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Research Article

A Note on Primitivity of Ideals in Skew Polynomial Rings of Automorphism Type

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We extend results about primitive ideals in polynomial rings over nil rings originally proved by Smoktunowicz (2005) for σ-primitive ideals in skew polynomial rings of automorphism type.

1. Introduction

Throughout this paper R denotes an associative ring but does not necessarily have an identity element and $\sigma : R \to R$ an automorphism of R, unless otherwise stated. We denote by $R[x;\sigma]$ the skew polynomial rings of automorphism type whose elements are polynomials $\sum_{i=0}^{n} a_i x^i$, $a_i \in R$, for every $i \geq 0$, with usual addition and the following multiplication: $xa = \sigma(a)x$ for all $a \in R$.

A ring R is said to be a Jacobson ring if every prime ideal of R is an intersection of (either left or right) primitive ideals of R. In [1], Smoktunowicz proved that if R is a nil ring and I an ideal of $R[x]$, then $R[x]/I$ is Jacobson radical if and only if $R[x]/I'[x]$ is Jacobson radical, where I' is the ideal of R generated by coefficients of polynomial from I. Also if R is a nil ring and I is a primitive ideal of $R[x]$, then $I = M[x]$ for some ideal M of R and affirmative answer to this question is equivalent to the Köthe conjecture. Our main results state that if R is a nil ring and I an ideal of $R[x;\sigma]$, then $R[x;\sigma]/I$ is σ-Jacobson radical if and only if $R[x;\sigma]/I'[x;\sigma]$ is σ-Jacobson radical, where I' is the ideal of R generated by coefficients of polynomial from I. Also if R is a nil ring and I is a σ-primitive ideal of $R[x;\sigma]$, then $I = M[x;\sigma]$ for some ideal M of R. This result includes, as particular cases, all the above results.

Now we recall some terminology and results; see [2–4]. A right ideal Q of a ring R is called modular in R if and only if there exists an element $b \in R$ such that $a - ba \in Q$ for every $a \in R$. An ideal I of a ring R is said to be a σ-invariant if and only if $\sigma(I) = I$. An ideal P of R is said to be a right σ-primitive in R if and only if there exists a modular maximal right ideal σ-invariant Q of R such that P is the maximal ideal contained in Q. For $f \in R[x;\sigma]$, $\deg(f)$ denotes the degree of f and $\mathrm{lc}(f)$ the leading coefficient of f.

2. Results

We begin with the following results that extend ([1, Lemma 1]) and the proof is also similar to the one in the paper.

Lemma 1. *Let R be a ring, J a right ideal of R, $f \in J[x;\sigma]$, Q a right ideal of $R[x;\sigma]$, and $b \in R[x;\sigma]$ such that $a - ba \in Q$ for every $a \in R[x;\sigma]$. If $b - fx \in Q$, then, for every $i \geq 1$, there are $f_i \in J[x;\sigma]$ such that $b - f_i x^i \in Q$ and $\deg(f_i) \leq \deg(f)$.*

Proof. We proceed by induction on n. If $n = 1$, we put $f_1 = f$. Suppose the lemma holds for some $n \geq 1$. Let

$$f_n = a_0 + a_1 x + \cdots + a_k x^k \in J[x;\sigma], \tag{1}$$

with $b - f_n x^n \in Q$ and $k \leq \deg(f)$. Consider

$$f_{n+1} = f\sigma(a_0) + a_1 + a_2 x + \cdots + a_k x^{k-1} \in J[x;\sigma]. \tag{2}$$

Since $b - fx \in Q$, then $fx = b + q$, $q \in Q$. Thus

$$b - f_{n+1} x^{n+1} = b - f_n x^n + (a_0 - ba_0) x^n - q a_0 x^n \in Q. \tag{3}$$

□

We denote by R^1 the usual extension of R to a ring with identity and by σ again the natural extension of σ to R^1.

The next lemma extends ([1, Lemma 2]).

Lemma 2. *Let I be an ideal of $R[x;\sigma]$ with $\sigma(I) = I$ and J a right ideal of R with $\sigma(J) = J$. Consider $p = a_0 + a_1x + \cdots + a_kx^k \in I$, $k > 0$, and*

$$U = \sum_{i\in\mathbf{Z}} J[x;\sigma]\,\sigma^i(a_k)\,R^1[x;\sigma]. \tag{4}$$

(i) *If $h \in U^l$, $l \geq 1$, and $\deg(h) \geq k$, then there exists $g \in U^{l-1}$ such that $h - g \in I$ and $\deg(g) < \deg(h)$.*

(ii) *Let Q be a right ideal of $R[x;\sigma]$, $b \in R[x;\sigma]$ such that $a - ba \in Q$ for every $a \in R[x;\sigma]$, and $I \subset Q$. If $b - fx \in Q$ with $f \in J[x;\sigma]$, $\deg(f) \geq 1$, and $b - g \in Q$, where $g \in U^{\deg(f)}$, then, for every $i > \deg(g)$, there exists $g_i \in J[x;\sigma]$ such that $b - g_ix^i \in Q$ and $\deg(g_i) < k$.*

Proof. (i) Let $h = c_0 + c_1x + \cdots + c_tx^t \in U^l$, $c_t \neq 0$, and $k \leq t$. We can write

$$c_t = \sum_{j=0}^{m'} \alpha_j\beta_j, \quad \alpha_j \in U,\ \beta_j \in U^{l-1}. \tag{5}$$

Then

$$\alpha_j = \sum_{i=0}^{n_j} t_{ji}\sigma^{q_{ji}}(a_k)\,u_{ji}, \quad t_{ji} \in J,\ u_{ji} \in R^1,\ q_{ij} \in \mathbb{Z}. \tag{6}$$

Hence

$$c_t = \sum_{j=0}^{m'}\sum_{i=0}^{n_j}\left(t_{ji}\sigma^{q_{ji}}(a_k)\,u_{ji}\beta_j\right) = \sum_{i=0}^{m} p_i\sigma^{l_i}(a_k)\,e_iq_i \tag{7}$$

with $p_i \in J$, $e_i, q_i \in R^1$, $q_i \in U^{l-1}$, and $l_i \in \mathbb{Z}$. Put

$$\begin{aligned} g &= h - c_tx^t \\ &\quad + \sum_{i=0}^{m} p_i\left(\sigma^{l_i}(p) - \sigma^{l_i}(a_k)\,x^k\right)\sigma^{-k}(e_iq_i)\,x^{t-k}. \end{aligned} \tag{8}$$

Therefore

$$g - h = \sum_{i=0}^{m} p_i\left(\sigma^{l_i}(p)\right)\sigma^{-k}(e_i)\,\sigma^{-k}(q_i)\,x^{t-k} \in I. \tag{9}$$

Since $J[x;\sigma]U^{l-1} \subset U^{l-1}$ and $h \in U^{l-1}$, then $g \in U^{l-1}$ and $\deg(g) < \deg(h)$.

(ii) By Lemma 1, for every $i \geq 1$, there exists $f_i \in J[x;\sigma]$ such that

$$b - f_ix^i \in Q, \quad \deg(f_i) \leq \deg(f). \tag{10}$$

Consider

$$g = \sum_{j=0}^{m} c_jx^j, \quad c_j \in U^{\deg(f)}. \tag{11}$$

For every $n > m$ denote

$$h_n = \sum_{j=0}^{m} f_{n-j}\sigma^{n-j}(c_j) \in J[x;\sigma] \cap U^{\deg(f)}. \tag{12}$$

Note that $\deg(h_n) \leq \deg(f_{n-j}) \leq \deg(f)$; thus for every $i \geq 1$

$$f_ix^i = b + q_i, \quad q_i \in Q. \tag{13}$$

Hence

$$b - h_nx^n = b - b\sum_{j=0}^{m} c_jx^j + \sum_{j=0}^{m} q_{n-j}c_jx^j \in Q. \tag{14}$$

Because $b - g \in Q$ and $g - bg \in Q$, then $b - bg \in Q$. We have that, for every $n > t$, there exists $h_n \in U^{\deg(f)} \subseteq J[x;\sigma]$ such that $b - h_nx^n \in Q$. If $\deg(h_n) < k$, then h_n is the g_n required. If $\deg(h_n) \geq k$, by first part of this lemma, there exists

$$\lambda_{n_1} \in U^{\deg(f)-1} \subseteq J[x;\sigma] \tag{15}$$

such that $h_n - \lambda_{n_1} \in I$ and $\deg(\lambda_{n_1}) < \deg(h_n)$. Thus $b - \lambda_{n_1}x^n \in Q$ for all $n > m$. If $\deg(\lambda_{n_1}) < k$, then λ_{n_1} is the g_n required. If $\deg(\lambda_{n_1}) \geq k$, using similar arguments as above, we can find $s \in \mathbb{N}$ such that

$$\lambda_{n_s} \in U^{\deg(f)-s} \subseteq J[x;\sigma] \tag{16}$$

with $b - \lambda_{n_s}x^n \in Q$ and $\deg(\lambda_{n_s}) \leq \deg(f) - s < k$ for every $n > m$. Hence λ_{n_s} is the g_n required. □

Let $r \in R$, Q a right ideal of $R[x;\sigma]$, $\sigma(Q) = Q$, and $b \in R[x;\sigma]$ such that $a - ba \in Q$ for all $a \in R[x;\sigma]$. Following [1] we have the following. We say that v is a "good number for r," if, for all sufficiently large n, there are $f_n \in R[x;\sigma]$ such that $b - rf_nx^n \in Q$ with $\deg(f_n) \leq v$. Let $A \subseteq R$; we denote

$$\widetilde{A} = \{a \in A - Q \mid \sigma(a) - a \in Q\}. \tag{17}$$

Lemma 3. *Let Q be a right ideal of $R[x;\sigma]$ maximal in the set of all right ideals σ-invariants with $b \in R[x;\sigma]$ such that $a - ba \in Q$ for all $a \in R[x;\sigma]$. Suppose $f \in R[x;\sigma]$ with $b - fx^j \in Q$ for some $j \geq 1$. If there is no right ideal J of R with $\sigma(J) = J$, $J \nsubseteq Q$, and $J \neq R$, then there exists a positive integer v and $r \in \widetilde{R}$ such that if $w \in rR[x;\sigma]$ with $b - wx^m \in Q$, $m \geq 0$, and $\deg(w) \leq v$, then $lc(w) \in \widetilde{rR}$, $lc(w)(Q \cap R) \subseteq Q$, and v is a good number for all $a \in \widetilde{rR}$.*

Proof. Let v be minimal positive integer such that there exists $w' = i_0 + i_1x + \cdots + rx^v \in R[x;\sigma]$ and $m \geq 1$ with $b - w'x^m \in Q$ and $\deg(w') = v$. It is clear that $r \notin Q$. If $c = \sigma(r) - r \notin Q$, put $g = \sigma(w') - w'$ and

$$A = \sum_{i\in\mathbf{Z}} \sigma^i(c)\,R^1. \tag{18}$$

Thus A is a right ideal of R with $\sigma(A) = A$ and $A \nsubseteq Q$. By assumption $A = R$, then $r = \sum_{i=0}^{s}\sigma^{q_i}(c)l_i$, where $l_i \in R^1$ and $q_i \in \mathbb{Z}$. Put

$$t = w' - \sum_{i=0}^{s}\sigma^{q_i}(g)\,\sigma^{-v}(l_i). \tag{19}$$

Comparing the leading coefficients of w' and $\sum_{i=0}^{s}\sigma^{q_i}(g)\sigma^{-\nu}(l_i)$, we have that

$$b - tx^m \in Q, \quad \deg(t) \le \nu - 1, \tag{20}$$

which contradicts the minimality of ν. Therefore $\sigma(r) - r \in Q$; consequently $r \in \widetilde{R}$.

Suppose that $rq \notin Q$ for some $q \in R \cap Q$. Put $g' = w'\sigma^{-\nu}(q) \in Q$; using similar arguments as above we can have a contradiction. Hence $r(Q \cap R) \subseteq Q$.

If there exists $w \in rR[x;\sigma]$ with $b - wx^j \in Q$, $j \ge 0$, and $\deg(w) \le \nu$, then using similar arguments as above we can show that $\mathrm{lc}(w) \in \widetilde{rR}$ and $\mathrm{lc}(w)(Q \cap R) \subseteqq Q$. Moreover, if $a \in \widetilde{rR}$, put $B = aR + Q \cap R$; we have that B is a right ideal of R with $\sigma(B) = B$ and $B \nsubseteq Q$.

By assumption $B = aR + Q \cap R = R$. Thus $w' = aw'' + q'$, where $w'' \in R[x;\sigma]$, $\deg(w'') \le \deg(w')$, and $q' \in Q$. Therefore $b - aw''x^m \in Q$. Consequently ν is a good number for all $a \in \widetilde{rR}$. □

Lemma 4. *Let J be a right ideal R with $\sigma(J) = J$, $J \nsubseteq Q$, and $J \ne R$ such that for all sufficiently large n there are $f_n \in J[x;\sigma]$ such that $b - f_n x^n \in Q$ and $\deg(f_n) \le k$, where Q is a right ideal of $R[x;\sigma]$ and $b \in R[x;\sigma]$ such that $a - ba \in Q$ for every $a \in R[x;\sigma]$. Then there exists a positive integer ν and $r \in \widetilde{R}$ such that if $w \in rR[x;\sigma]$ with $b - wx^m \in Q$, $m \ge 0$, and $\deg(w) \le \nu$ one has that $lc(w) \in \widetilde{rR}$, $lc(w)(Q \cap R) \subseteq Q$, and ν is a good number for all $a \in \widetilde{rR}$.*

Proof. Let ν be minimal positive integer such that for all sufficiently large n there are $f_n \in J[x;\sigma]$ such that $b - f_n x^n \in Q$ and $\deg(f_n) \le \nu$. Put

$$w' = i_0 + i_1 x + \cdots + i_{\nu-1}x^{\nu-1} + rx^{\nu} \in J[x;\sigma] \tag{21}$$

with $b - w'x^m \in Q$, $m \ge 0$, and $\deg(w') \le \nu$. By Lemma 1 and minimality of ν we have that $r \notin Q$. Using the same ideas of Lemma 3, we have that $r \in \widetilde{R}$ and $r(Q \cap R) \subseteq Q$. Since $rR \subseteq J$, we have that the first part of lemma is satisfied.

Let $a \in \widetilde{rR} \subseteq \widetilde{J}$; we denote by B the right ideal of R:

$$B = \sum_{i\in\mathbb{Z}} \sigma^i(a)R^1, \quad \sigma(B) = B, \ B \subseteq J, \ B \nsubseteq Q. \tag{22}$$

For sufficiently large n there are $g_n \in B[x;\sigma] \subseteq J[x;\sigma]$ such that $b - g_n x^n \in Q$ and $\deg(g_n) \le \nu$. Put

$$g_n = c_{n_0} + c_{n_1}x + \cdots + c_{n_\nu}x^{\nu} \in B[x;\sigma]. \tag{23}$$

For every $0 \le j \le \nu$ we have that $c_{n_j} = \sum_{i=0}^{m_j}\sigma^{q_{n_i}}(a)l_{n_i}$, where $l_{n_i} \in R^1$ and $q_{n_i} \in \mathbb{Z}$. Consequently

$$c_{n_j} = \sum_{i=0}^{m_j}\left(\sigma^{q_{n_i}}(a) - a\right)l_{n_i} + a\sum_{i=0}^{m_j} l_{n_i}. \tag{24}$$

Since $a \in \widetilde{rR}$, we can write

$$c_{n_j} = s_{n_j} + r_{n_j}, \quad s_{n_j} \in Q \cap R, \ r_{n_j} \in R. \tag{25}$$

Put $h_n = r_{n_0} + r_{n_1}x + \cdots + r_{n_\nu}x^{\nu}$; thus $b - ah_n x^n \in Q$. Therefore ν is a good number for all $a \in \widetilde{rR}$. □

Lemma 5. *Let Q be a right ideal of $R[x;\sigma]$, $b \in R[x;\sigma]$, such that $a - ba \in Q$ for all $a \in R[x;\sigma]$ and ν is good number for all $a \in \widetilde{rR}$, where $r \in \widetilde{R}$. Assume that for every $w \in rR[x;\sigma]$ with $b - wx^m \in Q$, $m \ge 0$, and $\deg(w) \le \nu$ one has that $lc(w) \in \widetilde{rR}$ and $lc(w)(Q \cap R) \subseteq Q$. If there are p and $p' \in \widetilde{rR}$ with*

$$\left(\widetilde{pR} + Q \cap rR\right) \cap \left(\widetilde{p'R} + Q \cap rR\right) \subseteq Q, \tag{26}$$

then $\nu - 1$ is a good number for r.

Proof. Since ν is a good number for p and p', then for every sufficiently large n there are $g_n \in pR[x;\sigma]$ and $g'_n \in p'R[x;\sigma]$ such that

$$b - g_n x^n \in Q, \tag{27}$$

$$b - g'_n x^n \in Q \tag{28}$$

with $\deg(g_n)$, $\deg(g'_n) \le \nu$. Consider

$$\begin{aligned} g_n &= p_{n_0} + p_{n_1}x + \cdots + p_{n_\nu}x^{\nu}, \quad p_{n_\nu} \in \widetilde{pR}, \\ g'_n &= p'_{n_0} + p'_{n_1}x + \cdots + p'_{n_\nu}x^{\nu}, \quad p'_{n_\nu} \in \widetilde{p'R}. \end{aligned} \tag{29}$$

Since $p_{n_\nu} - p'_{n_\nu} \in Q$, then

$$p_{n_\nu} \in \left(\widetilde{pR} + Q \cap rR\right) \cap \left(\widetilde{p'R} + Q \cap rR\right) \subseteq Q, \tag{30}$$

a contradiction.

Thus there exists sufficiently large $i \in \mathbb{N}$ such that $c = p_{i_\nu} - p'_{i_\nu} \in \widetilde{rR}$; hence ν is a good number for c. Then for all sufficiently large n there are $h_n \in R[x;\sigma]$ such that $b - ch_n x^n \in Q$ and $\deg(h_n) \le \nu$. We denote

$$h_n = r_{n_0} + r_{n_1}x + \cdots + r_{n_\nu}x^{\nu}. \tag{31}$$

Consider

$$k_n = ch_n + \left(g'_i - g_i\right)\sigma^{-\nu}\left(r_{n_\nu}\right) \in rR[x;\sigma]. \tag{32}$$

Since $g'_i - g_i \in Q$, then $b - k_n x^n \in Q$. Moreover

$$k_n = ch_n - cr_{n_\nu}x^{\nu} + \sum_{j=0}^{\nu-1}\left(p'_{i_j} - p_{i_j}\right)x^j\sigma^{-\nu}\left(r_{n_\nu}\right). \tag{33}$$

Consequently $\nu - 1$ is a good number for r. □

The following theorem extends ([1, Theorem 1]).

Theorem 6. *Let R be a nil ring and let I be a σ-primitive ideal in $R[x;\sigma]$. Then $I = I'[x;\sigma]$, where I' is an ideal σ-invariant of R.*

Proof. Assume by contradiction that there are $a_0, a_1, \ldots, a_k \in R$ with

$$a_0 + a_1x + \cdots + a_kx^k \in I, \quad a_k \notin I. \tag{34}$$

Since I is a σ-primitive ideal in $R[x;\sigma]$, there is a right ideal Q of $R[x;\sigma]$ with $\sigma(Q) = Q$ and $b \in R[x;\sigma]$ such that $a - ba \in Q$ for all $a \in R[x;\sigma]$. Moreover Q is a maximal in the set of right ideals σ-invariants and I is the maximal ideal contained in Q. We have that $R[x;\sigma]x \nsubseteq Q$; otherwise $b \in R$, which is impossible because R is a nil ring. By definition of Q it follows that $R[x;\sigma]x + Q = R[x;\sigma]$.

If $b - hx^i \in Q$ for some $i \geq 0$ with $h \in R[x;\sigma]$, then $\deg(h) \geq 1$. In fact, if $h \in R$, let $t \geq 1$ be the minimal positive integer with respect to $h^t \in Q$. Thus $(b - hx^i)\sigma^{-i}(h^{t-1}) \in Q$. Then $b\sigma^{-i}(h^{t-1}) \in Q$; hence $\sigma^{-i}(h^{t-1}) \in Q$. Consequently $h^{t-1} \in Q$, a contradiction.

Let J be a right ideal of R with $\sigma(J) = J$ and $J \nsubseteq Q$. We have that $J[x;\sigma]x + Q = R[x;\sigma]$. There exists $f \in J[x;\sigma]$ such that $b - fx \in Q$. Consider

$$U = \sum_{i\in\mathbb{Z}} J[x;\sigma]\,\sigma^i(a_k)\,R^1[x;\sigma]. \tag{35}$$

Since I is an ideal σ-prime and $a_k \notin I$, then $U \nsubseteq I$. Consequently $U \nsubseteq Q$, because I is the maximal ideal contained in Q. Then $U^{\deg(f)} + Q = R[x;\sigma]$. There exists $g' \in U^{\deg(f)}$ such that $b - g' \in Q$. By Lemma 2, for every $i \geq \deg(g')$, there are $g'_i \in J[x;\sigma]$ such that $b - g'_i x^i \in Q$ and $\deg(g'_i) < k$. Lemmas 3 and 4 imply that there are $r' \in \widetilde{R}$ and $v' \geq 1$ such that if $w \in r'R[x;\sigma]$ with $b - wx^m \in Q$, $m \geq 1$, and $\deg(w) \leq v'$, then $\mathrm{lc}(w) \in \widetilde{r'R}$ and $\mathrm{lc}\, w(Q \cap R) \subseteq Q$. Moreover v' is a good number for all $a \in \widetilde{r'R}$. Let v be minimal such that v is a good number for all $a \in \widetilde{r'R}$. We have that $v \leq v'$. Let $r \in \widetilde{r'R}$. Since v is a good number for r, then for sufficiently large n there are $h_n \in R[x;\sigma]$, such that

$$b - rh_n x^n \in Q, \quad \deg(h_n) \leq v. \tag{36}$$

Consider $f_n = rh_n$, then $b - f_n x^n \in Q$ and $\deg(f_n) \leq v$. For some $i \in \mathbb{N}$, there are $f_i, f_{i+1}, \ldots, f_{i+k} \in rR[x;\sigma]$, such that $b - f_j x^j \in Q$, $\deg(f_j) \leq v$, and $i \leq j \leq i + k$. Put

$$f_j = ra_{j_0} + ra_{j_1}x + \cdots + ra_{j_v}x^v = g_j + c_j x^v, \tag{37}$$

where $g_j = ra_{j_0} + ra_{j_1}x + \cdots + ra_{j_{v-1}}x^{v-1} \in rR[x;\sigma]$ and $c_j = ra_{j_v}$. Since $\deg(f_j) \leq v \leq v'$, then $c_j \notin Q$. Moreover,

$$\sigma(c_j) - c_j \in Q, \quad c_j(Q \cap R) \subseteq Q. \tag{38}$$

Since R is a nil ring, consider $e_j = c_j^{n_j}$, where n_j is a minimal with respect to the condition $c_j^{n_j} \notin Q$. Thus $\sigma(e_j) - e_j \in Q$ for all $i \geq 0$. We have that

$$\begin{aligned} f_j\sigma^j(e_j) &= g_j\sigma^j(e_j) + c_j\sigma^{j+v}(e_j)\,x^v \\ &= g_j\sigma^j(e_j) + c_j\left(\sigma^{j+v}(e_j) - e_j\right)x^v \\ &\quad + c_j e_j x^v. \end{aligned} \tag{39}$$

Put $t_j = g_j\sigma^j(e_j) \in rR[x;\sigma]$. Thus,

$$f_j\sigma^j(e_j) - t_j \in Q \quad \deg(t_j) \leq v - 1 \tag{40}$$

for every $i \leq j \leq i + k$. Since $e_j \in \widetilde{rR} \subseteq \widetilde{r'R}$, if $v - 1$ is not a good number for r, then Lemma 5 implies that

$$\bigcap_{j=1}^{i+k}\left(\widetilde{e_jR} + Q \cap rR\right) \nsubseteq Q. \tag{41}$$

In this case, there exists $s \in \bigcap_{j=1}^{i+k}(\widetilde{e_jR} + Q \cap rR)$ such that $s \notin Q$. Consequently $s - e_jd_j \in Q \cap rR$, $d_j \in R$, and $e_jd_j \in \widetilde{e_jR}$. Then $s \in \widetilde{rR} \subseteq \widetilde{r'R}$. Therefore v is a good number for s. Then for sufficiently large n there are $\overline{f}_n \in sR[x;\sigma]$, such that

$$b - \overline{f}_n x^n \in Q, \quad \deg(\overline{f}_n) \leq v. \tag{42}$$

Let

$$\overline{f}_n = \sum_{j=0}^{v} sb_{j_n}x^j. \tag{43}$$

Since $b - \overline{f}_j x^j \in Q$, $s - e_jd_j \in Q$, and $e_jd_j - be_jd_j \in Q$, then $(b - f_jx^j)e_jd_j \in Q$. Thus $be_jd_j - f_jx^je_jd_j \in Q$; hence $s - f_jx^je_jd_j \in Q$ for every $i \leq j \leq i + k$.

Let

$$\overline{g}_n = \sum_{j=0}^{v} f_{i+v-j}\sigma^{i+v-j}\left(e_{i+v-j}d_{i+v-j}b_{j_n}\right)x^{i+v} \in rR[x;\sigma]. \tag{44}$$

We have that $\overline{f}_n - \overline{g}_n \in Q$. Thus $b - \overline{g}_n x^n = (b - \overline{f}_n x^n) + (\overline{f}_n - \overline{g}_n)x^n \in Q$. Put

$$\overline{h}_n = \sum_{j=0}^{v} t_{i+v-j}\sigma^{i+v-j}\left(d_{i+v-j}b_{j_n}\right) \in rR[x;\sigma]. \tag{45}$$

We can write $b - \overline{h}_n x^{i+v+n}$ as

$$\begin{aligned} &b - \sum_{j=0}^{v}\left(t_{i+v-j} - f_{i+v-j}\sigma^{i+v-j}\left(e_{i+v-j}\right)\right)\sigma^{i+v-j}\left(d_{i+v-j}b_{j_n}\right) \\ &\quad \cdot x^{i+v+n} - \overline{g}_n x^n. \end{aligned} \tag{46}$$

Thus for all sufficient large n

$$b - \overline{h}_n x^{i+v+n} \in Q, \quad \deg(\overline{h}_n) \leq v - 1. \tag{47}$$

Then $v - 1$ is a good number for all $r \in \widetilde{r'R}$. This contradicts the minimality of v. □

Recall that the σ-Jacobson radical $J_\sigma(R)$ of a ring R is defined as the intersection of all σ-primitive ideals of R. A ring R is a σ-Jacobson radical if $J_\sigma(R) = R$.

Theorem 7. *Let R be a nil ring and let I be an ideal of $R[x;\sigma]$. Consider $\overline{I}$ the ideal of R generated by coefficients of polynomial from I. Then $R[x;\sigma]/\overline{I}[x;\sigma]$ is σ-Jacobson radical if and only if $R[x;\sigma]/I$ is σ-Jacobson radical.*

Proof. Assume by contradiction that $R[x;\sigma]/I$ is not σ-Jacobson radical. Then there is a σ-primitive ideal P of $R[x;\sigma]/I$ such that $P \neq R[x;\sigma]/I$. We have that there is an ideal K of $R[x;\sigma]$ such that $P = K/I$. Therefore K is a σ-primitive ideal of $R[x;\sigma]$. By Theorem 6, there is an ideal $\overline{P}$ of R such that $K = \overline{P}[x;\sigma]$. It is clear that $\overline{I} \subseteq \overline{P}$. Since

$$\frac{\left(R[x;\sigma]/\overline{I}[x;\sigma]\right)}{\left(\overline{P}[x;\sigma]/\overline{I}[x;\sigma]\right)} \simeq \frac{R[x;\sigma]}{K}, \tag{48}$$

then $\overline{P}[x;\sigma]/\overline{I}[x;\sigma]$ is a σ-primitive ideal, a contradiction. Using the fact that $I \subseteq \overline{I}[x;\sigma]$, the converse follows. □

Corollary 8. *If R is a nil ring, then the polynomial ring of type automorphism $R[x;\sigma]$ can not be homomorphically mapped onto a σ-simple σ-primitive ring.*

Competing Interests

The author declares that they have no competing interests.

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