Research Article

Modal Logic Axioms Valid in Quotient Spaces of Finite CW-Complexes and Other Families of Topological Spaces

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Received 24 August 2016; Accepted 8 December 2016

Academic Editor: Ram U. Verma

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In this paper we consider the topological interpretations of $\mathcal{L}_{\Box}$, the classical logic extended by a “box” operator $\Box$ interpreted as interior. We present extensions of **S4** that are sound over some families of topological spaces, including particular point topological spaces, excluded point topological spaces, and quotient spaces of finite CW-complexes.

1. Introduction

It was shown by McKinsey and Tarski in [1] that every finite well-connected topological space is an open image of a metric separable dense-in-itself space. This implies that the modal logic **S4** is complete in any metric separable dense-in-itself space, for example, $\mathbb{R}^n$. The original proof of McKinsey and Tarski was quite tedious and technical; simpler, shorter, more geometric proofs were given in [2–5]. It follows that **S4** is sound and complete over all topological spaces; that is, a formula is derivable from the axiom system of **S4** if and only if it is valid in all topological spaces. Note that if a formula is valid in some topological space, it does not follow that this formula is derivable from **S4**.

In Section 2 we recall all necessary definitions, and in Section 3 we consider two axioms, denoted by *M* and *G*, each not derivable from **S4** and the other one, and for each of them we give necessary and sufficient conditions under which it is valid in a quotient space of a finite CW-complex, a particular point topological space, and an excluded point topological space.

2. Preliminaries

2.1. CW-Complexes and Their Quotient Spaces, Particular Point Topological Spaces, and Excluded Point Topological Spaces

Definition 1. Let X be a CW-complex. Its quotient space $Q(X)$ is a topological space whose points are in one-to-one

correspondence with cells of X , and a subset of $Q(X)$ is open if and only if the union of the corresponding cells is open in X .

Definition 2. If X is a CW-complex, its cell c is called a top cell if it is not in the boundary of any other cell.

Remark 3. A single point in $Q(X)$ is open if and only if it corresponds to a top cell.

Definition 4. Let X be any nonempty set and $p \in X$. Collection $T_p = \{S \subseteq X \mid p \in S \text{ or } S = \emptyset\}$ of subsets of X is called the *particular point topology* on X ([6, p. 44]).

Remark 5. Since, for any particular point topological space X with the particular point $p \in X$, any nonempty open subset includes p , it follows that the only closed subset that includes p is X .

Definition 6. Let X be any nonempty set and $e \in X$. Collection $T_e = \{S \subseteq X \mid e \notin S \text{ or } S = X\}$ of subsets of X is called the *excluded point topology* on X ([6, p. 47]).

Remark 7. Since, for any excluded point topological space X with the excluded point $e \in X$, every proper open subset of X excludes e , it follows that every nonempty closed set includes e .

Example 8. The quotient space of a bouquet of spheres (of any positive dimension) has excluded point topology.

FIGURE 1

FIGURE 2

Example 9. Sierpiński space, a topological space with two points only one of which is open ([6, p. 44]), is a topological space with both particular point and excluded point topology. It is the quotient space of the standard CW-complex for $\mathbb{R}P^1$ (real projective plane) (see Figure 1).

Example 10. More generally, for each $n \in \mathbb{N}$, let $X_n = \{x_1, x_2, \dots, x_n\}$ with topology (see Figure 2).

$$T_s = \{S \subseteq X \mid S = \emptyset \text{ or } S = \{x_1, x_2, \dots, x_k\} \text{ for } 1 \leq k \leq n\}. \quad (1)$$

For each $n \in \mathbb{N}$, X_n with topology T_s is the quotient space of the standard CW-complex for $\mathbb{R}P^{n-1}$.

Remark 11. Topology T_s is a subtopology of both a particular point topology (with the particular point being x_1) and an excluded point topology (with the excluded point being x_n).

2.2. Modal Logic Language $\mathcal{L}_\square$ and Its Interpretation in Topological Spaces

Definition 12. Let $\mathcal{L}_\square$ be a modal logic language consisting of propositional variables, conjunction $\wedge$, disjunction $\vee$, negation $\neg$, and modality $\square$. Modality $\diamond$ is defined as the dual to $\square$; that is, $\diamond P = \neg \square \neg P$.

Definition 13. **S4** denotes the subset of $\mathcal{L}_\square$ consisting of all formulas that can be derived from the following axioms:

- (i) All axioms of the classical propositional logic
- (ii) $\square(A \rightarrow B) \rightarrow (\square A \rightarrow \square B)$
- (iii) $\square A \rightarrow A$
- (iv) $\square A \rightarrow \square \square A$

using the following rules:

- (i) $(A, A \rightarrow B)/B$ (Modus ponens)
- (ii) $A/\square A$ (necessitation).

Definition 14. A topological model of $\mathcal{L}_\square$ is a pair $\langle X, \|\cdot\| \rangle$, where

- (1) X is a topological space
- (2) $\|\cdot\|$ is a valuation function mapping formulas in $\mathcal{L}_\square$ to subsets of X . It assigns each propositional variable P an arbitrary subset $\|P\| \subseteq X$ and satisfies the following conditions for every $F, G \in \mathcal{L}_\square$:

$$\begin{aligned} \|F \wedge G\| &= \|F\| \cap \|G\|, \\ \|F \vee G\| &= \|F\| \cup \|G\|, \\ \|\neg F\| &= \|F\|^C = X \setminus \|F\|, \\ \|\square F\| &= \text{int}(\|F\|), \end{aligned} \quad (2)$$

where $\text{int}(\cdot)$ is the topological interior operator.

Remark 15. Note that the modality $\diamond$ then maps to the topological closure operator; that is, $\|\diamond F\| = \text{cl}(\|F\|)$.

Definition 16. Let $F \in \mathcal{L}_\square$ be any formula. One says that F is *valid* in a topological space X if, for any topological model $\langle X, \|\cdot\| \rangle$, one has $\|F\| = X$.

Theorem 17 (topological completeness of **S4**, [1–5]). *For any formula $F \in \mathcal{L}_\square$, the following two statements are equivalent:*

- (i) F is derivable in **S4**.
- (ii) F is valid in all topological spaces.

3. Axioms M and G

Consider the following two axioms (the first form of each axiom is the one appearing most often in the literature, but the second one will be more convenient for us):

$$\begin{aligned} M : (\square \diamond P) &\longrightarrow (\diamond \square \neg P) \equiv (\diamond \square P) \vee (\diamond \square \neg P), \\ G : (\diamond \square P) &\longrightarrow (\square \diamond \neg P) \equiv (\square \diamond P) \vee (\square \diamond \neg P). \end{aligned} \quad (3)$$

Theorem 18. Both axioms M and G are valid in any particular point topological space.

Proof. Let X be a particular point topological space with particular point p , and let $\|P\| = A$ be any subset of X .

Case 1 ($p \in A$). Then A is open. So $\text{int}(A) = A$. Since the only closed set that includes p is X , $\text{cl}(\text{int}(A)) = \text{cl}(A) = X$. Therefore $\|(\diamond \square P) \vee (\diamond \square \neg P)\| = \text{cl}(\text{int}(A)) \cup \text{cl}(\text{int}(A^C)) = X$.

Also, since the only closed set that includes p is X , $\text{cl}(A) = X$. Then $\text{int}(\text{cl}(A)) = \text{int}(X) = X$. So $\|(\square \diamond P) \vee (\square \diamond \neg P)\| = \text{int}(\text{cl}(A)) \cup \text{int}(\text{cl}(A^C)) = X$.

Case 2 ($p \notin A$). Then $p \in A^C$. The rest is similar to case 1, with the roles of A and A^C switched. $\square$

Theorem 19. (1) Axiom M is valid in any excluded point topological space.

(2) Axiom G is valid in an excluded point topological space if and only if the space has only 1 or 2 points.

Proof. (1) Let X be an excluded point topological space with excluded point e , and let $\|P\| = A$ be any subset of X .

Case 1 ($A = X$). Then $\text{cl}(\text{int}(A)) = \text{cl}(\text{int}(X)) = X$.

Case 2 ($A = \emptyset$). Then $A^C = X$, so $\text{cl}(\text{int}(A^C)) = \text{cl}(\text{int}(X)) = X$.

Case 3 ($A = X \setminus \{e\}$, $A \neq \emptyset$). Then $\text{cl}(\text{int}(A)) = \text{cl}(A) = X$.

Case 4 ($A = \{e\}$, $A \neq X$). Then $A^C = X \setminus \{e\}$, $A^C \neq \emptyset$; thus $\text{cl}(\text{int}(A^C)) = X$.

Case 5 ($e \in A \subset X$ and $A \neq \{e\}$). Then $\text{cl}(\text{int}(A)) = \text{cl}(A \setminus \{e\}) = A$ and $\text{cl}(\text{int}(A^C)) = \text{cl}(A^C) = A^C \cup \{e\}$.

Case 6 ($e \notin A$, $A \neq \emptyset$, and $A \neq X \setminus \{e\}$). Then $e \in A^C \subset X$ and $A^C \neq \{e\}$. Thus this case is similar to case 5, with the roles of A and A^C switched.

In each of these cases, we have

$$\|(\Diamond\Box P) \vee (\Diamond\Box\neg P)\| = \text{cl}(\text{int}(A)) \cup \text{cl}(\text{int}(A^C)) = X. \quad (4)$$

(2) An excluded point topological space with only one or two points is also a particular point topological space; thus axiom G is valid by Theorem 18. If there are at least three points, let e be the excluded point and let a be any other point. Consider $A = \{a, e\}$. Then $\text{int}(\text{cl}(A)) = \text{int}(\text{cl}(\{a, e\})) = \text{int}(\{a, e\}) = \{a\}$ and $\text{int}(\text{cl}(A^C)) = \text{int}(\text{cl}(X \setminus \{a, e\})) = \text{int}(X \setminus \{a\}) = X \setminus \{a, e\}$, so $\|(\Diamond\Box P) \vee (\Diamond\Box\neg P)\| = \text{int}(\text{cl}(A)) \cup \text{int}(\text{cl}(A^C)) = X \setminus \{e\} \neq X$. $\square$

Remark 20. It is known (see, e.g., [7]) that axioms M and G are independent from each other and **S4**. Theorem 19 provides another proof of independency of G from **S4** + Axiom M , traditionally denoted by **S4.1**.

Theorem 21. (1) Axiom M is valid in the quotient space of any finite CW-complex.

(2) Axiom G is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.

Proof. (1) Let X be any finite CW-complex, and let $c \in Q(X)$ be any point. Then c is in the closure of some point t corresponding to a top cell, so t is open.

Case 1 ($t \in \|P\|$). Then $c \in \text{cl}(\{t\}) = \text{cl}(\text{int}(\{t\})) \subseteq \text{cl}(\text{int}(\|P\|))$.

Case 2 ($t \notin \|P\|$). Then $t \in \|P\|^C$, so $c \in \text{cl}(\text{int}(\|P\|^C))$.

(2) ($\Leftarrow$) If t corresponds to a top cell, then its entire connected component is in $\text{int}(\text{cl}(\{t\}))$. Since each connected component of a finite CW-complex contains a top cell and the point in $Q(X)$ corresponding to this top cell is either in $\|P\|$ or in $\|P\|^C$, $\|(\Diamond\Box P) \vee (\Diamond\Box\neg P)\| = X$.

($\Rightarrow$) If a connected component of a finite CW-complex contains at least two top cells, then there two top cells whose closures have nonempty intersection, so the intersection contains at least one cell. Let t_1 and t_2 be points in the quotient space corresponding to such two top cells and let c be the point corresponding to a cell in the intersection of their

closures. Consider a validation mapping such that $t_1 \in \|P\|$ and $t_2 \notin \|P\|$. Since t_2 corresponds to a top cell, it is not in the boundary of any other point; therefore $t_2 \notin \text{cl}(\|P\|)$, so $t_2 \in (\text{cl}(\|P\|))^C$. Then $c \in \text{cl}(\{t_2\}) \subseteq \text{cl}((\text{cl}(\|P\|))^C)$; therefore $c \notin (\text{cl}((\text{cl}(\|P\|))^C))^C = \|\neg\Diamond\neg P\| = \|\Box\Diamond P\|$. Since $t_1 \notin \text{cl}(\|P\|)$, in a similar way, we have $c \notin \|\Box\Diamond\neg P\|$. Thus $\|(\Box\Diamond P) \vee (\Box\Diamond\neg P)\| \neq X$. $\square$

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

The authors are very grateful to Fred Cohen for suggesting the study of quotient spaces of CW-complexes and to the College of Science and Mathematics of the California State University, Fresno for supporting this work.

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