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Research Article

Some Remarks on Quasi-Generalized CR-Null Geometry in Indefinite Nearly Cosymplectic Manifolds

Fortuné Massamba and Samuel Ssekajja

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, South Africa

Correspondence should be addressed to Fortuné Massamba; massfort@yahoo.fr

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Attention is drawn to some distributions on ascreen Quasi-Generalized Cauchy-Riemannian (QGCR) null submanifolds in an indefinite nearly cosymplectic manifold. We characterize totally umbilical and irrotational ascreen QGCR-null submanifolds. We finally discuss the geometric effects of geodesity conditions on such submanifolds.

1. Introduction

One of the current interesting research areas in semi-Riemannian geometry is the theory of null (or lightlike) submanifolds. An intrinsic approach to the theory of null submanifolds was advanced by Kupeli [1], yet an extrinsic counterpart had to wait for Duggal and Bejancu [2], and later by Duggal and Sahin [3]. Since then, many researchers have laboured to extend their theories with evidence from the following few selected papers: [3–11] and other references therein. The rapid increase in research on this topic, since 1996, is inspired by the numerous applications of the theory to mathematical physics, particularly in general relativity. More precisely, in general relativity, null submanifolds represent different models of black hole horizons (see [2, 3] for details).

In [5], the authors initiated the study of generalized CR- (GCR-) null submanifolds of an indefinite Sasakian manifold, which are *tangent* to the structure vector field, ξ, of the almost contact structure $(\overline{\phi}, \xi, \eta)$. Moreover, when ξ is tangent to the submanifold, Calin [12] proved that it belongs to its screen distribution. This assumption is widely accepted and it has been applied in many papers on null contact geometry, for instance, [3, 5, 8–11, 13]. It is worth mentioning that ξ is a *global vector field* defined on the entire tangent bundle of the ambient almost contact manifold. Thus, restricting it to the screen distribution is only one of those cases in which it can be placed. In the study of Riemannian CR-submanifolds of Sasakian manifolds, Yano and Kon [14, p. 43] proved that making ξ a normal vector field in such scenario leads to an anti-invariant submanifold, and hence ξ was kept tangent to the CR-submanifold. Their proof leans against the fact that the shape operator on such CR-submanifold is naturally *symmetric* with respect to the induced Riemannian metric g. On the other hand, the shape operators of any r-null submanifold are generally *not* symmetric with respect to the induced degenerate metric g (see [2, 3] for details).

In an attempt to generalize ξ, we introduced a special class of CR-null submanifolds of a nearly Sasakian manifold, known as *quasi-generalized CR- (QGCR-)* null submanifold [15], for which the classical GCR-null submanifolds [3] form part. Among other benefits, generalizing ξ leads to QGCR-null submanifolds of lower dimensions and with quite different geometric properties compared to respective GCR-null submanifolds.

The purpose of this paper is to investigate the geometry of distributions on ascreen QGCR-null submanifolds of indefinite nearly cosymplectic manifolds. A null submanifold M of an indefinite nearly cosymplectic manifold is called ascreen if the structure vector field ξ belongs to $\mathrm{Rad}(TM) \oplus l\mathrm{tr}(TM)$ [16]. The paper is organized as follows. In Section 2, we present the basic notions of null submanifolds and nearly cosymplectic manifolds. More details can be found in [17–22]. In Section 3, we review the basic notions of QGCR-null submanifolds and we give an example of ascreen QGCR-null

submanifold. In Section 4, we discuss totally umbilical, totally geodesic, and irrotational ascreen QGCR-null submanifolds of an indefinite nearly cosymplectic space form $\overline{M}(\overline{c})$. Finally, in Section 5 we investigate the geodesity of the distributions D and $\widehat{D}$.

2. Preliminaries

Let M^m be a codimension n submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant index ν, $1 \leq \nu \leq m+n$, with $m, n \geq 1$. Then, M is said to be a *null submanifold* of $\overline{M}$ if the tangent and normal bundles of M have a nontrivial intersection. This intersection defines a smooth distribution on M, called the *radical* distribution [2]. More precisely, consider $p \in M$; one defines the orthogonal complement $T_pM^{\perp}$ of the tangent space T_pM by

$$T_pM^{\perp} = \left\{X \in T_pM : \overline{g}(X, Y) = 0, \ \forall Y \in T_pM\right\}. \quad (1)$$

If we denote the radical distribution on M by $\operatorname{Rad} T_pM$, then $\operatorname{Rad} T_pM = \operatorname{Rad} T_pM^{\perp} = T_pM \cap T_pM^{\perp}$. The submanifold M of $\overline{M}$ is said to be *r-null submanifold* (one supposes that the index of $\overline{M}$ is $\nu \geq r$), if the mapping $\operatorname{Rad} TM : p \in M \to \operatorname{Rad} T_pM$ defines a *smooth distribution* on M of rank $r > 0$.

In this paper, r-null submanifold will simply be called a *null submanifold* and $g = \overline{g}|_{TM}$ is a *null metric*, unless we need to specify r.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\operatorname{Rad} TM$ in TM; that is,

$$TM = \operatorname{Rad} TM \perp S(TM). \quad (2)$$

Consider a screen transversal bundle $S(TM^{\perp})$, which is semi-Riemannian and complementary to $\operatorname{Rad} TM$ in $TM^{\perp}$. For any local basis $\{E_1, \ldots, E_r\}$ of $\operatorname{Rad} TM$, there exists a local null frame $\{N_1, \ldots, N_r\} \subset S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $g(E_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$. It follows that there exists a null transversal vector bundle $l\operatorname{tr}(TM)$ locally spanned by $\{N_1, \ldots, N_r\}$ (see details in [2, 3]). If $\operatorname{tr}(TM)$ denotes the complementary (but not orthogonal) vector bundle to TM in $T\overline{M}$, then,

$$\operatorname{tr}(TM) = l\operatorname{tr}(TM) \perp S(TM^{\perp}), \quad (3)$$

$$\begin{aligned} T\overline{M} &= S(TM) \perp S(TM^{\perp}) \\ &\quad \perp \{\operatorname{Rad} TM \oplus l\operatorname{tr}(TM)\}. \end{aligned} \quad (4)$$

It is important to note that the screen distribution $S(TM)$ is not unique and is canonically isomorphic to the factor vector bundle $TM/\operatorname{Rad} TM$ [1].

Given a null submanifold M, then the following classifications of M are well-known [2]: (i) M is r-null if $1 \leq r < \min\{m, n\}$; (ii) M is coisotropic if $1 \leq r = n < m$, $S(TM^{\perp}) = \{0\}$; (iii) M is isotropic if $1 \leq r = m < n$, $S(TM) = \{0\}$; (iv) M is totally null if $r = n = m$, $S(TM) = S(TM^{\perp}) = \{0\}$.

Where necessary, the following range of indices will be used

$$\begin{aligned} &i, j, k \in \{1, \ldots, r\}, \\ &\alpha, \beta, \gamma \in \{r+1, \ldots, n\}. \end{aligned} \quad (5)$$

Consider a local quasi-orthonormal fields of frames of $\overline{M}$ along M as

$$\{E_1, \ldots, E_r, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m, W_{1+r}, \ldots, W_n\}, \quad (6)$$

where $\{X_{r+1}, \ldots, X_m\}$ and $\{W_{1+r}, \ldots, W_n\}$ are, respectively, orthonormal bases of $\Gamma(S(TM))$ and $\Gamma(S(TM^{\perp}))$.

Throughout the paper we consider $\Gamma(\Xi)$ to be a set of smooth sections of the vector bundle Ξ.

Let P be the projection morphism of TM onto $S(TM)$. Then, the following are the Gauss-Weingarten equations of r-null submanifold M and $S(TM)$ (see [2, 3] for detailed explanations):

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^l(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X, Y) W_\alpha, \quad (7)$$

$$\overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) W_\alpha, \quad (8)$$

$$\overline{\nabla}_X W_\alpha = -A_{W_\alpha} X + \sum_{i=1}^{r} \varphi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X) W_\beta, \quad (9)$$

$$\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h_i^*(X, PY) E_i, \quad (10)$$

$$\nabla_X E_i = -A_{E_i}^* X - \sum_{j=1}^{r} \tau_{ji}(X) E_j, \quad \forall X, Y \in \Gamma(TM), \quad (11)$$

where ∇ and ∇^* are the induced connections on TM and $S(TM)$, respectively and h_i^l and h_α^s are symmetric bilinear forms known as *local null* and *screen fundamental* forms of TM, respectively. Furthermore, h_i^* are the *second fundamental forms* of $S(TM)$. A_{N_i}, $A_{E_i}^*$, and A_{W_α} are linear operators on TM while τ_{ij}, $\rho_{i\alpha}$, $\varphi_{\alpha i}$, and $\sigma_{\alpha\beta}$ are 1-forms on TM. Note that the *second fundamental tensor* of M is given by

$$h(X, Y) = \sum_{i=1}^{r} h_i^l(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X, Y) W_\alpha, \quad (12)$$

for any $X, Y \in \Gamma(TM)$. The connection ∇^* is a metric connection on $S(TM)$ while ∇ is generally not a metric connection and is given by

$$\begin{aligned} &(\nabla_X g)(Y, Z) \\ &\quad = \sum_{i=1}^{r} \left\{h_i^l(X, Y) \lambda_i(Z) + h_i^l(X, Z) \lambda_i(Y)\right\}, \end{aligned} \quad (13)$$

for any $X, Y \in \Gamma(TM)$ and λ_i are 1-forms given by $\lambda_i(X) = \overline{g}(X, N_i)$, for all $X \in \Gamma(TM)$. By using (7), (8), and (9), the

curvature tensors $\overline{R}$ and R of $\overline{M}$ and M, respectively, are related as, for any $X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned}\overline{R}(X,W,Z,Y) = {} & \overline{g}(R(X,W)Z,Y) \\ & + \overline{g}\left(A_{h^l(X,Z)}W,Y\right) \\ & - \overline{g}\left(A_{h^l(W,Z)}X,Y\right) \\ & + \overline{g}\left(A_{h^s(X,Z)}W,Y\right) \\ & - \overline{g}\left(A_{h^s(W,Z)}X,Y\right) \\ & + \overline{g}\left(\left(\nabla_X h^l\right)(W,Z),Y\right) \\ & - \overline{g}\left(\left(\nabla_W h^l\right)(X,Z),Y\right) \\ & + \overline{g}\left(D^l\left(X,h^s(W,Z)\right),Y\right) \\ & - \overline{g}\left(D^l\left(W,h^s(X,Z)\right),Y\right) \\ & + \overline{g}\left(\left(\nabla_X h^s\right)(W,Z),Y\right) \\ & - \overline{g}\left(\left(\nabla_W h^s\right)(X,Z),Y\right) \\ & + \overline{g}\left(D^s\left(X,h^l(W,Z)\right),Y\right) \\ & - \overline{g}\left(D^s\left(W,h^l(X,Z)\right),Y\right).\end{aligned} \tag{14}$$

A null submanifold (M, g) of an indefinite manifold $(\overline{M}, \overline{g})$ is said to be totally umbilical in $\overline{M}$ [3] if there is a smooth transversal vector field $\mathscr{H} \in \Gamma(\mathrm{tr}(TM))$, called the transversal curvature vector of M such that

$$h = g \otimes \mathscr{H}. \tag{15}$$

Moreover, it is easy to see that M is totally umbilical in $\overline{M}$, if and only if on each coordinate neighborhood U there exist smooth vector fields $\mathscr{H}^l \in \Gamma(l\mathrm{tr}(TM))$ and $\mathscr{H}^s \in \Gamma(S(TM^{\perp}))$ and smooth functions $\mathscr{H}^l_i \in F(l\mathrm{tr}(TM))$ and $\mathscr{H}^s_\alpha \in F(S(TM^{\perp}))$ such that,

$$\begin{aligned} h^l(X,Y) &= \mathscr{H}^l g(X,Y), \\ h^s(X,Y) &= \mathscr{H}^s g(X,Y), \\ h^l_i(X,Y) &= \mathscr{H}^l_i g(X,Y), \\ h^s_\alpha(X,Y) &= \mathscr{H}^s_\alpha g(X,Y), \end{aligned} \tag{16}$$

for all $X, Y \in \Gamma(TM)$.

Let us now consider $\overline{M}$ to be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$; that is, $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\begin{aligned} \overline{\phi}^2 &= -\mathbb{I} + \eta \otimes \xi, \\ \eta(\xi) &= 1, \\ \eta \circ \overline{\phi} &= 0, \\ \overline{\phi}(\xi) &= 0. \end{aligned} \tag{17}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an *indefinite almost contact metric structure* on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that [19], for any vector field $\overline{X}, \overline{Y}$ on $\overline{M}$,

$$\begin{aligned} \overline{g}\left(\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}\right) &= \overline{g}\left(\overline{X}, \overline{Y}\right) - \eta\left(\overline{X}\right)\eta\left(\overline{Y}\right), \\ \eta\left(\overline{X}\right) &= \overline{g}\left(\xi, \overline{X}\right). \end{aligned} \tag{18}$$

An indefinite almost contact metric manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ is said to be *nearly cosymplectic* if

$$\left(\overline{\nabla}_{\overline{X}}\overline{\phi}\right)\overline{Y} + \left(\overline{\nabla}_{\overline{Y}}\overline{\phi}\right)\overline{X} = 0, \quad \forall \overline{X}, \overline{Y} \in \Gamma\left(T\overline{M}\right), \tag{19}$$

where $\overline{\nabla}$ is the Levi-Civita connection for $\overline{g}$. Taking $\overline{Y} = \xi$ in (19), we get

$$\overline{\nabla}_{\overline{X}}\xi = -\overline{H}\,\overline{X}, \quad \forall \overline{X} \in \Gamma\left(T\overline{M}\right). \tag{20}$$

It is easy to verify the following properties of $\overline{H}$:

$$\begin{aligned} \overline{H}\,\overline{\phi} + \overline{\phi}\,\overline{H} &= 0, \\ \overline{H}\xi &= 0, \\ \eta \circ \overline{H} &= 0, \\ \left(\overline{\nabla}_X\overline{\phi}\right)\xi &= \overline{\phi}\,\overline{H}X, \\ \overline{g}\left(\overline{H}\,\overline{X}, \overline{Y}\right) &= -\overline{g}\left(\overline{X}, \overline{H}\,\overline{Y}\right) \\ & \qquad \left(\text{i.e., } \overline{H} \text{ is skew-symmetric}\right), \end{aligned} \tag{21}$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$. Let Ω denote the fundamental 2-form of $\overline{M}$ defined by

$$\Omega\left(\overline{X}, \overline{Y}\right) = \overline{g}\left(\overline{X}, \overline{\phi}\overline{Y}\right), \quad \overline{X}, \overline{Y} \in \Gamma\left(T\overline{M}\right). \tag{22}$$

Then the 1-form η and tensor $\overline{H}$ are related as follows.

Lemma 1. *Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be indefinite nearly cosymplectic. Then,*

$$d\eta\left(\overline{X}, \overline{Y}\right) = \overline{g}\left(\overline{X}, \overline{H}\,\overline{Y}\right), \quad \forall \overline{X}, \overline{Y} \in \Gamma\left(T\overline{M}\right). \tag{23}$$

Moreover, $\overline{M}$ is cosymplectic if and only if $\overline{H}$ vanishes identically on $\overline{M}$.

Note that, for all $\overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M})$,

$$\overline{g}\left(\left(\overline{\nabla}_{\overline{Z}}\overline{\phi}\right)\overline{X}, \overline{Y}\right) = -\overline{g}\left(\overline{X}, \left(\overline{\nabla}_{\overline{Z}}\overline{\phi}\right)\overline{Y}\right), \tag{24}$$

which means that the tensor $\overline{\nabla}\,\overline{\phi}$ is skew-symmetric. The following lemma is fundamental to the sequel.

Lemma 2. *Let $\overline{M}$ be a nearly cosymplectic manifold. Then*

$$\left(\overline{\nabla}_{\overline{X}}\overline{\phi}\right)\overline{\phi}\,\overline{Y} = -\overline{\phi}\left(\overline{\nabla}_{\overline{X}}\overline{\phi}\right)\overline{Y} - \overline{g}\left(\overline{Y}, \overline{H}\,\overline{X}\right)\xi - \eta\left(\overline{Y}\right)\overline{H}\,\overline{X}, \tag{25}$$

$$\left(\overline{\nabla}_{\overline{\phi}\,\overline{X}}\overline{\phi}\right)\overline{\phi}\,\overline{Y} = -\left(\overline{\nabla}_{\overline{X}}\overline{\phi}\right)\overline{Y} - \eta\left(\overline{X}\right)\overline{\phi}\,\overline{H}\,\overline{Y} + \eta\left(\overline{Y}\right)\overline{\phi}\,\overline{H}\,\overline{X}, \tag{26}$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$.

Proof. The proof follows from a straightforward calculation. □

3. Quasi-Generalized CR-Null Submanifolds

We recall some basic notions on QGCR-null submanifolds (see [15] for details).

The structure vector field ξ of an indefinite almost contact manifold $(\overline{M}, \overline{g})$ can be written according to decomposition (4) as follows:

$$\xi = \xi_S + \sum_{i=1}^{r} a_i E_i + \sum_{i=1}^{r} b_i N_i + \sum_{\alpha=r+1}^{n} c_\alpha W_\alpha, \tag{27}$$

where ξ_S is a smooth vector field of $S(TM)$ while $a_i = \eta(N_i)$, $b_i = \eta(E_i)$, and $c_\alpha = \epsilon_\alpha \eta(W_\alpha)$, all smooth functions on $\overline{M}$. Here $\epsilon_\alpha = \overline{g}(W_\alpha, W_\alpha)$.

We adopt the definition of quasi-generalized CR- (QGCR-) null submanifolds given in [15] for indefinite nearly cosymplectic manifolds.

Definition 3. Let $(M, g, S(TM), S(TM^\perp))$ be a null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. We say that M is quasi-generalized CR- (QGCR-) null submanifold of $\overline{M}$ if the following conditions are satisfied.

(i) There exist two distributions D_1 and D_2 of $\operatorname{Rad} TM$ such that

$$\operatorname{Rad} TM = D_1 \oplus D_2, \quad \overline{\phi} D_1 = D_1, \quad \overline{\phi} D_2 \subset S(TM). \tag{28}$$

(ii) There exist vector bundles D_0 and $\overline{D}$ over $S(TM)$ such that

$$S(TM) = \left\{\overline{\phi} D_2 \oplus \overline{D}\right\} \perp D_0, \quad \text{with } \overline{\phi} D_0 \subseteq D_0,\ \overline{D} = \overline{\phi}\mathcal{S} \oplus \overline{\phi}\mathcal{L}, \tag{29}$$

where D_0 is a nondegenerate distribution on M and $\mathcal{L}$ and $\mathcal{S}$ are, respectively, vector subbundles of $l\mathrm{tr}(TM)$ and $S(TM^\perp)$.

If $D_1 \neq \{0\}$, $D_0 \neq \{0\}$, $D_2 \neq \{0\}$, and $\mathcal{S} \neq \{0\}$, then M is called a *proper QGCR-null submanifold*.

A proof of the following proposition uses similar arguments as in [15].

Proposition 4. *A QGCR-null submanifold M of an indefinite nearly cosymplectic manifold $\overline{M}$ tangent to the structure vector field ξ is a GCR-null submanifold.*

Using (2), the tangent bundle of any QGCR-null submanifold, TM, can be decomposed as

$$TM = D \oplus \widehat{D}, \quad \text{with } D = D_0 \perp D_1,\ \widehat{D} = \left\{D_2 \perp \overline{\phi} D_2\right\} \oplus \overline{D}. \tag{30}$$

Unlike a GCR-null submanifold, in a QGCR-null submanifold, D is invariant with respect to $\overline{\phi}$ while $\widehat{D}$ is not generally anti-invariant.

Throughout this paper, we suppose that $(M, g, S(TM), S(TM^\perp))$ is a proper QGCR-null submanifold. From the above definition, we can easily deduce the following:

(1) Condition (i) implies that $\dim(\operatorname{Rad} TM) = s \geq 3$.

(2) Condition (ii) implies that $\dim(D) \geq 4l \geq 4$ and $\dim(D_2) = \dim(\mathcal{L})$.

Definition 5 (see [16]). A null submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be *ascreen* if the structure vector field, ξ, belongs to $\operatorname{Rad} TM \oplus l\mathrm{tr}(TM)$.

From Definition 3.3, Lemma 3.6, and Theorem 3.7 of [15], we have the following.

Theorem 6. *Let $(M, g, S(TM), S(TM^\perp))$ be an ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $\overline{M}$; then $\xi \in \Gamma(D_2 \oplus \mathcal{L})$. If M is a 3-null QGCR submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$, then M is ascreen null submanifold if and only if $\overline{\phi}\mathcal{L} = \overline{\phi} D_2$.*

Proof. The proof follows from straightforward calculation as in [15]. □

It is crucial to note the following aspects with ascreen QGCR-null submanifold: item (2) of Definition 3 implies that $\dim(D) \geq 4l \geq 4$ and $\dim(D_2) = \dim(\mathcal{L})$. Thus $\dim(M) \geq 7$ and $\dim(\overline{M}) \geq 11$, and any 7-dimensional ascreen QGCR-null submanifold is 3-null.

In what follows, we construct an ascreen QGCR-null submanifold of a special nearly cosymplectic manifold with $\overline{H} = 0$ (i.e., $\overline{M}$ is a cosymplectic manifold). Thus, let $(\mathbb{R}_q^{2m+1}, \overline{\phi}_0, \xi, \eta, \overline{g})$ denote the manifold $\mathbb{R}_q^{2m+1}$ with its usual cosymplectic structure given by

$$\eta = dz, \quad \xi = \partial z, \quad \overline{g} = \eta \otimes \eta - \sum_{i=1}^{q/2}\left(dx^i \otimes dx^i + dy^i \otimes dy^i\right)$$

$$+\sum_{i=q+1}^{m}\left(dx^i\otimes dx^i+dy^i\otimes dy^i\right),$$
$$\phi_0\left(\sum_{i=1}^{m}\left(X_i\partial x^i+Y_i\partial y^i\right)+Z\partial z\right)=\sum_{i=1}^{m}\left(Y_i\partial x^i-X_i\partial y^i\right), \tag{31}$$

where (x^i, y^i, z) are Cartesian coordinates and $\partial t_k = \partial/\partial t^k$, for $t \in \mathbb{R}^{2m+1}$.

Now, we use the above structure to construct the following example.

Example 7. Let $\overline{M} = (\mathbb{R}_4^{11}, \overline{g})$ be a semi-Euclidean space, with $\overline{g}$ being of signature $(-,-,+,+,+,-,-,+,+,+,+)$ with respect to the canonical basis

$$\left(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z\right). \tag{32}$$

Let (M, g) be a submanifold of $\overline{M}$ given by

$$\begin{aligned} x^1 &= y^4,\\ y^1 &= -x^4,\\ z &= x^2\sin\theta + y^2\cos\theta,\\ y^5 &= \left(x^5\right)^{1/2}, \end{aligned} \tag{33}$$

where $\theta \in (0, \pi/2)$. By direct calculations, we can see that the vector fields

$$\begin{aligned} E_1 &= \partial x_4 + \partial y_1,\\ E_2 &= \partial x_1 - \partial y_4,\\ E_3 &= \sin\theta\partial x_2 + \cos\theta\partial y_2 + \partial z,\\ X_1 &= 2y^5\partial x_5 + \partial y_5,\\ X_2 &= -\cos\theta\partial x_2 + \sin\theta\partial y_2,\\ X_3 &= \partial y_3,\\ X_4 &= \partial x_3, \end{aligned} \tag{34}$$

form a local frame of TM. Then $\operatorname{Rad}TM$ is spanned by $\{E_1, E_2, E_3\}$, and, therefore, M is 3-null. Further, $\overline{\phi}_0 E_1 = E_2$; therefore we set $D_1 = \operatorname{Span}\{E_1, E_2\}$. Also $\overline{\phi}_0 E_3 = -X_2$ and thus $D_2 = \operatorname{Span}\{E_3\}$. It is easy to see that $\overline{\phi}_0 X_3 = X_4$, so we set $D_0 = \operatorname{Span}\{X_3, X_4\}$. On the other hand, following direct calculations, we have

$$\begin{aligned} N_1 &= \frac{1}{2}\left(\partial x_4 - \partial y_1\right),\\ N_2 &= \frac{1}{2}\left(-\partial x_1 - \partial y_4\right),\\ N_3 &= \frac{1}{2}\left(-\sin\theta\partial x_2 - \cos\theta\partial y_2 + \partial z\right),\\ W &= \partial x_5 - 2y^5\partial y_5, \end{aligned} \tag{35}$$

from which $l\operatorname{tr}(TM) = \operatorname{Span}\{N_1, N_2, N_3\}$ and $S(TM^{\perp}) = \operatorname{Span}\{W\}$. Clearly, $\overline{\phi}_0 N_2 = -N_1$. Further, $\overline{\phi}_0 N_3 = (1/2)X_2$ and thus $\mathscr{L} = \operatorname{Span}\{N_3\}$. Notice that $\overline{\phi}_0 N_3 = -(1/2)\overline{\phi}_0 E_3$ and therefore $\overline{\phi}_0\mathscr{L} = \overline{\phi}_0 D_2$. Also, $\overline{\phi}_0 W = -X_1$ and therefore $\mathscr{S} = \operatorname{Span}\{W\}$. Finally, we calculate ξ as follows. Using Theorem 6 we have $\xi = aE_3 + bN_3$. Applying $\overline{\phi}_0$ to this equation we obtain $a\overline{\phi}_0 E_3 + b\overline{\phi}_0 N_3 = 0$. Now, substituting for $\overline{\phi}_0 E_3$ and $\overline{\phi}_0 N_3$ in this equation we get $2a = b$, from which we get $\xi = (1/2)(E_3 + 2N_3)$. Since $\overline{\phi}_0\xi = 0$ and $\overline{g}(\xi,\xi) = 1$, we conclude that (M, g) is an ascreen QGCR-null submanifold of $\overline{M}$.

Proposition 8. *There exist no coisotropic, isotropic, or totally null proper QGCR-null submanifolds of an indefinite nearly cosymplectic manifold.*

4. Umbilical and Geodesic Ascreen QGCR-Null Submanifolds

In this section, we prove two main theorems concerning totally umbilical, totally geodesic and irrotational ascreen QGCR-null submanifolds of $\overline{M}$. An indefinite nearly cosymplectic manifold $\overline{M}$ is called an *indefinite nearly cosymplectic space form*, denoted by $\overline{M}(\overline{c})$, if it has the constant $\overline{\phi}$-sectional curvature $\overline{c}$. The curvature tensor $\overline{R}$ of the indefinite nearly cosymplectic space form $\overline{M}(\overline{c})$ is given by [21]:

$$\begin{aligned} 4\overline{R}\left(\overline{X},\overline{W},\overline{Z},\overline{Y}\right) &= \overline{g}\left(\left(\overline{\nabla}_{\overline{W}}\overline{\phi}\right)\overline{Z}, \left(\overline{\nabla}_{\overline{X}}\overline{\phi}\right)\overline{Y}\right)\\ &- \overline{g}\left(\left(\overline{\nabla}_{\overline{W}}\overline{\phi}\right)\overline{Y}, \left(\overline{\nabla}_{\overline{X}}\overline{\phi}\right)\overline{Z}\right)\\ &- 2\overline{g}\left(\left(\overline{\nabla}_{\overline{W}}\overline{\phi}\right)\overline{X}, \left(\overline{\nabla}_{\overline{Y}}\overline{\phi}\right)\overline{Z}\right) + \overline{g}\left(\overline{H}\,\overline{W},\overline{Z}\right)\\ &\cdot \overline{g}\left(\overline{H}\,\overline{X},\overline{Y}\right) - \overline{g}\left(\overline{H}\,\overline{W},\overline{Y}\right)\overline{g}\left(\overline{H}\,\overline{X},\overline{Z}\right)\\ &- 2\overline{g}\left(\overline{H}\,\overline{W},\overline{X}\right)\overline{g}\left(\overline{H}\,\overline{Y},\overline{Z}\right) - \eta\left(\overline{W}\right)\eta\left(\overline{Y}\right)\\ &\cdot \overline{g}\left(\overline{H}\,\overline{X},\overline{H}\,\overline{Z}\right) + \eta\left(\overline{W}\right)\eta\left(\overline{Z}\right)\overline{g}\left(\overline{H}\,\overline{X},\overline{H}\,\overline{Y}\right)\\ &+ \eta\left(\overline{X}\right)\eta\left(\overline{Y}\right)\overline{g}\left(\overline{H}\,\overline{W},\overline{H}\,\overline{Z}\right) - \eta\left(\overline{X}\right)\eta\left(\overline{Z}\right)\\ &\cdot \overline{g}\left(\overline{H}\,\overline{W},\overline{H}\,\overline{Y}\right) + \overline{c}\left\{\overline{g}\left(\overline{X},\overline{Y}\right)\overline{g}\left(\overline{Z},\overline{W}\right)\right.\\ &- \overline{g}\left(\overline{Z},\overline{X}\right)\overline{g}\left(\overline{Y},\overline{W}\right) + \eta\left(\overline{Z}\right)\eta\left(\overline{X}\right)\overline{g}\left(\overline{Y},\overline{W}\right)\\ &- \eta\left(\overline{Y}\right)\eta\left(\overline{X}\right)\overline{g}\left(\overline{Z},\overline{W}\right) + \eta\left(\overline{Y}\right)\eta\left(\overline{W}\right)\overline{g}\left(\overline{Z},\overline{X}\right)\\ &- \eta\left(\overline{Z}\right)\eta\left(\overline{W}\right)\overline{g}\left(\overline{Y},\overline{X}\right) + \overline{g}\left(\overline{\phi}\,\overline{Y},\overline{X}\right)\overline{g}\left(\overline{\phi}\,\overline{Z},\overline{W}\right)\\ &- \overline{g}\left(\overline{\phi}\,\overline{Z},\overline{X}\right)\overline{g}\left(\overline{\phi}\,\overline{Y},\overline{W}\right)\\ &\left.- 2\overline{g}\left(\overline{\phi}\,\overline{Z},\overline{Y}\right)\overline{g}\left(\overline{\phi}\,\overline{X},\overline{W}\right)\right\}, \end{aligned} \tag{36}$$

for all $\overline{X}, \overline{Y}, \overline{Z}, \overline{W} \in \Gamma(T\overline{M})$.

Notice that D_0 and $\overline{\phi}\mathcal{S}$ are orthogonal and nondegenerate subbundles of TM and that when M is ascreen QGCR-null submanifold, we observe that

$$\eta(X) = \eta(Z) = 0, \quad \forall X \in \Gamma(D_0), \; Z \in \Gamma(\overline{\phi}\mathcal{S}). \tag{37}$$

Theorem 9. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical or totally geodesic ascreen QGCR-null submanifold of an indefinite nearly cosymplectic space form $\overline{M}(\overline{c})$, of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$, such that D_0 and $\overline{\phi}\mathcal{S}$ are space-like and parallel distributions with respect to ∇. Then, $\overline{c} \geq 0$. Equality occurs when $\overline{M}(\overline{c})$ is an indefinite cosymplectic space form.*

Proof. Let X and Z be vector fields in D_0 and $\overline{\phi}\mathcal{S}$, respectively. Replacing $\overline{W}$ with $\overline{\phi}X$ and $\overline{Y}$ with $\overline{\phi}Z$ in (36), we get

$$\begin{aligned} 4\overline{R}(X, \overline{\phi}X, Z, \overline{\phi}Z) &= \overline{g}\left(\left(\overline{\nabla}_{\overline{\phi}X}\overline{\phi}\right)Z, \left(\overline{\nabla}_X\overline{\phi}\right)\overline{\phi}Z\right) \\ &\quad - \overline{g}\left(\left(\overline{\nabla}_{\overline{\phi}X}\overline{\phi}\right)\overline{\phi}Z, \left(\overline{\nabla}_X\overline{\phi}\right)Z\right) \\ &\quad - 2\overline{g}\left(\left(\overline{\nabla}_{\overline{\phi}X}\overline{\phi}\right)X, \left(\overline{\nabla}_{\overline{\phi}Z}\overline{\phi}\right)Z\right) \\ &\quad + \overline{g}\left(\overline{H}\overline{\phi}X, Z\right)\overline{g}\left(\overline{H}X, \overline{\phi}Z\right) \\ &\quad - \overline{g}\left(\overline{H}\overline{\phi}X, \overline{\phi}Z\right)\overline{g}\left(\overline{H}X, Z\right) \\ &\quad - 2\overline{g}\left(\overline{H}\overline{\phi}X, X\right)\overline{g}\left(\overline{H}\overline{\phi}Z, Z\right) \\ &\quad - 2\overline{c}g\left(\overline{\phi}Z, \overline{\phi}Z\right)g\left(\overline{\phi}X, \overline{\phi}X\right). \end{aligned} \tag{38}$$

Considering the first three terms on the right hand side of (38), we have

$$\begin{aligned} &\overline{g}\left(\left(\overline{\nabla}_{\overline{\phi}X}\overline{\phi}\right)Z, \left(\overline{\nabla}_X\overline{\phi}\right)\overline{\phi}Z\right) \\ &\quad = -\overline{g}\left(\left(\overline{\nabla}_Z\overline{\phi}\right)\overline{\phi}X, \left(\overline{\nabla}_X\overline{\phi}\right)\overline{\phi}Z\right). \end{aligned} \tag{39}$$

Applying (25) of Lemma 2 on (39) we derive

$$\begin{aligned} &\overline{g}\left(\left(\overline{\nabla}_{\overline{\phi}X}\overline{\phi}\right)Z, \left(\overline{\nabla}_X\overline{\phi}\right)\overline{\phi}Z\right) \\ &\quad = -\overline{g}\left(\left(\overline{\nabla}_Z\overline{\phi}\right)\overline{\phi}X, \left(\overline{\nabla}_X\overline{\phi}\right)\overline{\phi}Z\right) \\ &\quad = \overline{g}\left(\left(\overline{\nabla}_X\overline{\phi}\right)Z, \left(\overline{\nabla}_X\overline{\phi}\right)Z\right) - \overline{g}\left(\overline{\phi}Z, \overline{H}X\right)^2 \\ &\qquad + \overline{g}\left(Z, \overline{H}X\right)^2. \end{aligned} \tag{40}$$

In a similar way, using (26) of Lemma 2, we get

$$\begin{aligned} &-\overline{g}\left(\left(\overline{\nabla}_{\overline{\phi}X}\overline{\phi}\right)\overline{\phi}Z, \left(\overline{\nabla}_X\overline{\phi}\right)Z\right) \\ &\quad = \overline{g}\left(\left(\overline{\nabla}_X\overline{\phi}\right)Z, \left(\overline{\nabla}_X\overline{\phi}\right)Z\right), \\ &-2\overline{g}\left(\left(\overline{\nabla}_{\overline{\phi}X}\overline{\phi}\right)X, \left(\overline{\nabla}_{\overline{\phi}Z}\overline{\phi}\right)Z\right) = 0. \end{aligned} \tag{41}$$

Now substituting (40) and (41) into (38), we get

$$\begin{aligned} 4\overline{R}(X, \overline{\phi}X, Z, \overline{\phi}Z) &= 2\overline{g}\left(\left(\overline{\nabla}_X\overline{\phi}\right)Z, \left(\overline{\nabla}_X\overline{\phi}\right)Z\right) \\ &\quad - \overline{g}\left(\overline{\phi}Z, \overline{H}X\right)^2 + \overline{g}\left(Z, \overline{H}X\right)^2 \\ &\quad + \overline{g}\left(\overline{H}\overline{\phi}X, Z\right)\overline{g}\left(\overline{H}X, \overline{\phi}Z\right) \\ &\quad - \overline{g}\left(\overline{H}\overline{\phi}X, \overline{\phi}Z\right)\overline{g}\left(\overline{H}X, Z\right) \\ &\quad - 2\overline{g}\left(\overline{H}\overline{\phi}X, X\right)\overline{g}\left(\overline{H}\overline{\phi}Z, Z\right) \\ &\quad - 2\overline{c}g\left(\overline{\phi}Z, \overline{\phi}Z\right)g\left(\overline{\phi}X, \overline{\phi}X\right), \end{aligned} \tag{42}$$

from which we obtain

$$\begin{aligned} 2\overline{R}(X, \overline{\phi}X, Z, \overline{\phi}Z) &= \overline{g}\left(\left(\overline{\nabla}_X\overline{\phi}\right)Z, \left(\overline{\nabla}_X\overline{\phi}\right)Z\right) \\ &\quad + \overline{g}\left(Z, \overline{H}X\right)^2 \\ &\quad - \overline{c}g(Z, Z)g(X, X). \end{aligned} \tag{43}$$

Then using the facts that D_0 and $\overline{\phi}\mathcal{S}$ are space-like and parallel with respect to ∇, we have $(\overline{\nabla}_Z\overline{\phi})X = (\nabla_Z\overline{\phi})X \in \Gamma(D_0)$, and (43) reduces to

$$\begin{aligned} 2\overline{R}(X, \overline{\phi}X, Z, \overline{\phi}Z) &= \left\|\left(\nabla_Z\overline{\phi}\right)X\right\|^2 + \overline{g}\left(Z, \overline{H}X\right)^2 \\ &\quad - \overline{c}\,\|X\|^2\,\|Z\|^2, \end{aligned} \tag{44}$$

where $\|\cdot\|$ denotes the norm on $D_0 \perp \overline{\phi}\mathcal{S}$ with respect to g.

On the other hand, if we set $\overline{W} = \overline{\phi}X$ and $\overline{Y} = \overline{\phi}Z$ in (14), we have

$$\begin{aligned} \overline{R}(X, \overline{\phi}X, Z, \overline{\phi}Z) &= \overline{g}\left(\left(\nabla_X h^s\right)\left(\overline{\phi}X, Z\right), \overline{\phi}Z\right) \\ &\quad - \overline{g}\left(\left(\nabla_{\overline{\phi}X} h^s\right)(X, Z), \overline{\phi}Z\right), \end{aligned} \tag{45}$$

where

$$\begin{aligned} \left(\nabla_X h^s\right)\left(\overline{\phi}X, Z\right) &= \nabla^s_X h^s\left(\overline{\phi}X, Z\right) - h^s\left(\nabla_X\overline{\phi}X, Z\right) \\ &\quad - h^s\left(\overline{\phi}X, \nabla_X Z\right). \end{aligned} \tag{46}$$

By the fact that M is totally umbilical in $\overline{M}$, we have $h^s(\overline{\phi}X, Z) = 0$. Thus using (16), (46) becomes

$$\begin{aligned} \left(\nabla_X h^s\right)\left(\overline{\phi}X, Z\right) &= -h^s\left(\nabla_X\overline{\phi}X, Z\right) - h^s\left(\overline{\phi}X, \nabla_X Z\right) \\ &= -g\left(\nabla_X\overline{\phi}X, Z\right)\mathcal{H}^s \\ &\quad - g\left(\overline{\phi}X, \nabla_X Z\right)\mathcal{H}^s. \end{aligned} \tag{47}$$

Differentiating $\overline{g}(\overline{\phi}X, Z) = 0$ covariantly with respect to X and then applying (7), we obtain

$$g\left(\nabla_X\overline{\phi}X, Z\right) + g\left(\overline{\phi}X, \nabla_X Z\right) = 0. \tag{48}$$

Substituting (48) into (47) gives

$$\left(\nabla_X h^s\right)\left(\overline{\phi}X, Z\right) = 0. \tag{49}$$

Similarly,

$$\left(\nabla_{\overline{\phi}X} h^s\right)(X, Z) = 0. \tag{50}$$

Then, substituting (49) and (50) into (45), we get

$$\overline{R}\left(X, \overline{\phi}X, Z, \overline{\phi}Z\right) = 0. \tag{51}$$

Substituting (51) into (44), gives

$$\overline{c}\,\|X\|^2\,\|Z\|^2 = \left\|\left(\nabla_Z\overline{\phi}\right)X\right\|^2 + \overline{g}\left(Z, \overline{H}X\right)^2 \geq 0, \tag{52}$$

which implies that $\overline{c} \geq 0$. When the ambient manifold is cosymplectic, then $\nabla\overline{\phi} = 0$ and $d\eta = 0$ [19] and in this case $\overline{c} = 0$. □

Example 10. Let M be an ascreen QGCR-null submanifold in Example 7. Applying (7) and Koszul's formula (see [2]) to Example 7 we obtain

$$\begin{aligned} &h_i^l(X, Y) = 0 \\ &\qquad\qquad \forall X, Y \in \Gamma(TM), \text{ where } i = 1, 2, 3, \\ &\epsilon_4 h_4^s(X_1, X_1) = 2, \\ &h_4^s(X, Y) = 0, \\ &\qquad\qquad \forall X \neq X_1,\ Y \neq X_1. \end{aligned} \tag{53}$$

Using (12), (53), and $\epsilon_4 = \overline{g}(W, W) = 1+4(y^5)^2$, we also derive

$$h(X_1, X_1) = \frac{2}{1 + 4\left(y^5\right)^2}W. \tag{54}$$

We remark that M is not totally geodesic. From (54) and (15) we note that M is totally umbilical with

$$\mathscr{H} = \frac{2}{\left(1 + 4\left(y^5\right)^2\right)^2}W. \tag{55}$$

By straightforward calculations we also have

$$\begin{aligned} &\nabla_{X_1}X_1 = 4y^5X_1, \\ &\nabla_{X_i}X_j = 0 \quad \forall i, j \neq 1. \end{aligned} \tag{56}$$

Thus, D_0 and $\overline{\phi}\mathcal{S}$ are parallel distributions with respect to ∇. Hence, M satisfies Theorem 9 and $\overline{c} = 0$.

Corollary 11. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical or totally geodesic ascreen QGCR-null submanifold of an indefinite cosymplectic space form $\overline{M}(\overline{c})$ of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$. Then, $\overline{c} = 0$.*

A null submanifold M of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called irrotational [3] if $\overline{\nabla}_X E \in \Gamma(TM)$, for any $E \in \Gamma(\operatorname{Rad} TM)$ and $X \in \Gamma(TM)$. Equivalently, M is irrotational if

$$h^l(X, E) = h^s(X, E) = 0, \tag{57}$$

for all $X \in \Gamma(TM)$ and $E \in \Gamma(\operatorname{Rad} TM)$.

Theorem 12. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an irrotational ascreen QGCR-null submanifold of an indefinite nearly cosymplectic space form $\overline{M}(c)$ of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$. Then, $\overline{c} \leq 0$ or $\overline{c} \geq 0$. Equality holds when $\overline{M}(\overline{c})$ is an indefinite cosymplectic space form.*

Proof. By setting $Y = E$, $Z = \overline{\phi}E$, and $W = E$ in (14), we get

$$\begin{aligned} \overline{R}\left(X, E, \overline{\phi}E, E\right) = {} & \overline{g}\left(\left(\nabla_X h^l\right)\left(E, \overline{\phi}E\right), E\right) \\ & - \overline{g}\left(\left(\nabla_E h^l\right)\left(X, \overline{\phi}E\right), E\right) \\ & + \overline{g}\left(\left(\nabla_X h^s\right)\left(E, \overline{\phi}E\right), E\right) \\ & - \overline{g}\left(\left(\nabla_E h^s\right)\left(X, \overline{\phi}E\right), E\right) \end{aligned} \tag{58}$$

for any $X \in \Gamma(TM)$ and $E \in \Gamma(\operatorname{Rad} TM)$. Then, using the fact that M is irrotational, (58) reduces to

$$\overline{R}\left(X, E, \overline{\phi}E, E\right) = 0, \quad \forall X \in \Gamma(TM). \tag{59}$$

On the other hand, setting $\overline{Y} = \overline{W} = E$ and $\overline{Z} = \overline{\phi}E$ in (36) and simplifying, we get

$$\begin{aligned} \overline{R}\left(X, E, \overline{\phi}E, E\right) = {} & -3\overline{g}\left(\left(\overline{\nabla}_E\overline{\phi}\right)\overline{\phi}E, \left(\overline{\nabla}_E\overline{\phi}\right)X\right) \\ & - \eta(E)^2\,\overline{g}\left(\overline{H}X, \overline{H}\,\overline{\phi}E\right) \\ & + 4\overline{c}\eta(E)^2\,\overline{g}\left(X, \overline{\phi}E\right). \end{aligned} \tag{60}$$

Now, using (59) and (60), we get

$$\begin{aligned} 4\overline{c}\eta(E)^2\,\overline{g}\left(X, \overline{\phi}E\right) = {} & 3\overline{g}\left(\left(\overline{\nabla}_E\overline{\phi}\right)\overline{\phi}E, \left(\overline{\nabla}_E\overline{\phi}\right)X\right) \\ & + \eta(E)^2\,\overline{g}\left(\overline{H}X, \overline{H}\,\overline{\phi}E\right). \end{aligned} \tag{61}$$

Replacing X with $\overline{\phi}E$ in (61) and using (25) of Lemma 2 to the resulting equation gives

$$\overline{c}\eta(E)^2\,\overline{g}\left(\overline{\phi}E, \overline{\phi}E\right) = \eta(E)^2\,\overline{g}\left(\overline{H}\,\overline{\phi}E, \overline{H}\,\overline{\phi}E\right). \tag{62}$$

Since M is ascreen QGCR-null submanifold, there exists $E \in \Gamma(D_2)$ such that $\eta(E) = b \neq 0$, and thus (62) simplifies to

$$\overline{c} = -\frac{1}{b^2}\overline{g}\left(\overline{H}E, \overline{H}E\right) = \frac{1}{b^2}d\eta\left(E, \overline{H}E\right). \tag{63}$$

We observe that $\overline{c} = 0$ if either $d\eta = 0$ (i.e., $\overline{M}(c)$ is cosymplectic space form [19]) or $\overline{H}E$ is a null vector field. The second case implies that $\overline{H}E$ belongs to $\operatorname{Rad} TM$ or $l\operatorname{tr}(TM)$. If

$\overline{H}E \in \Gamma(\mathrm{Rad}\,TM)$, then there exists a nonzero smooth function κ such that $\overline{H}E = \kappa E$, for some arbitrary $E \in \Gamma(\mathrm{Rad}\,TM)$. Taking the $\overline{g}$-product of $\overline{H}E = \kappa E$ with ξ leads to $0 = \kappa\eta(E)$, from which $\eta(E) = 0$. Since M is ascreen QGCR-null submanifold, then, there is $E \in \Gamma(D_2)$ such that $\eta(E) \neq 0$, hence a contradiction. Similar reasoning can be applied if $\overline{H}E \in \Gamma(l\mathrm{tr}(TM))$. Therefore, $\overline{c} = 0$ only if $\overline{H}E = 0$ (i.e., $d\eta = 0$) which occurs when $\overline{M}(c)$ is cosymplectic space form [19]. It turns out that $\overline{c} \leq 0$ or $\overline{c} \geq 0$ depending on whether $\overline{H}E$ is space-like or time-like vector field, respectively. □

Corollary 13. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an irrotational ascreen QGCR-null submanifold of an indefinite cosymplectic space form $\overline{M}(\overline{c})$ of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$. Then, $\overline{c} = 0$.*

It is easy to see from (54) that $h^l(X, E) = h^s(X, E) = 0$ and hence M given in Example 10 is an irrotational ascreen QGCR-null submanifold of an indefinite cosymplectic space form $\overline{M}(\overline{c})$. As is proved in that example $\overline{c} = 0$.

5. Mixed Totally Geodesic QGCR-Null Submanifolds

Definition 14. A QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$ is called *mixed totally geodesic* if its second fundamental form, h, satisfies $h(X, Y) = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(\widehat{D})$.

We will need the following lemma in the next theorem.

Lemma 15. *Let $(M, g, S(TM), S(TM^{\perp}))$ be any 3-null proper ascreen QG CR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. Then,*

$$2\eta(E)\,\eta(N) = 1, \tag{64}$$

for any $E \in \Gamma(D_2)$ and $N \in \Gamma(\mathscr{L})$.

Proof. The proof follows from straightforward calculations using $\overline{g}(\xi, \xi) = 1$ and $\xi = \eta(N)E + \eta(E)N$. □

Theorem 16. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a 3-null proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. Then, M is mixed totally geodesic if and only if $h^s_\alpha(X, Y) = 0$ and $A^*_{E_i}X = 0$, for all $X \in \Gamma(D)$, $Y \in \Gamma(\widehat{D})$, $W_\alpha \in \Gamma(S(TM^{\perp}))$, and $E_i \in \Gamma(\mathrm{Rad}\,TM)$.*

Proof. By the definition of ascreen QGCR-null submanifold, M is mixed geodesic if

$$\overline{g}\left(h(X, Y), W_\alpha\right) = \overline{g}\left(h(X, Y), E_i\right) = 0, \tag{65}$$

for all $X \in \Gamma(D)$, $Y \in \Gamma(\widehat{D})$, $W_\alpha \in \Gamma(S(TM^{\perp}))$, and $E_i \in \Gamma(\mathrm{Rad}\,TM)$. Now, by virtue of (12) and the first equation of (65), we have

$$0 = \overline{g}\left(h(X, Y), W_\alpha\right) = \epsilon_\alpha h^s_\alpha(X, Y), \tag{66}$$

from which $h^s_\alpha(X, Y) = 0$, since $\epsilon_\alpha \neq 0$. On the other hand, using (65), (7), and (11) we derive

$$\begin{aligned}\overline{g}\left(h(X, Y), E_i\right) &= \overline{g}\left(\overline{\nabla}_X Y, E_i\right) = -\overline{g}\left(Y, \overline{\nabla}_X E_i\right)\\ &= g\left(Y, A^*_{E_i}X\right) = 0.\end{aligned} \tag{67}$$

Since $D = D_0 \perp D_1$ and $\widehat{D} = \{D_2 \perp \overline{\phi}D_2\} \oplus \overline{D}$, we observe that $A^*_{E_i}X \notin \Gamma(\overline{\phi}D_2)$ or $\overline{\phi}\mathscr{L}$. In fact, suppose that $A^*_{E_i}X \notin \Gamma(\overline{\phi}D_2)$; then there exists a nonvanishing smooth function ℓ such that $A^*_{E_i}X = \ell\overline{\phi}E$, for $E \in \Gamma(D_2)$. Thus,

$$0 = g\left(Y, A^*_{E_i}X\right) = \ell g\left(Y, \overline{\phi}E\right), \quad \forall Y \in \Gamma\left(\widehat{D}\right). \tag{68}$$

Taking $Y = \overline{\phi}N$ in (68), where $N \in \Gamma(\mathscr{L})$ and using Lemma 15, we have

$$\begin{aligned}0 &= g\left(Y, A^*_{E_i}X\right) = \ell g\left(\overline{\phi}N, \overline{\phi}E\right) = \ell\left(1 - \eta(E)\,\eta(N)\right)\\ &= \frac{1}{2}\ell,\end{aligned} \tag{69}$$

which is a contradiction, since $\ell \neq 0$. Hence $A^*_{E_i}X \notin \Gamma(\overline{\phi}D_2 \oplus \overline{\phi}\mathscr{L})$. Moreover, $A^*_{E_i}X \notin \Gamma(\overline{\phi}\mathscr{S})$ since if $A^*_{E_i}X \in \Gamma(\overline{\phi}\mathscr{S})$, then there is a nonvanishing smooth function ω such that $A^*_{E_i}X = \omega\overline{\phi}W_\alpha$. Taking the $\overline{g}$-product of this equation with respect to $Y = \overline{\phi}W_\alpha$ and using the fact that $\eta(W_\alpha) = 0$, we get

$$\begin{aligned}0 &= g\left(Y, A^*_{E_i}X\right) = \omega\overline{g}\left(\overline{\phi}W_\alpha, \overline{\phi}W_\alpha\right) = \omega\overline{g}\left(W_\alpha, W_\alpha\right)\\ &= \omega\epsilon_\alpha,\end{aligned} \tag{70}$$

which is a contradiction, since $\epsilon_\alpha \neq 0$ and $\omega \neq 0$. Hence, $A^*_{E_i}X \notin \Gamma(\{\overline{\phi}D_2 \oplus \overline{\phi}\mathscr{L}\} \perp \overline{\phi}\mathscr{S})$, which implies that $A^*_{E_i}X \in \Gamma(D_0)$. Since $A^*_{E_i}X \in \Gamma(D_0)$, then the nondegeneracy of D_0 implies that there exists some $Z \in \Gamma(D_0)$ such that $g(A^*_{E_i}X, Z) \neq 0$. But using (11) and (7), together with the fact that M is mixed geodesic, we derive

$$\begin{aligned}g\left(A^*_{E_i}X, Z\right) &= -g\left(\nabla_X E_i, Z\right) = \overline{g}\left(E_i, \overline{\nabla}_X Z\right)\\ &= \overline{g}\left(E_i, \nabla_X Z\right) = 0,\end{aligned} \tag{71}$$

which is a contradiction. Thus $A^*_{E_i}X \notin \Gamma(\{\overline{\phi}D_2 \oplus \overline{\phi}\mathscr{L}\} \perp \overline{\phi}\mathscr{S} \perp D_0)$; that is, $A^*_{E_i}X = 0$. The converse is obvious. □

Corollary 17. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. Then, if M is mixed totally geodesic then $h^l_i(X, E_i) = 0$ and $\varphi_{\alpha i}(X) = 0$, for all $X \in \Gamma(D)$ and $E_i \in \Gamma(D_2)$.*

Definition 18. A QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$ is called D-totally geodesic if its second fundamental form h satisfies $h(X, Y) = 0$, $\forall X, Y \in \Gamma(D)$.

Since M is ascreen QGCR-null submanifold, we have $\overline{g}(X,\xi) = 0$, for all $X \in \Gamma(D)$. Applying $\overline{\nabla}_Y$ to $\overline{g}(X,\xi) = 0$ we get

$$\eta\left(\overline{\nabla}_Y X\right) = -\overline{g}\left(X, \overline{\nabla}_Y \xi\right) = \overline{g}\left(X, \overline{H}Y\right). \tag{72}$$

Interchanging X and Y in (72) and then adding the resulting equation to (72) gives

$$\eta\left(\overline{\nabla}_X Y\right) + \eta\left(\overline{\nabla}_Y X\right) = \overline{g}\left(Y, \overline{H}X\right) + \overline{g}\left(X, \overline{H}Y\right) = 0. \tag{73}$$

Theorem 19. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. Then, M is D-totally geodesic if and only if $\overline{\phi}h^l(X, \overline{\phi}E)$ and $\overline{\phi}h^s(X, \overline{\phi}W)$, respectively, have no components along $l\mathrm{tr}(TM)$ and $S(TM^{\perp})$, while both $\nabla_X\overline{\phi}E$ and $\nabla_X\overline{\phi}W \notin \Gamma(D_0)$ for all $X \in \Gamma(D)$, $E \in \Gamma(\mathrm{Rad}\,TM)$, and $W \in \Gamma(\mathcal{S})$.*

Proof. By the definition of ascreen QGCR-null submanifold, M is D geodesic if and only if $\overline{g}(h(X,Y),E) = \overline{g}(h(X,Y),W) = 0$, for all $X, Y \in \Gamma(D)$, $W_\alpha \in \Gamma(S(TM^{\perp}))$, and $E \in \Gamma(\mathrm{Rad}\,TM)$.

Using (7) and (18), we derive

$$\begin{aligned}\overline{g}\left(h\left(X,Y\right),E\right) &= \overline{g}\left(\overline{\nabla}_X Y, E\right)\\ &= \overline{g}\left(\overline{\phi}\,\overline{\nabla}_X Y, \overline{\phi}E\right)\\ &\quad - \overline{g}\left(Y, \overline{\nabla}_X \xi\right)\overline{g}\left(E,\xi\right),\end{aligned} \tag{74}$$

from which when we apply (20) we get

$$\begin{aligned}\overline{g}\left(h\left(X,Y\right),E\right) &= \overline{g}\left(\overline{\phi}\,\overline{\nabla}_X Y, \overline{\phi}E\right)\\ &\quad + \overline{g}\left(Y, \overline{H}X\right)\overline{g}\left(E,\xi\right).\end{aligned} \tag{75}$$

Interchanging X and Y in (75) and considering the fact that h is symmetric we get

$$\begin{aligned}\overline{g}\left(h\left(X,Y\right),E\right) &= \overline{g}\left(\overline{\phi}\,\overline{\nabla}_Y X, \overline{\phi}E\right)\\ &\quad + \overline{g}\left(X, \overline{H}Y\right)\overline{g}\left(E,\xi\right).\end{aligned} \tag{76}$$

Summing (75) and (76) and then applying (73), we have

$$2\overline{g}\left(h\left(X,Y\right),E\right) = \overline{g}\left(\overline{\phi}\,\overline{\nabla}_X Y, \overline{\phi}E\right) + \overline{g}\left(\overline{\phi}\,\overline{\nabla}_Y X, \overline{\phi}E\right). \tag{77}$$

Now, applying the nearly cosymplectic condition in (19) to (77) leads to

$$2\overline{g}\left(h\left(X,Y\right),E\right) = \overline{g}\left(\overline{\nabla}_X\overline{\phi}Y, \overline{\phi}E\right) + \overline{g}\left(\overline{\nabla}_Y\overline{\phi}X, \overline{\phi}E\right). \tag{78}$$

From (78) and (7) we derive

$$\begin{aligned}2\overline{g}\left(h\left(X,Y\right),E\right) &= \overline{g}\left(\overline{\nabla}_X\overline{\phi}Y, \overline{\phi}E\right) + \overline{g}\left(\overline{\nabla}_Y\overline{\phi}X, \overline{\phi}E\right)\\ &\quad - \overline{g}\left(\overline{\phi}Y, h\left(X, \overline{\phi}E\right)\right)\\ &\quad - \overline{g}\left(\overline{\phi}X, h\left(Y, \overline{\phi}E\right)\right).\end{aligned} \tag{79}$$

If we let $X, Y \in \Gamma(D_1)$ in (79), we obtain

$$\begin{aligned}2\overline{g}\left(h\left(X,Y\right),E\right) &= \overline{g}\left(Y, \overline{\phi}h\left(X, \overline{\phi}E\right)\right)\\ &\quad + \overline{g}\left(X, \overline{\phi}h\left(Y, \overline{\phi}E\right)\right).\end{aligned} \tag{80}$$

On the other hand, for $X, Y \in \Gamma(D_0)$, we get

$$2\overline{g}\left(h\left(X,Y\right),E\right) = -\overline{g}\left(\overline{\phi}Y, \nabla_X\overline{\phi}E\right) - \overline{g}\left(\overline{\phi}X, \nabla_Y\overline{\phi}E\right). \tag{81}$$

It is easy to see from (80) and (81) that if $\overline{\phi}h(X, \overline{\phi}E) \notin \Gamma(l\mathrm{tr}(TM))$ and $\nabla_X\overline{\phi}E \notin \Gamma(D_0)$, then $\overline{g}(h(X,Y),E) = 0$. The other assertions follows in the same way. The converse is obvious. □

Corollary 20. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. If M is D-totally geodesic then $\nabla^*_X\overline{\phi}E$ and $\nabla^*_X\overline{\phi}W \notin \Gamma(D_0)$, for all $X \in \Gamma(D)$, $E \in \Gamma(D_2)$, and $W \in \Gamma(\mathcal{S})$.*

Corollary 21. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. If M is D-totally geodesic, then D defines a totally geodesic foliation in M.*

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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