We extend results about primitive ideals in polynomial rings over nil rings originally proved by Smoktunowicz (2005) for \( \sigma \)-primitive ideals in skew polynomial rings of automorphism type.

1. Introduction

Throughout this paper \( R \) denotes an associative ring but does not necessarily have an identity element and \( \sigma : R \to R \) an automorphism of \( R \), unless otherwise stated. We denote by \( R[x; \sigma] \) the skew polynomial rings of automorphism type whose elements are polynomials \( \sum_{i=0}^{n} a_i x^i \), \( a_i \in R \), for every \( i \geq 0 \), with usual addition and the following multiplication: \( xa = \sigma(a)x \) for all \( a \in R \).

An ideal \( I \) is said to be a Jacobson ring if every prime ideal of \( R \) is an intersection of (either left or right) primitive ideals of \( R \). In [1], Smoktunowicz proved that if \( R \) is a nil ring and \( I \) an ideal of \( R[x] \), then \( R[x]/I \) is Jacobson radical if and only if \( R[x]/I' \) is Jacobson radical, where \( I' \) is the ideal of \( R \) generated by coefficients of polynomial from \( I \). Also if \( R \) is a nil ring and \( I \) is a primitive ideal of \( R[x] \), then \( I = M[x] \) for some ideal \( M \) of \( R \) and affirmative answer to this question is equivalent to the Kôthe conjecture. Our main results state that if \( R \) is a nil ring and \( I \) an ideal of \( R[x; \sigma] \), then \( R[x; \sigma]/I \) is \( \sigma \)-Jacobson radical if and only if \( R[x; \sigma]/I' \) is \( \sigma \)-Jacobson radical, where \( I' \) is the ideal of \( R \) generated by coefficients of polynomial from \( I \). Also if \( R \) is a nil ring and \( I \) is a \( \sigma \)-primitive ideal of \( R[x; \sigma] \), then \( I = M[x; \sigma] \) for some ideal \( M \) of \( R \). This result includes, as particular cases, all the above results.

Now we recall some terminology and results; see [2–4]. A right ideal \( Q \) of a ring \( R \) is called modular in \( R \) if and only if there exists an element \( b \in R \) such that \( a - ba \in Q \) for every \( a \in R \). An ideal \( I \) of a ring \( R \) is said to be a \( \sigma \)-invariant if and only if \( \sigma(I) = I \). An ideal \( P \) of \( R \) is said to be a right \( \sigma \)-primitive in \( R \) if and only if there exists a modular maximal right ideal \( \sigma \)-invariant \( Q \) of \( R \) such that \( P \) is the maximal ideal contained in \( Q \). For \( f \in R[x; \sigma] \), \( \deg(f) \) denotes the degree of \( f \) and \( \text{lc}(f) \) the leading coefficient of \( f \).

2. Results

We begin with the following results that extend ([1, Lemma 1]) and the proof is also similar to the one in the paper.

**Lemma 1.** Let \( R \) be a ring, \( I \) a right ideal of \( R \), \( f \in J[x; \sigma] \), \( Q \) a right ideal of \( R[x; \sigma] \), and \( b \in R[x; \sigma] \) such that \( a - ba \in Q \) for every \( a \in R[x; \sigma] \). If \( b - fx \), then, for every \( i \geq 1 \), there are \( f_i \in J[x; \sigma] \) such that \( b - f_x^i \in Q \) and \( \deg(f_i) \leq \deg(f) \).

**Proof.** We proceed by induction on \( n \). If \( n = 1 \), we put \( f_1 = f \). Suppose the lemma holds for some \( n \geq 1 \). Let

\[
  f_n = a_0 + a_1 x + \cdots + a_k x^k \in J[x; \sigma],
\]

with \( b - f_n x^n \in Q \) and \( k \leq \deg(f) \). Consider

\[
  f_{n+1} = f \sigma(a_0) + a_1 + a_2 x + \cdots + a_k x^{k-1} \in J[x; \sigma].
\]

Since \( b - fx \in Q \), then \( fx = b + q, q \in Q \). Thus

\[
  b - f_{n+1} x^{n+1} = b - f_n x^n + (a_0 - ba_0) x^n - qa_0 x^n \in Q.
\]
We denote by $R'$ the usual extension of $R$ to a ring with identity and by $\sigma$ again the natural extension of $\sigma$ to $R'$.

The next lemma extends ([1, Lemma 2]).

**Lemma 2.** Let $I$ be an ideal of $R[x;\sigma]$ with $\sigma(I) = 1$ and $I$ a right ideal of $R$ with $\sigma(J) = J$. Consider $p = a_0 + a_1 x + \cdots + a_k x^k \in I$, $k > 0$, and

$$ U = \sum_{i \in \mathbb{Z}} [x;\sigma] \sigma^i (a_k) R^1 [x;\sigma]. \quad (4) $$

(i) If $h \in U^l$, $l \geq 1$, and $\deg(h) \geq k$, then there exists $g \in U^{l-1}$ such that $h - g \in I$ and $\deg(g) < \deg(h)$.

(ii) Let $Q$ be a right ideal of $R[x;\sigma]$, $b \in R[x;\sigma]$ such that $a - ba = b - ab \in Q$ for every $a \in R[x;\sigma]$, and $l \in Q$. If $b - fx = Q$ with $f \in [x;\sigma]$, $\deg(f) \geq 1$, and $b - g \in Q$, where $g \in U^{\deg(f)}$, then, for every $i > \deg(g)$, there exists $g_i \in [x;\sigma]$ such that $b - g_i x^i \in Q$ and $\deg(g_i) < k$.

**Proof.** (i) Let $h = c_0 + c_1 x + \cdots + c_t \in U^l$, $c_t \neq 0$, and $k \leq t$. We can write

$$ c_i = \sum_{j=0}^{m'} \alpha_j \beta_j, \quad \alpha_j \in U, \quad \beta_j \in U^{l-1}. \quad (5) $$

Then

$$ \alpha_j = \sum_{i=0}^{n_1} \left( t_{i,j} \sigma^i (a_k) u_{i,j} \right), \quad t_{i,j} \in I, \quad u_{i,j} \in R^1, \quad q_{i,j} \in \mathbb{Z}. \quad (6) $$

Hence

$$ c_i = \sum_{j=0}^{m'} \sum_{i=0}^{n_1} \left( t_{i,j} \sigma^i (a_k) u_{i,j} \beta_j \right) = \sum_{i=0}^{m} p_i \sigma^i (a_k) c_i q_i. \quad (7) $$

with $p_i \in I$, $e_i, q_i \in R^1$, $q_i \in U^{l-1}$, and $l_i \in \mathbb{Z}$. Put

$g = h - c_t x^t$

$$ + \sum_{i=0}^{m} p_i \left( \sigma^i (p) - \sigma^i (a_k) x^k \right) \sigma^{-k} (e_i q_i) x^{t-k}. \quad (8) $$

Therefore

$$ g - h = \sum_{i=0}^{m} p_i \left( \sigma^i (p) \right) \sigma^{-k} (e_i) \sigma^{-k} (q_i) x^{t-k} \in I. \quad (9) $$

Since $I[x;\sigma] U^{l-1} \subset U^{l-1}$ and $h \in U^{l-1}$, then $g \in U^{l-1}$ and $\deg(g) < \deg(h)$.

(ii) By Lemma 1, for every $i \geq 1$, there exists $f_i \in [x;\sigma]$ such that

$$ b - f_i x^i \in Q, \quad \deg(f_i) \leq \deg(f). \quad (10) $$

Consider

$$ g = \sum_{j=0}^{m} c_j x^j, \quad c_j \in U^{\deg(f)}. \quad (11) $$

For every $n > m$ denote

$$ h_n = \sum_{j=0}^{m} f_{n-j} \sigma^{n-j} (e_i) \in I[x;\sigma] \cap U^{\deg(f)}. \quad (12) $$

Note that $\deg(h_n) \leq \deg(f_{n-j}) \leq \deg(f)$; thus for every $i \geq 1$

$$ f_i x^i = b + q_i, \quad q_i \in Q. \quad (13) $$

Hence

$$ b - h_n x^n = b - \sum_{j=0}^{m} c_j x^j + \sum_{j=0}^{m} q_{n-j} x^j \in Q. \quad (14) $$

Because $b - g \in Q$ and $b - bg \in Q$, we have that, for every $n > t$, there exists $h_n \in U^{\deg(f) - 1} \subset I[x;\sigma]$ such that $b - h_n x^n \in Q$. If $\deg(h_n) \leq k$, then $h_n$ is the $g_0$ required. If $\deg(h_n) \geq k$, by first part of this lemma, there exists

$$ \lambda_{n-1} \in U^{\deg(f) - 1} \subset I[x;\sigma] \quad (15) $$

such that $h_n - \lambda_{n-1} \in I$ and $\deg(\lambda_{n-1}) < \deg(h_n)$. Thus $b - \lambda_{n-1} x^n \in Q$ for all $n > m$. If $\deg(\lambda_{n-1}) < k$, then $\lambda_{n-1}$ is the $g_0$ required. If $\deg(\lambda_{n-1}) \geq k$, using similar arguments as above, we can find $s \in \mathbb{N}$ such that

$$ \lambda_n \in U^{\deg(f) - 1} \subset I[x;\sigma] \quad (16) $$

with $b - \lambda_s x^s \in Q$ and $\deg(\lambda_s) \leq \deg(f) - s < k$ for every $n > m$. Hence $\lambda_s$ is the $g_0$ required.

**Lemma 3.** Let $Q$ be a right ideal of $R[x;\sigma]$ maximal in the set of all right ideals $\sigma$-invariants with $b \in R[x;\sigma]$ such that $a - ba = b - ab \in Q$ for all $a \in R[x;\sigma]$. Suppose $f \in R[x;\sigma]$ with $b - fx \in Q$ for some $j \geq 1$. If there is no right ideal $I$ of $R$ with $\sigma(I) = 1, J \subseteq Q$, and $J \neq I$, then there exists a positive integer $s$ and $r \in R$ such that $if \in rR[x;\sigma]$ with $b - wx^m \in Q$, $m \geq 0$, and $\deg(w) \leq v$, then $lc(w) \in rR, lc(w)(Q \cap R) \subseteq Q$, and $v$ is a good number for all $a \in rR$.

**Proof.** Let $v$ be minimal positive integer such that there exists $w' = i_j + i_1 x + \cdots + i_x x^x \in R[x;\sigma]$ and $m \geq 1$ with $b - w' x^m \in Q$ and $\deg(w') = v$. It is clear that $r \notin Q$. If $\sigma = \sigma(r) - r \notin Q$, put $g = \sigma(w') - w'$ and

$$ A = \sum_{i \in \mathbb{Z}} \sigma^i (c) R^1. \quad (18) $$

Thus $A$ is a right ideal of $R$ with $\sigma(A) = A$ and $A \notin Q$. By assumption $A = R$, then $r = \sum_{i=0}^{x} \sigma^i (c) l_i$, where $l_i \in R^1$ and $q_i \in \mathbb{Z}$. Put

$$ t = w' + \sum_{i=0}^{x} \sigma^i (g) \sigma^{-v} (l_i). \quad (19) $$
Comparing the leading coefficients of \( w \)' and
\[
\sum_{i=0}^{k} \sigma^i(g)(\sigma^{-1}(t)),
\]
we have that
\[
b - tx^m \in Q, \quad \deg(t) \leq v - 1,
\]
which contradicts the minimality of \( v \). Therefore \( \sigma(r) - r \in Q \); consequently \( r \in \overline{R} \).

Suppose that \( rq \notin Q \) for some \( q \in R \cap Q \). Put \( g' = w' \sigma^{-1}(q) \in Q \); using similar arguments as above we can have a contradiction. Hence \( r(Q \cap R) \subseteq Q \).

If there exists \( w \in rR[x;\sigma] \) with \( b - wx^m \in Q \), \( j \geq 0 \), and \( \deg(w) \leq v \), then using similar arguments as above we can show that \( lc(w) \in \overline{R} \) and \( lc(w)(Q \cap R) \subseteq Q \). Moreover, if \( a \in \overline{R} \), then \( B = aR + Q \cap R \); we have that \( B \) is a right ideal of \( R \) with \( \sigma(B) = B \) and \( B \notin Q \).

By assumption \( B = aR + Q \cap R = R \). Thus \( w' = aw' + q' \), where \( w' \in R[x;\sigma] \), \( \deg(w') \leq \deg(w') \), and \( q' \in Q \). Therefore \( b - aw'x^m \in Q \). Consequently \( v \) is a good number for all \( a \in \overline{R} \).

**Lemma 4.** Let \( J \) be a right ideal \( R \) with \( \sigma(J) = J, J \notin Q \), and \( J \notin R \) such that for all sufficiently large \( n \) there are \( f_n \in ]J[x;\sigma] \) such that \( b - f_nx^n \in Q \) and \( \deg(f_n) \leq v \). Put
\[
w' = i_0 + i_1x + \cdots + i_{v-1}x^{v-1} + rx^v \in ]J[x;\sigma];
\]
with \( b - w'x^m \in Q, m \geq 0 \), and \( \deg(w') \leq v \). By Lemma 1 and minimality of \( v \) we have that \( r \notin Q \). Using the same ideas of Lemma 3, we have that \( r \in \overline{R} \) and \( r(Q \cap R) \subseteq Q \). Since \( r \in J \), we have that the first part of lemma is satisfied.

Let \( a \in \overline{R} \cap J \); we denote by \( B \) the right ideal of \( R \):
\[
B = \sum_{i \in \mathbb{Z}} a^i \sigma^i(B), \quad \sigma(B) = B, \quad B \subseteq J, \quad B \notin Q.
\]

For sufficiently large \( n \) there are \( g_n \in B[x;\sigma] \subseteq J[x;\sigma] \) such that \( b - g_nx^n \in Q \) and \( \deg(g_n) \leq v \). Put
\[
g_n = c_{n_0} + c_{n_1}x + \cdots + c_{n_v}x^v \in B[x;\sigma].
\]

For every \( 0 \leq j \leq v \) we have that \( c_{n_j} = \sum_{i=0}^{m_j} \sigma^{l_i}a(l_i)h_i \), where \( h_i \in R \) and \( c_{n_j} \in \mathbb{Z} \). Consequently
\[
c_{n_j} = \sum_{i=0}^{m_j} (\sigma^{l_i}a - a)l_i + \sum_{i=0}^{m_j} l_i h_i.
\]

Since \( a \in \overline{R} \), we can write
\[
c_{n_j} = s_{n_j} + r_{n_j}, \quad s_{n_j} \in Q \cap R, \quad r_{n_j} \in R.
\]

Put \( h_n = r_{n_0} + r_{n_1}x + \cdots + r_{n_v}x^v \); therefore \( b - aha_{n}x^n \in Q \). Therefore \( v \) is a good number for all \( a \in \overline{R} \).

**Lemma 5.** Let \( Q \) be a right ideal of \( R[x;\sigma], b \in R[x;\sigma] \), such that \( a - ba \in Q \) for all \( a \in R[x;\sigma] \) and \( v \) is good number for all \( a \in \overline{R} \), where \( r \in \overline{R} \). Assume that for every \( w \in rR[x;\sigma] \) with \( b - wx^m \in Q, m \geq 0 \), and \( \deg(w) \leq v \) one has that \( lc(w) \in \overline{R} \) and \( lc(w)(Q \cap R) \subseteq Q \). If there are \( p \) and \( p' \in \overline{R} \) with
\[
(pR + Q \cap R) \cap (pR + Q \cap R) \subseteq Q,
\]
then \( v - 1 \) is a good number for \( r \).

**Proof.** Since \( v \) is a good number for \( p \) and \( p' \), then for every sufficiently large \( n \) there are \( g_n \in pR[x;\sigma] \) and \( g'_n \in p'R[x;\sigma] \) such that
\[
b - g_nx^n \in Q, \quad b - g'_nx^n \in Q.
\]
with \( \deg(g_n), \deg(g'_n) \leq v \). Consider
\[
g_n = p_n + p_n'x + \cdots + p_nx^v, \quad p_n \in pR,
\]
\[
g'_n = p'_n + p'_nx + \cdots + p'_nx^v, \quad p'_n \in p'R.
\]

Since \( p_n - p'_n \in Q \), then
\[
\left(pR + Q \cap R\right) \cap \left(p'R + Q \cap R\right) \subseteq Q,
\]
a contradiction.

Thus there exists sufficiently large \( i \in \mathbb{N} \) such that \( c = p_i - p'_i \in \overline{R} \); hence \( v \) is a good number for \( c \). Then for all sufficiently large \( n \) there are \( h_n \in R[x;\sigma] \) such that \( b - ch_nx^n \in Q \) and \( \deg(h_n) \leq v \). We denote
\[
h_n = r_{n_0} + r_{n_1}x + \cdots + r_{n_v}x^v.
\]

Consider
\[
k_n = c_{n_0} + (g_i - g_i) \sigma^{-1}(r_n) \in rR[x;\sigma].
\]

Since \( g_i - g_i \in Q \), then \( b - k_nx^v \in Q \). Moreover
\[
k_n = c_{n_0} - cr_{n_1}x + \sum_{j=0}^{v-1} \left(p_{i,j} - p_{i,j}ight)x^j \sigma^{-1}(r_n).
\]

Consequently \( v - 1 \) is a good number for \( r \).

**Theorem 6.** Let \( R \) be a nil ring and let \( I \) be a \( \sigma \)-primitive ideal in \( R[x;\sigma] \). Then \( I = I'[x;\sigma] \), where \( I' \) is an ideal \( \sigma \)-invariant of \( R \).

**Proof.** Assume by contradiction that there are \( a_0, a_1, \ldots, a_k \in R \) with
\[
a_0 + a_1x + \cdots + a_kx^k \in I, \quad a_k \notin I.
\]
Since $I$ is a $\sigma$-primitive ideal in $R[x;\sigma]$, there is a right ideal $Q$ of $R[x;\sigma]$ with $\sigma(Q) = Q$ and $b \in R[x;\sigma]$ such that $a - ba \in Q$ for all $a \in R[x;\sigma]$. Moreover, $Q$ is a maximal in the set of right ideals $\sigma$-invariants and $J$ is the maximal ideal contained in $Q$. We have that $R[x;\sigma]x \notin Q$; otherwise $b \in R$, which is impossible because $R$ is a nil ring. By definition of $Q$ it follows that $R[x;\sigma]x + Q = R[x;\sigma]$.

If $b - \Delta x^i \in Q$ for some $i \geq 0$ with $h \in R[x;\sigma]$, then $\deg(h^i) \geq 1$. In fact, if $h \in R$, let $i \geq 1$ be the minimal positive integer with respect to $h^i \in Q$. Thus $(b - \Delta x^i)\sigma^{-i}(h^i - 1) \in Q$. Then $b\sigma^{-i}(h^i - 1) \in Q$; hence $\sigma^{-i}(h^i - 1) \in Q$. Consequently $h^i - 1 \in Q$, a contradiction.

Let $J$ be a right ideal of $R$ with $\sigma(J) = J$ and $J \notin Q$. We have that $J[x;\sigma]x + Q = R[x;\sigma]$. There exists $f \in J[x;\sigma]$ such that $b - fx \in Q$. Consider

$$U = \sum_{i \in \mathbb{Z}} \left( J[x;\sigma] \sigma^{-i}(a_k) R^1[x;\sigma] \right).$$

Since $I$ is an ideal $\sigma$-prime and $a_k \notin I$, then $U \notin I$. Consequently $U \notin Q$, because $I$ is the maximal ideal contained in $Q$. Then $U_{\deg(f)} + Q = R[x;\sigma]$. There exists $g^i \in U_{\deg(f)}$ such that $b - g^i \in Q$. By Lemma 2, for every $i \geq \deg(g^i)$, there are $g_i \in J[x;\sigma]$ such that $b - g_i x^i \in Q$ and $\deg(g_i) < k$. Lemmas 3 and 4 imply that there are $r^i \in R$ and $v' \geq 1$ such that if $w \in r' R[x;\sigma]$ with $b - uw^m \in Q$, $m \geq 1$, and $\deg(w) \leq v'$, then $l(w) \in r'' R$ and $l(w)(Q \cap R) \subseteq Q$. Moreover $v'$ is a good number for all $a \in r'' R$. Let $v$ be minimal such that $v$ is a good number for all $a \in r'' R$. We have that $v \leq v'$. Let $r \in r'' R$. Since $v$ is a good number for $r$, then for sufficiently large $n$ there are $h_n \in R[x;\sigma]$, such that

$$b - rh_n x^n \in Q, \quad \deg(h_n) \leq v.$$  

Consider $f_n = rh_n$, then $b - f_n x^n \in Q$ and $\deg(f_n) \leq v$. For some $i \in \mathbb{N}$, there are $f_i, f_{i+1}, \ldots, f_{i+k} \in R[x;\sigma]$, such that $b - f_j x^j \in Q$, $\deg(f_j) \leq v$, and $i \leq j \leq i + k$. Put

$$f_j = ra_j x + r\Delta, j \geq j, c_j = c_j x^j,$$

where $c_j = ra_j + r\Delta, j \geq j, c_j = c_j x^j \in R[x;\sigma]$ and $c_j = ra_j$. Since $\deg(f_j) \leq v \leq v'$, then $c_j \notin Q$. Moreover,

$$\sigma(c_j) - c_j \in Q, \quad c_j (Q \cap R) \subseteq Q.$$  

Since $R$ is a nil ring, consider $e_j = c_j x^j$, where $n_j$ is a minimal with respect to the condition $c_j x^j \notin Q$. Thus $\sigma(e_j) - e_j \in Q$ for all $i \geq 0$. We have that

$$f_j \sigma^i(e_j) = g^i \sigma^i(e_j) + c_j \sigma^i \sigma^v(e_j) x^v$$

$$= g^i \sigma^i(e_j) + c_j \sigma^i \sigma^v(e_j) - e_j \sigma^v + c_j e_j x^v.$$  

Put $t_j = g^i \sigma^i(e_j) \in R[x;\sigma]$, thus,

$$f_j \sigma^i(e_j) - t_j \in Q \quad \deg(t_j) \leq v - 1$$  

for every $i \leq j \leq i + k$. Since $e_j \in \overline{r} R \subseteq \overline{r'' R}$, if $v - 1$ is not a good number for $r$, then Lemma 5 implies that

$$\bigcap_{j=1}^{i+k} \left( e_j R + Q \cap r'' R \right) \notin Q.$$  

In this case, there exists $s \in \bigcap_{j=1}^{i+k} (e_j R + Q \cap r'' R)$ such that $s \notin Q$. Consequently $s - e_j d_j \in Q \cap r'' R$, $d_j \in R$, and $e_j d_j \in e_j R$. Then $s \in \overline{r'' R} \subseteq \overline{r'' R}$. Therefore $v$ is a good number for $s$. Then for sufficiently large $n$ there are $\overline{f}_n \in s R[x;\sigma]$, such that

$$b - \overline{f}_n x^n \in Q, \quad \deg(\overline{f}_n) \leq v.$$  

Let

$$\overline{f}_n = \sum s b_{j_n} x^j.$$  

Since $b - \overline{f}_n x^n \in Q$, $s - e_j d_j \in Q$, and $e_j d_j - be_j d_j \in Q$, then $(b - f_n x^n) e_j d_j \in Q$. Thus $be_j d_j - f_n x^n e_j d_j \in Q$; hence $s - f_n x^n e_j d_j \in Q$ for every $i \leq j \leq i + k$.

Let

$$\overline{g}_n = \sum_{i=0}^{v} \left( f_{i+1} \sigma^{i+1} \sigma^v \left( e_{i+1} \sigma^v d_i + h_i \right) x^{i+1} \right) \in r R[x;\sigma].$$  

We have that $\overline{f}_n - \overline{g}_n x^n = (b - \overline{f}_n x^n) + (\overline{f}_n - \overline{g}_n)x^n \in Q$. Put

$$\overline{h}_n = \sum_{j=0}^{v} t_{i+j} \sigma^{i+j} \sigma^v \left( d_{i+j} + h_j \right) \in r R[x;\sigma].$$  

We can write $b - \overline{h}_n x^{i+j} + v$ as

$$b - \sum_{j=0}^{v} \left( t_{i+j} - f_j \sigma^j \sigma^v \left( e_{i+j} \right) \sigma^v \left( d_{i+j} + h_j \right) \right) x^{i+j} \in r R[x;\sigma].$$  

Thus for all sufficient large $n$

$$b - \overline{h}_n x^{i+j} \in Q, \quad \deg(\overline{h}_n) \leq v - 1.$$  

Then $v - 1$ is a good number for all $r \in r'' R$. This contradicts the minimality of $v$. \hfill $\Box$

Recall that the $\sigma$-Jacobson radical $J_\sigma(R)$ of a ring $R$ is defined as the intersection of all $\sigma$-primitive ideals of $R$. A ring $R$ is a $\sigma$-Jacobson radical if and only if $R[x;\sigma]/I$ is $\sigma$-Jacobson radical.

**Theorem 7.** Let $R$ be a nil ring and let $I$ be an ideal of $R[x;\sigma]$. Consider $\overline{I}$ the ideal of $R$ generated by coefficients of polynomial from $I$. Then $R[x;\sigma]/\overline{I}[x;\sigma]$ is $\sigma$-Jacobson radical if and only if $R[x;\sigma]/I$ is $\sigma$-Jacobson radical.
Proof. Assume by contradiction that $R[x;\sigma]/I$ is not $\sigma$-Jacobson radical. Then there is a $\sigma$-primitive ideal $P$ of $R[x;\sigma]/I$ such that $P \neq R[x;\sigma]/I$. We have that there is an ideal $K$ of $R[x;\sigma]$ such that $P = K/I$. Therefore $K$ is a $\sigma$-primitive ideal of $R[x;\sigma]$. By Theorem 6, there is an ideal $\overline{P}$ of $R$ such that $K = \overline{P}[x;\sigma]$. It is clear that $\overline{I} \subseteq \overline{P}$. Since

$$\frac{(R[x;\sigma]/\overline{I}[x;\sigma])}{(\overline{P}[x;\sigma]/\overline{I}[x;\sigma])} \cong \frac{R[x;\sigma]}{K},$$

then $\overline{P}[x;\sigma]/\overline{I}[x;\sigma]$ is a $\sigma$-primitive ideal, a contradiction. Using the fact that $I \subseteq \overline{I}[x;\sigma]$, the converse follows.

Corollary 8. If $R$ is a nil ring, then the polynomial ring of type automorphism $R[x;\sigma]$ can not be homomorphically mapped onto a $\sigma$-simple $\sigma$-primitive ring.

Competing Interests

The author declares that they have no competing interests.

References


