Research Article
Some Remarks on Quasi-Generalized CR-Null Geometry in Indefinite Nearly Cosymplectic Manifolds

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Attention is drawn to some distributions on a screen Quasi-Generalized Cauchy-Riemannian (QGCR) null submanifolds in an indefinite nearly cosymplectic manifold. We characterize totally umbilical and irrotational ascreen QGCR-null submanifolds. We finally discuss the geometric effects of geodesity conditions on such submanifolds.

1. Introduction

One of the current interesting research areas in semi-Riemannian geometry is the theory of null (or lightlike) submanifolds. An intrinsic approach to the theory of null submanifolds was advanced by Kupeli [1], yet an extrinsic counterpart had to wait for Duggal and Bejancu [2], and later by Duggal and Sahin [3]. Since then, many researchers have laboured to extend their theories with evidence from the following few selected papers: [3–11] and other references therein. The rapid increase in research on this topic, since 1996, is inspired by the numerous applications of the theory to mathematical physics, particularly in general relativity. More precisely, in general relativity, null submanifolds represent different models of black hole horizons (see [2, 3] for details).

In [5], the authors initiated the study of generalized CR-(GCR-) null submanifolds of an indefinite Sasakian manifold, which are tangent to the structure vector field, $\xi$, of the almost contact structure $(\phi, \xi, \eta)$. Moreover, when $\xi$ is tangent to the submanifold, Calin [12] proved that it belongs to its screen distribution. This assumption is widely accepted and it has been applied in many papers on null contact geometry, for instance, [3, 5, 8–11, 13]. It is worth mentioning that $\xi$ is a global vector field defined on the entire tangent bundle of the ambient almost contact manifold. Thus, restricting it to the screen distribution is only one of those cases in which it can be placed. In the study of Riemannian CR-submanifolds of Sasakian manifolds, Yano and Kon [14, p. 43] proved that making $\xi$ a normal vector field in such scenario leads to an anti-invariant submanifold, and hence $\xi$ was kept tangent to the CR-submanifold. Their proof leans against the fact that the shape operator on such CR-submanifold is naturally symmetric with respect to the induced Riemannian metric $g$. On the other hand, the shape operators of any $r$-null submanifold are generally not symmetric with respect to the induced degenerate metric $g$ (see [2, 3] for details).

In an attempt to generalize $\xi$, we introduced a special class of CR-null submanifolds of a nearly Sasakian manifold, known as quasi-generalized CR- (QGCR-) null submanifold [15], for which the classical GCR-null submanifolds [3] form part. Among other benefits, generalizing $\xi$ leads to QGCR-null submanifolds of lower dimensions and with quite different geometric properties compared to respective GCR-null submanifolds.

The purpose of this paper is to investigate the geometry of distributions on a screen QGCR-null submanifolds of indefinite nearly cosymplectic manifolds. A null submanifold $M$ of an indefinite nearly cosymplectic manifold is called ascreen if the structure vector field $\xi$ belongs to $\text{Rad}(TM) \oplus \text{ltr}(TM)$ [16]. The paper is organized as follows. In Section 2, we present the basic notions of null submanifolds and nearly cosymplectic manifolds. More details can be found in [17–22]. In Section 3, we review the basic notions of QGCR-null submanifolds and we give an example of ascreen QGCR-null...
submanifold. In Section 4, we discuss totally umbilical, totally geodesic, and irrotational ascreen QGCR-null submanifolds of an indefinite nearly cosymplectic space form \( \overline{M}(\mathcal{C}) \). Finally, in Section 5 we investigate the geodesity of the distributions \( D \) and \( \overline{D} \).

2. Preliminaries

Let \( M^m \) be a codimension \( n \) submanifold of a semi-Riemannian manifold \((\overline{M}, \overline{g})\) of constant index \( \nu \), \( 1 \leq \nu \leq m+n \), with \( m, n \geq 1 \). Then, \( M \) is said to be a null submanifold of \( \overline{M} \) if the tangent and normal bundles of \( M \) have a nontrivial intersection. This intersection defines a smooth distribution on \( M \), called the radical distribution \([2]\). More precisely, consider \( p \in M \); one defines the orthogonal complement \( T_p M^\perp \) of the tangent space \( T_p M \) by

\[
T_p M^\perp = \{ X \in T_p M : \overline{g}(X, Y) = 0, \forall Y \in T_p M \} .
\]

If we denote the radical distribution on \( M \) by \( \text{Rad}_M \), then \( \text{Rad}_M \) is a null submanifold of \( \overline{M} \) if the mapping \( \text{Rad} : p \in M \mapsto \text{Rad}_p M \) defines a smooth distribution on \( M \) of rank \( r > 0 \).

Throughout the paper we consider \( \Gamma(\overline{M}) \) to be a set of smooth sections of the vector bundle \( \overline{E} \).

Consider a null transversal bundle \( S(TM^\perp) \), which is semi-Riemannian and complementary to \( \text{Rad}_M \) in \( TM^\perp \). For any local basis \( \{E_i, \ldots, E_r\} \) of \( TM^\perp \), there exists a local null frame \( \{N_1, \ldots, N_r\} \subset S(TM^\perp) \) in \( S(TM^\perp) \) such that \( g(E_i, N_j) = \delta_{ij} \) and \( \overline{g}(N_i, N_j) = 0 \). It follows that there exists a null transversal vector bundle \( \text{It}r(TM) \) locally spanned by \( \{N_1, \ldots, N_r\} \) (see details in \([2, 3]\)). If \( \text{tr}(TM) \) denotes the complementary (but not orthogonal) vector bundle to \( TM \) in \( T\overline{M} \), then

\[
\text{tr}(TM) = \text{It}r(TM) \perp S(TM^\perp) .
\]

Consider a screen transversal bundle \( S(TM) \), which is semi-Riemannian and complementary to \( \text{Rad}_M \) in \( TM^\perp \). For any local basis \( \{E_1, \ldots, E_r\} \) of \( TM^\perp \), there exists a local null frame \( \{N_1, \ldots, N_r\} \subset S(TM) \) in \( S(TM) \) such that \( g(E_i, N_j) = \delta_{ij} \) and \( \overline{g}(N_i, N_j) = 0 \). It follows that there exists a null transversal vector bundle \( \text{It}r(TM) \) locally spanned by \( \{N_1, \ldots, N_r\} \) (see details in \([2, 3]\)). If \( \text{tr}(TM) \) denotes the complementary (but not orthogonal) vector bundle to \( TM \) in \( T\overline{M} \), then

\[
\text{tr}(TM) = \text{It}r(TM) \perp S(TM) .
\]

It is important to note that the screen distribution \( S(TM) \) is not unique and is canonically isomorphic to the factor vector bundle \( TM/\text{Rad}_M \) \([1]\).

Given a null submanifold \( M \), then the following classifications of \( M \) are well-known \([2]\): (i) \( M \) is \( r \)-null if \( 1 \leq r < \min(m, n) \); (ii) \( M \) is coisotropic if \( 1 \leq r = n < m \), \( S(TM^\perp) = \{0\} \); (iii) \( M \) is isotropic if \( 1 \leq r = m < n \), \( S(TM) = \{0\} \); (iv) \( M \) is totally null if \( r = n = m \), \( S(TM) = S(TM^\perp) = \{0\} \).

Where necessary, the following range of indices will be used

\[
i, j, k \in \{1, \ldots, r\}, \quad \alpha, \beta, \gamma \in \{r + 1, \ldots, n\} .
\]

Consider a local quasi-orthonormal fields of frames of \( T\overline{M} \) along \( M \) as

\[
\{ E_1, \ldots, E_r, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m, W_{1r}, \ldots, W_{nr} \} ,
\]

where \( \{X_{r+1}, \ldots, X_m\} \) and \( \{W_{1r}, \ldots, W_{nr}\} \) are, respectively, orthonormal bases of \( \Gamma(S(TM)) \) and \( \Gamma(S(TM^\perp)) \).

The connection \( \nabla \) is a metric connection on \( S(TM) \) while \( \nabla^* \) is generally not a metric connection and is given by

\[
(\nabla_X g)(Y, Z) = \sum_{i=1}^r \left[ h_i^*(X, Y) \lambda_i(Z) + h_i^*(X, Z) \lambda_i(Y) \right] ,
\]

for any \( X, Y \in \Gamma(T\overline{M}) \) and \( \lambda_i \) are 1-forms given by \( \lambda_i(X) = \overline{g}(X, N_i) \), for all \( X \in \Gamma(TM) \). By using (7), (8), and (9), the
curvature tensors $\bar{R}$ and $R$ of $\bar{M}$ and $M$, respectively, are related as, for any $X, Y, Z, W \in \Gamma(TM)$,
\[
\bar{R}(X, W, Z, Y) = \bar{g}(R(X, W)Z, Y)
\]
\[
+ \bar{g}\left( A_h'(XZ), W, Y \right)
\]
\[
- \bar{g}\left( A_h'(WZ), X, Y \right)
\]
\[
+ \bar{g}\left( A_h'(XZ), W, Y \right)
\]
\[
- \bar{g}\left( A_h'(WZ), X, Y \right)
\]
\[
+ \bar{g}\left( (\nabla_h h')(W, Z), Y \right)
\]
\[
- \bar{g}\left( (\nabla_W h')(X, Z), Y \right)
\]
\[
+ \bar{g}\left( (\nabla_x h')(W, Z), Y \right)
\]
\[
- \bar{g}\left( (\nabla_W h')(X, Z), Y \right)
\].

(14)

A null submanifold $(M, g)$ of an indefinite manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ [3] if there is a smooth transversal vector field $\mathcal{H} \in \Gamma(tr(TM))$, called the transversal curvature vector of $M$ such that
\[
h = g \otimes \mathcal{H}.
\]

Moreover, it is easy to see that $M$ is totally umbilical in $\bar{M}$, if and only if on each coordinate neighborhood $U$ there exist smooth vector fields $\mathcal{H}^i \in \Gamma(tr(TM))$ and $\mathcal{H}^a \in \Gamma(S(TM^\perp))$ and smooth functions $\mathcal{H}_i^a \in F(tr(TM))$ and $\mathcal{H}_a \in F(S(TM^\perp))$ such that,
\[
h^i(X, Y) = \mathcal{H}_i^a g(X, Y),
\]
\[
h^a(X, Y) = \mathcal{H}_a g(X, Y),
\]
\[
\bar{g}\left( (\nabla_x h')(W, Z), Y \right)
\]
\[
- \bar{g}\left( (\nabla_W h')(X, Z), Y \right)
\]
\[
+ \bar{g}\left( (\nabla_x h')(W, Z), Y \right)
\]
\[
- \bar{g}\left( (\nabla_W h')(X, Z), Y \right)
\].

(15)

Let us now consider $\bar{M}$ to be a $(2n + 1)$-dimensional manifold endowed with an almost contact structure $(\bar{\phi}, \bar{\xi}, \eta, \bar{g})$; that is, $\bar{\phi}$ is a tensor field of type $(1, 1)$, $\bar{\xi}$ is a vector field, and $\eta$ is a 1-form satisfying
\[
\bar{\phi}^2 = -\bar{g} + \eta \otimes \xi,
\]
\[
\eta(\xi) = 1,
\]
\[
\eta \circ \bar{\phi} = 0,
\]
\[
\bar{\phi}(\xi) = 0.
\]

(17)

Then $(\bar{\phi}, \bar{\xi}, \eta, \bar{g})$ is called an indefinite almost contact metric structure on $\bar{M}$ if $(\bar{\phi}, \bar{\xi}, \eta)$ is an almost contact structure on $\bar{M}$ and $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$ such that [19], for any vector field $X, Y$ on $\bar{M}$,
\[
\bar{g}(\nabla_X \bar{\phi}, \bar{\xi}) = \bar{g}(X, Y) - \eta(X) \eta(Y),
\]
\[
\eta(X) = \bar{g}(\xi, X).
\]

(18)

An indefinite almost contact metric manifold $(\bar{M}, \bar{\phi}, \bar{\xi}, \eta, \bar{g})$ is said to be nearly cosymplectic if
\[
(\nabla_X \bar{\phi})Y + (\nabla_Y \bar{\phi})X = 0, \quad \forall X, Y \in \Gamma(T\bar{M}),
\]

(19)

where $\nabla$ is the Levi-Civita connection for $\bar{g}$. Taking $Y = \xi$ in (19), we get
\[
\nabla_X \xi = -\bar{H}X, \quad \forall X \in \Gamma(T\bar{M}).
\]

(20)

It is easy to verify the following properties of $\bar{H}$:
\[
\bar{H} \bar{\phi} + \bar{\phi} \bar{H} = 0,
\]
\[
\bar{H} \xi = 0,
\]
\[
\eta \circ \bar{H} = 0,
\]
\[
(\nabla_X \bar{\phi}) \xi = \bar{H}X,
\]
\[
\bar{g}(\bar{H}X, Y) = -\bar{g}(X, \bar{H}Y)
\]

(21)

(i.e., $\bar{H}$ is skew-symmetric),

for all $X, Y \in \Gamma(T\bar{M})$. Let $\Omega$ denote the fundamental 2-form of $\bar{M}$ defined by
\[
\Omega(X, Y) = \bar{g}(X, \bar{\phi}Y), \quad X, Y \in \Gamma(T\bar{M}).
\]

(22)

Then the 1-form $\eta$ and tensor $\bar{H}$ are related as follows.

Lemma 1. Let $(\bar{M}, \bar{\phi}, \bar{\xi}, \eta, \bar{g})$ be indefinite nearly cosymplectic. Then,
\[
d \eta(X, Y) = \bar{g}(\bar{H}X, Y), \quad \forall X, Y \in \Gamma(T\bar{M}).
\]

(23)

Moreover, $\bar{M}$ is cosymplectic if and only if $\bar{H}$ vanishes identically on $\bar{M}$.

Note that, for all $X, Y, Z \in \Gamma(T\bar{M})$,
\[
\bar{g}\left( (\nabla_X \bar{\phi})X, Y \right) = -\bar{g}(X, (\nabla_Y \bar{\phi})Y),
\]

(24)

which means that the tensor $\nabla \bar{\phi}$ is skew-symmetric. The following lemma is fundamental to the sequel.
Lemma 2. Let $\overline{M}$ be a nearly cosymplectic manifold. Then
\begin{align}
(\nabla_X \phi) Y &= -\phi (\nabla_X \phi) Y - \phi (Y, H X) X \\
&+ \phi (\phi H X, Y) X,
\end{align}
(25)
for all $X, Y \in \Gamma(T\overline{M})$.

Proof. The proof follows from a straightforward calculation. \hfill \Box

3. Quasi-Generalized CR-Null Submanifolds

We recall some basic notions on QGCR-null submanifolds (see [15] for details).

The structure vector field $\xi$ of an indefinite almost contact manifold $(\overline{M}, \overline{g})$ can be written according to decomposition (4) as follows:
\begin{align}
\xi = \xi_S + \sum_{i=1}^{r} a_i E_i + \sum_{i=1}^{r} b_i N_i + \sum_{\alpha=r+1}^{n} c_\alpha W_\alpha,
\end{align}
(27)
where $\xi_S$ is a smooth vector field of $S(TM)$ while $a_i = \eta(N_i)$, $b_i = \eta(E_i)$, and $c_\alpha = \varepsilon_\alpha \eta(W_\alpha)$, all smooth functions on $\overline{M}$. Here $\varepsilon_\alpha = \overline{g}(W_\alpha, W_\alpha)$.

We adopt the definition of quasi-generalized CR-(QGCR-) null submanifolds given in [15] for indefinite nearly cosymplectic manifolds.

Definition 3. Let $(M, g, S(TM), S(TM^+))$ be a null submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$. We say that $M$ is quasi-generalized CR- (QGCR-) null submanifold of $\overline{M}$ if the following conditions are satisfied.

(i) There exist two distributions $D_1$ and $D_2$ of $Rad TM$ such that
\begin{align}
Rad TM &= D_1 \oplus D_2, \\
\overline{\phi} D_1 &= D_1, \\
\overline{\phi} D_2 &= S(TM).
\end{align}
(28)
(ii) There exist vector bundles $D_0$ and $\overline{D}$ over $S(TM)$ such that
\begin{align}
S(TM) &= \{ \overline{\phi} D_2 \oplus \overline{D} \} \perp D_0, \\
\text{with } \overline{\phi} D_0 &= D_0, \quad \overline{D} = \overline{\phi} S \oplus \overline{\phi} \mathcal{L},
\end{align}
(29)
where $D_0$ is a nondegenerate distribution on $M$ and $\mathcal{L}$ and $S$ are, respectively, vector subbundles of $Itr(TM)$ and $S(TM^+)$. If $D_1 \neq \{0\}$, $D_0 \neq \{0\}$, $D_2 \neq \{0\}$, and $\mathcal{L} \neq \{0\}$, then $M$ is called a proper QGCR-null submanifold.

A proof of the following proposition uses similar arguments as in [15].

Proposition 4. A QGCR-null submanifold $M$ of an indefinite nearly cosymplectic manifold $\overline{M}$ tangent to the structure vector field $\xi$ is a GCR-null submanifold.

Using (2), the tangent bundle of any QGCR-null submanifold, $TM$, can be decomposed as
\begin{align}
TM = D \oplus \overline{D},
\end{align}
(30)
with $D = D_0 \perp D_1$, $\overline{D} = \{ D_2 \perp \overline{\phi} D_2 \} \oplus \overline{D}$.

Unlike a GCR-null submanifold, in a QGCR-null submanifold, $D$ is invariant with respect to $\overline{\phi}$ while $\overline{D}$ is not generally anti-invariant.

Throughout this paper, we suppose that $(M, g, S(TM), S(TM^+))$ is a proper QGCR-null submanifold. From the above definition, we can easily deduce the following:

1. Condition (i) implies that $\dim(Rad TM) = s \geq 3$.
2. Condition (ii) implies that $\dim(D) \geq 4l \geq 4$ and $\dim(D_2) = \dim(\mathcal{L})$.

Definition 5 (see [16]). A null submanifold $M$ of a semi-Riemannian manifold $\overline{M}$ is said to be a screen if the structure vector field, $\xi$, belongs to $Rad TM \oplus Itr(TM)$.

From Definition 3.3, Lemma 3.6, and Theorem 3.7 of [15], we have the following.

Theorem 6. Let $(M, g, S(TM), S(TM^+))$ be an screen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $\overline{M}$; then $\xi \in \Gamma(D_2 \perp \mathcal{L})$. If $M$ is a 3-null QGCR submanifold of an indefinite nearly cosymplectic manifold $(\overline{M}, \overline{g})$, then $M$ is screen null submanifold if and only if $\overline{\phi} \mathcal{L} = \overline{\phi} D_2$.

Proof. The proof follows from straightforward calculation as in [15]. \hfill \Box
\[ M = (M_{\xi} M_{\gamma} M_{\chi} \eta (g_{\chi}) M_{\gamma} M_{\chi}) \eta (W) g (Y, X) - \eta (Z) \eta (W) g (X, Y) + g (\phi Y, X) g (\phi Z, W) - g (\phi Z, X) g (\phi Y, W) - 2 g (\phi Z, Y) g (\phi X, W), \]
\[ (36) \]
for all \( X, Y, Z, W \in \Gamma (TM) \).

from which \( ltr(TM) = \text{Span}[N_1, N_2, N_3] \) and \( S(TM^+) = \text{Span}[W] \). Clearly, \( \phi_0 N_2 = -N_1 \). Further, \( \phi_0 N_3 = (1/2)X_2 \) and thus \( \mathcal{L} = \text{Span}[N_3] \). Notice that \( \phi_0 N_3 = -(1/2)\phi_0 E_3 \) and therefore \( \phi_0 \mathcal{L} = \phi_0 D_2 \). Also, \( \phi_0 W = -X_1 \) and therefore \( \delta = \text{Span}[W] \). Finally, we calculate \( \xi \) as follows. Using Theorem 6 we have \( \xi = aE_3 + bN_3 \). Applying \( \phi_0 \) to this equation we obtain \( a\phi_0 E_3 + b\phi_0 N_3 = 0 \). Now, substituting for \( \phi_0 E_3 \) and \( \phi_0 N_3 \) in this equation we get \( 2a = b \), from which we get \( \xi = (1/2)(E_3 + 2N_3) \). Since \( \phi_0 \xi = 0 \) and \( \phi_0 (\xi, \xi) = 1 \), we conclude that \( (M, g) \) is an ascreen QGCR-null submanifold of \( \overline{M} \).

**Proposition 8.** There exist no coisotropic, isotropic, or totally null proper QGCR-null submanifolds of an indefinite nearly cosymplectic manifold.

### 4. Umbilical and Geodesic Ascreen QGCR-Null Submanifolds

In this section, we prove two main theorems concerning totally umbilical, totally geodesic and irrotational ascreen QGCR-null submanifolds of \( \overline{M} \). An indefinite nearly cosymplectic manifold \( \overline{M} \) is called an indefinite nearly cosymplectic space form, denoted by \( \overline{M}(\mathcal{C}) \), if it has the constant \( \phi \)-sectional curvature \( \mathcal{C} \). The curvature tensor \( \overline{R} \) of the indefinite nearly cosymplectic space form \( \overline{M}(\mathcal{C}) \) is given by \([21]\):

\[ 4\overline{R}(X, W, Z, Y) = \overline{g}((\nabla_{\nu Y} \phi) Z, (\nabla_{\nu X} \phi) \overline{Y}) - \overline{g}((\nabla_{\nu X} \phi) Y, (\nabla_{\nu Z} \phi) \overline{Y}) - 2\overline{g}((\nabla_{\nu X} \phi) Z, (\nabla_{\nu Y} \phi) \overline{Z}) + \overline{g}(H W, Z) \cdot \overline{g}(H X, Y) - \overline{g}(H W, Y) \overline{g}(H X, Z) - 2\overline{g}(H W, X) \overline{g}(H Y, \overline{Z}) - \eta(\overline{W}) \eta(\overline{Y}) \cdot \overline{g}(H X, H Z) + \eta(\overline{W}) \eta(\overline{Z}) \overline{g}(H X, H Y) + \eta(\overline{X}) \eta(\overline{Y}) \overline{g}(H W, H Z) - \eta(\overline{X}) \eta(\overline{Z}) \overline{g}(H W, H Y) + \overline{g}(H W, H Y) + \mathcal{C} \cdot \overline{g}(\overline{X}, \overline{Y}) \overline{g}(Z, W) - \overline{g}(Z, X) \overline{g}(Y, W) + \eta(\overline{Z}) \eta(\overline{X}) \overline{g}(Y, W) - \eta(\overline{Y}) \eta(\overline{X}) \overline{g}(Z, W) + \eta(\overline{Y}) \eta(\overline{W}) \overline{g}(Z, X) - \eta(\overline{Z}) \eta(\overline{W}) \overline{g}(Y, X) + \overline{g}(\phi Y, X) \overline{g}(\phi Z, W) - \overline{g}(\phi Z, X) \overline{g}(\phi Y, W) - 2\overline{g}(\phi Z, Y) \overline{g}(\phi X, W), \]

for all \( X, Y, Z, W \in \Gamma (TM) \).

**Example 7.** Let \( M = (\mathbb{R}^4, g) \) be a semi-Euclidean space, with \( g \) being of signature \((-,-,+,-,-,+,-,+,-,+,+,+)\) with respect to the canonical basis

\[ (dx_1, dx_2, dx_3, dx_4, dy_1, dy_2, dy_3, dy_4, dz). \]

(32)

Let \((M, g)\) be a submanifold of \(\overline{M}\) given by

\[ x^1 = y^4, \]
\[ y^1 = -x^4, \]
\[ z = x^2 \sin \theta + y^2 \cos \theta, \]
\[ y^5 = (x^5)^{1/2}, \]

where \( \theta \in (0, \pi/2) \). By direct calculations, we can see that the vector fields

\[ E_1 = \partial x_4 + \partial y_1, \]
\[ E_2 = \partial x_1 - \partial y_4, \]
\[ E_3 = \sin \theta \partial x_2 + \cos \theta \partial y_2 + \partial z, \]
\[ X_1 = 2y^4 \partial x_2 + \partial y_5, \]
\[ X_2 = -\cos \theta \partial x_2 + \sin \theta \partial y_2, \]
\[ X_3 = \partial y_3, \]
\[ X_4 = \partial x_3, \]

form a local frame of \( TM \). Then \( Rad(TM) \) is spanned by \( \{E_1, E_2, E_3\} \), and, therefore, \( M \) is 3-null. Further, \( \phi_0 E_1 = E_2 \); therefore we set \( D_1 = \text{Span}[E_1, E_2] \). Also \( \phi_0 E_3 = -X_2 \) and thus \( D_2 = \text{Span}[E_3] \). It is easy to see that \( \phi_0 X_3 = X_4 \); so we set \( D_3 = \text{Span}[X_3, X_4] \). On the other hand, following direct calculations, we have

\[ N_1 = \frac{1}{2} (\partial x_4 - \partial y_1), \]
\[ N_2 = \frac{1}{2} (-\partial x_1 - \partial y_4), \]

**4. Umbilical and Geodesic Ascreen QGCR-Null Submanifolds**
Notice that $D_0$ and $\bar{\phi}'\mathcal{S}$ are orthogonal and nondegenerate subbundles of $TM$ and that when $M$ is ascreen QGCR-null submanifold, we observe that

$$\eta(X) = \eta(Z) = 0, \quad \forall X \in \Gamma(D_0), \quad Z \in \Gamma(\bar{\phi}'\mathcal{S}).$$

(37)

**Theorem 9.** Let $(M, g, S(TM), S(TM'))$ be a totally umbilical or totally geodesic ascreen QGCR-null submanifold of an indefinite nearly cosymplectic space form $\bar{M}(\bar{\tau})$, of pointwise constant $\bar{\phi}$-sectional curvature $\bar{\tau}$, such that $D_0$ and $\bar{\phi}'\mathcal{S}$ are space-like and parallel distributions with respect to $\bar{\nabla}$. Then, $\bar{\tau} \geq 0$. Equality occurs when $\bar{M}(\bar{\tau})$ is an indefinite cosymplectic space form.

**Proof.** Let $X$ and $Z$ be vector fields in $D_0$ and $\bar{\phi}'\mathcal{S}$, respectively. Replacing $\bar{\nabla}$ with $\bar{\phi}X$ and $\bar{Y}$ with $\bar{\phi}Z$ in (36), we get

$$4\bar{R}(X, \bar{\phi}X, Z, \bar{\phi}Z) = \bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) - \bar{g} \big( (\nabla_{\bar{\phi}Z}\bar{\phi})Z, (\nabla_{\bar{\phi}Z}\bar{\phi})Z \big) - 2\bar{g} \big( (\nabla_{\bar{\phi}Z}\bar{\phi})X, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) + \bar{g} \big( \bar{H}\bar{\phi}X, Z \big) \bar{g} \big( \bar{H}X, \bar{\phi}Z \big) - \bar{g} \big( \bar{H}\bar{\phi}X, \bar{\phi}Z \big) \bar{g} \big( \bar{H}X, Z \big) - 2\bar{g} \big( \bar{H}\bar{\phi}X, X \big) \bar{g} \big( \bar{H}\bar{\phi}Z, Z \big) - 2\bar{c}g \big( \bar{\phi}Z, \bar{\phi}Z \big) \bar{g} \big( \bar{\phi}X, \bar{\phi}X \big).$$

(38)

Considering the first three terms on the right hand side of (38), we have

$$\bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) = -\bar{g} \big( (\nabla_{\bar{\phi}Z}\bar{\phi})X, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big).$$

(39)

Applying (25) of Lemma 2 on (39) we derive

$$\bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) = -\bar{g} \big( (\nabla_{\bar{\phi}Z}\bar{\phi})X, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) = \bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) - \bar{g} \big( \bar{\phi}Z, \bar{H}X \big)^2 + \bar{g} \big( Z, \bar{H}X \big)^2. \quad \text{(40)}$$

In a similar way, using (26) of Lemma 2, we get

$$-\bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) = \bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big),$$

(41)

$$-2\bar{g} \big( (\nabla_{\bar{\phi}Z}\bar{\phi})X, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) = 0.$$  

Now substituting (40) and (41) into (38), we get

$$4\bar{R}(X, \bar{\phi}X, Z, \bar{\phi}Z) = 2\bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) - \bar{g} \big( \bar{\phi}Z, \bar{H}X \big)^2 + \bar{g} \big( Z, \bar{H}X \big)^2 + \bar{g} \big( \bar{H}\bar{\phi}X, Z \big) \bar{g} \big( \bar{H}X, \bar{\phi}Z \big) - \bar{g} \big( \bar{H}\bar{\phi}X, \bar{\phi}Z \big) \bar{g} \big( \bar{H}X, Z \big) - 2\bar{g} \big( \bar{H}\bar{\phi}X, X \big) \bar{g} \big( \bar{H}\bar{\phi}Z, Z \big) - 2\bar{c}g \big( \bar{\phi}Z, \bar{\phi}Z \big) \bar{g} \big( \bar{\phi}X, \bar{\phi}X \big),$$

from which we obtain

$$2\bar{R}(X, \bar{\phi}X, Z, \bar{\phi}Z) = \bar{g} \big( (\nabla_{\bar{\phi}X}\bar{\phi})Z, (\nabla_{\bar{\phi}X}\bar{\phi})Z \big) + \bar{g} \big( Z, \bar{H}X \big)^2 - \bar{c}g \big( Z, Z \big) \bar{g} \big( X, X \big).$$

(42)

Then using the facts that $D_0$ and $\bar{\phi}'\mathcal{S}$ are space-like and parallel with respect to $\bar{\nabla}$, we have $(\bar{\nabla}_{\bar{\phi}X}\bar{\phi})X = (\bar{\nabla}_{\bar{\phi}Z}\bar{\phi})X \in \Gamma(D_0)$, and (43) reduces to

$$2\bar{R}(X, \bar{\phi}X, Z, \bar{\phi}Z) = \| (\nabla_{\bar{\phi}X}\bar{\phi})X \|^2 + \bar{g} \big( Z, \bar{H}X \big)^2 - \bar{c}g \big( Z, Z \big) \| X \|^2,$$

(44)

where $\cdot \parallel$ denotes the norm on $D_0 \perp \bar{\phi}'\mathcal{S}$ with respect to $g$. On the other hand, if we set $\bar{\nabla} = \bar{\phi}X$ and $\bar{Y} = \bar{\phi}Z$ in (14), we have

$$\bar{R}(X, \bar{\phi}X, Z, \bar{\phi}Z) = \bar{g} \big( (\nabla_{\bar{\phi}X}'\bar{\phi})' \bar{\phi}X, \bar{\phi}Z \big) - \bar{g} \big( (\nabla_{\bar{\phi}Z}'\bar{\phi}) \bar{\phi}X, \bar{\phi}Z \big),$$

(45)

where

$$\bar{R}(X, \bar{\phi}X, Z, \bar{\phi}Z) = \| (\nabla_{\bar{\phi}X}'\bar{\phi})' \bar{\phi}X, \bar{\phi}Z \|^2 - \bar{g} \big( (\nabla_{\bar{\phi}X}'\bar{\phi}) \bar{\phi}X, \bar{\phi}Z \big),$$

(46)

By the fact that $M$ is totally umbilical in $\bar{M}$, we have $h'(\bar{\phi}X, Z) = 0$. Thus using (16), (46) becomes

$$\| (\nabla_{\bar{\phi}X}'\bar{\phi})' \bar{\phi}X, \bar{\phi}Z \|^2 = -\bar{g} \big( (\nabla_{\bar{\phi}X}'\bar{\phi}) \bar{\phi}X, \bar{\phi}Z \big) - \bar{g} \big( (\nabla_{\bar{\phi}X}'\bar{\phi}) \bar{\phi}X, \bar{\phi}Z \big),$$

(47)

Differentiating $\bar{g}(\bar{\phi}X, Z) = 0$ covariantly with respect to $X$ and then applying (7), we obtain

$$\bar{g} \big( (\nabla_{\bar{\phi}X}'\bar{\phi}) X, \bar{\phi}X, \bar{\phi}Z \big) = 0.$$
Substituting (48) into (47) gives
\[
(\nabla_X h^i) \left( \phi X, Z \right) = 0. 
\]
(49)

Similarly,
\[
(\nabla_{\phi X} h^i) (X, Z) = 0. 
\]
(50)

Then, substituting (49) and (50) into (45), we get
\[
\mathcal{R}(X, \phi X, Z, \phi Z) = 0. 
\]
(51)

Substituting (51) into (44), gives
\[
\bar{\varepsilon} \|X\|^2 \|Z\|^2 = \left\| (\nabla_{\phi X} \Phi) X \right\|^2 + \bar{g}(Z, \mathcal{H} X)^2 \geq 0, 
\]
(52)

which implies that \( \bar{\varepsilon} \geq 0 \). When the ambient manifold is cosymplectic, then \( \nabla \phi = 0 \) and \( d\eta = 0 \) [19] and in this case \( \bar{\varepsilon} = 0 \). \( \square \)

Example 10. Let \( M \) be an ascreen QGCR-null submanifold in Example 7. Applying (7) and Koszul’s formula (see [2]) to Example 7 we obtain
\[
h'_i(X, Y) = 0 
\]
(53)

\[ \forall X, Y \in \Gamma(TM), \text{ where } i = 1, 2, 3, \]
\[
e_4 h'_i(X_1, X_1) = 2, \]
\[ h'_i(X, Y) = 0, \] \( \forall X \neq X_1, Y \neq X_1. \)

Using (12), (53), and \( e_4 = \bar{g}(W, W) = 1 + 4(y^2)^2 \), we also derive
\[
h(X_1, X_1) = \frac{2}{1 + 4(y^2)^2} W. 
\]
(54)

We remark that \( M \) is not totally geodesic. From (54) and (15) we note that \( M \) is totally umbilical with
\[
\mathcal{H} = \frac{2}{1 + 4(y^2)^2} W. 
\]
(55)

By straightforward calculations we also have
\[
\nabla_X X_1 = 4y^2 X_1, 
\]
\[
\nabla_X X_j = 0 \quad \forall i, j \neq 1. 
\]
(56)

Thus, \( D_\phi \) and \( \Phi \) are parallel distributions with respect to \( \nabla \). Hence, \( M \) satisfies Theorem 9 and \( \bar{\varepsilon} = 0 \).

Corollary 11. Let \( (M, g, S(TM), S(TM^+) ) \) be a totally umbilical or totally geodesic ascreen QGCR-null submanifold of an indefinite cosymplectic space form \( \overline{M}(\bar{\varepsilon}) \) of pointwise constant \( \phi \)-sectional curvature \( \bar{\varepsilon} \). Then, \( \bar{\varepsilon} = 0 \).

A null submanifold \( M \) of a semi-Riemannian manifold \((\overline{M}, \overline{g})\) is called irrotational [3] if \( \nabla_X \phi E \in \Gamma(TM) \), for any \( E \in \Gamma(Rad TM) \) and \( X \in \Gamma(TM) \). Equivalently, \( M \) is irrotational if
\[
h^i(X, E) = h^i(X, E) = 0, 
\]
(57)

for all \( X \in \Gamma(TM) \) and \( E \in \Gamma(Rad TM) \).

Theorem 12. Let \((M, g, S(TM), S(TM^+) )\) be an irrotational ascreen QGCR-null submanifold of an indefinite nearly cosymplectic space form \( \overline{M}(\bar{\varepsilon}) \) of pointwise constant \( \phi \)-sectional curvature \( \bar{\varepsilon} \). Then, \( \bar{\varepsilon} \leq 0 \) or \( \bar{\varepsilon} \geq 0 \). Equality holds when \( \overline{M}(\bar{\varepsilon}) \) is an indefinite cosymplectic space form.

Proof. By setting \( Y = E, Z = \phi E, \) and \( W = E \) in (14), we get
\[
\mathcal{R}(X, E, \phi E, E) = \frac{3}{\bar{g}} \left( (\nabla_X h^i) (E, \phi E), E \right) 
\]
\[
- \eta(E) \left( (\nabla_{\phi E} h^i) (X, \phi E), E \right) 
\]
\[
+ 4\eta(E) \left( (\nabla_{\phi E} h^i) (X, \phi E), E \right) 
\]
(58)

for any \( X \in \Gamma(TM) \) and \( E \in \Gamma(Rad TM) \). Then, using the fact that \( M \) is irrotational, (58) reduces to
\[
\mathcal{R}(X, E, \phi E, E) = 0, \quad \forall X \in \Gamma(TM). 
\]
(59)

On the other hand, setting \( Y = W = E \) and \( Z = \phi E \) in (36) and simplifying, we get
\[
\mathcal{R}(X, E, \phi E, E) = -\frac{3}{\bar{g}} \left( (\nabla_X h^i) (E, \phi E), (\nabla_{\phi E} h^i) X \right) 
\]
\[
- \eta(E)^2 \frac{2}{\bar{g}} (H X, H \phi E) 
\]
\[
+ 4\eta(E)^2 \frac{2}{\bar{g}} (X, \phi E) \] 
(60)

Now, using (59) and (60), we get
\[
4\bar{\varepsilon} \eta(E)^2 \frac{2}{\bar{g}} (X, \phi E) = 3\bar{g} \left( (\nabla_{\phi E} h^i) \phi E, (\nabla_{\phi E} h^i) X \right) 
\]
\[
+ \eta(E)^2 \frac{2}{\bar{g}} (H X, H \phi E) \]. 
(61)

Replacing \( X \) with \( \phi E \) in (61) and using (25) of Lemma 2 to the resulting equation gives
\[
\bar{\varepsilon} \eta(E)^2 \frac{2}{\bar{g}} (\phi E, \phi E) = \eta(E)^2 \frac{2}{\bar{g}} (H \phi E, H \phi E). 
\]
(62)

Since \( M \) is ascreen QGCR-null submanifold, there exists \( E \in \Gamma(D_\phi) \) such that \( \eta(E) = b \neq 0 \), and thus (62) simplifies to
\[
\bar{\varepsilon} = -\frac{1}{b^2} \frac{2}{b^2} (H \phi E, H \phi E) \]
(63)

We observe that \( \bar{\varepsilon} = 0 \) if either \( b = 0 \) (i.e., \( \overline{M}(\bar{\varepsilon}) \) is cosymplectic space form [19]) or \( H \phi E \) is a null vector field. The second case implies that \( H \phi E \) belongs to \( Rad TM \) or \( \text{Itr}(TM) \). If
\( HE \in \Gamma(\text{Rad} TM) \), then there exists a nonzero smooth function \( \kappa \) such that \( HE = \kappa E \), for some arbitrary \( E \in \Gamma(\text{Rad} TM) \). Taking the \( \tilde{g} \)-product of \( HE = \kappa E \) with \( \xi \) leads to \( 0 = \kappa \eta(E) \), from which \( \eta(E) = 0 \). Since \( M \) is ascreen QGCR-null submanifold, then, there is \( E \in \Gamma(D_2) \) such that \( \eta(E) \neq 0 \), hence a contradiction. Similar reasoning can be applied if \( HE \in \Gamma(\text{tr}(TM)) \). Therefore, \( \tilde{c} = 0 \) only if \( HE = 0 \) (i.e., \( d\eta = 0 \)) which occurs when \( M(c) \) is cosymplectic space form [19]. It turns out that \( \tilde{c} \leq 0 \) or \( \tilde{c} \geq 0 \) depending on whether \( HE \) is space-like or time-like vector field, respectively. 

**Corollary 13.** Let \((M, g, S(TM), S(TM^+))\) be an irrotational ascreen QGCR-null submanifold of an indefinite cosymplectic space form \( \tilde{M}(\tilde{c}) \) of pointwise constant \( \tilde{f} \)-sectional curvature \( \tilde{c} \). Then, \( \tilde{c} = 0 \).

It is easy to see from (54) that \( h^i(X, E) = h^i(X, E) = 0 \) and hence \( M \) given in Example 10 is an irrotational ascreen QGCR-null submanifold of an indefinite cosymplectic space form \( \tilde{M}(\tilde{c}) \). As is proved in that example \( \tilde{c} = 0 \).

### 5. Mixed Totally Geodesic QGCR-Null Submanifolds

**Definition 14.** A QGCR-null submanifold of an indefinite nearly cosymplectic manifold \((\tilde{M}, \tilde{g})\) is called mixed totally geodesic if its second fundamental form, \( h \), satisfies \( h(X, Y) = 0 \), for any \( X \in \Gamma(D) \) and \( Y \in \Gamma(\tilde{D}) \).

We will need the following lemma in the next theorem.

**Lemma 15.** Let \((M, g, S(TM), S(TM^+))\) be any 3-null proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold \((\tilde{M}, \tilde{g})\). Then,

\[
2\eta(E) \eta(N) = 1,
\]

for any \( E \in \Gamma(D_2) \) and \( N \in \Gamma(\tilde{\mathcal{L}}) \).

**Proof.** The proof follows from straightforward calculations using \( \tilde{g}(\xi, \xi) = 1 \) and \( \xi = \eta(E) E + \eta(N) N \).

**Theorem 16.** Let \((M, g, S(TM), S(TM^+))\) be a 3-null proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold \((\tilde{M}, \tilde{g})\). Then, \( M \) is mixed totally geodesic if and only if \( h_a^i(X, Y) = 0 \) and \( A^a_i X = 0 \), for all \( X \in \Gamma(D) \), \( Y \in \Gamma(\tilde{D}) \), \( W_a \in \Gamma(S(TM^+)) \), and \( E_i \in \Gamma(\text{Rad} TM) \).

**Proof.** By the definition of ascreen QGCR-null submanifold, \( M \) is mixed geodesic if

\[
\tilde{g}(h(X, Y), W_a) = \tilde{g}(h(X, E), E) = 0,
\]

for all \( X \in \Gamma(D) \), \( Y \in \Gamma(\tilde{D}) \), \( W_a \in \Gamma(S(TM^+)) \), and \( E_i \in \Gamma(\text{Rad} TM) \). Now, by virtue of (12) and the first equation of (65), we have

\[
0 = \tilde{g}(h(X, Y), W_a) = e_a h_a^i(X, Y),
\]

from which \( h_a^i(X, Y) = 0 \), since \( e_a \neq 0 \). On the other hand, using (65), (7), and (11) we derive

\[
\tilde{g}(h(X, Y), E_i) = \tilde{g}(\nabla_X Y, E_i) = \tilde{g}(Y, \nabla_X E_i)
\]

\[= g(Y, A^a_i X) = 0. \tag{67}\]

Since \( D = D_0 \perp D_1 \) and \( \tilde{D} = \{D_2 \perp \tilde{\phi}D_2\} \perp \tilde{D} \), we observe that \( A^a_i X \neq \tilde{\phi}\tilde{D}_2 \). In fact, suppose that \( A^a_i X \in \Gamma(\tilde{\phi}\tilde{D}_2) \); then there exists a nonvanishing smooth function \( \ell \) such that \( A^a_i X = \ell \tilde{\phi} E \), for \( E \in \Gamma(D_2) \). Thus,

\[
0 = g(Y, A^a_i X) = \ell g(Y, \tilde{\phi} E), \quad \forall Y \in \Gamma(\tilde{D}). \tag{68}\]

Taking \( Y = \tilde{\phi} N \) in (68), where \( N \in \Gamma(\tilde{\mathcal{L}}) \) and using Lemma 15, we have

\[
0 = g(Y, A^a_i X) = \ell g(\tilde{\phi} N, \tilde{\phi} E) = \ell (1 - \eta(E) \eta(N))
\]

\[= \frac{1}{2} \ell. \tag{69}\]

which is a contradiction, since \( \ell \neq 0 \). Hence \( A^a_i X \neq \Gamma(\tilde{\phi}\tilde{D}_2 \perp \tilde{\phi}\tilde{D}_2) \). Moreover, \( A^a_i X \neq \Gamma(\tilde{\phi}\tilde{\mathcal{L}}) \) since if \( A^a_i X \in \Gamma(\tilde{\phi}\tilde{\mathcal{L}}) \), then there is a nonvanishing smooth function \( \omega \) such that \( A^a_i X = \omega \tilde{\phi} W_a \). Taking the \( \tilde{g} \)-product of this equation with respect to \( Y = \tilde{\phi} W_a \) and using the fact that \( \eta(W_a) = 0 \), we get

\[
0 = g(Y, A^a_i X) = \omega \tilde{g}(\tilde{\phi} W_a, \tilde{\phi} W_a) = \omega \tilde{g}(W_a, W_a)
\]

\[= \omega e_a, \tag{70}\]

which is a contradiction, since \( e_a \neq 0 \) and \( \omega \neq 0 \). Hence, \( A^a_i X \neq \Gamma(\tilde{\phi}\tilde{D}_2 \perp \tilde{\phi}\tilde{D}_2) \), which implies that \( A^a_i X \in \Gamma(D_0) \). Since \( A^a_i X \in \Gamma(D_0) \), then the nondegeneracy of \( D_0 \) implies that there exists some \( Z \in \Gamma(D_0) \) such that \( g(A^a_i X, Z) \neq 0 \). But using (11) and (7), together with the fact that \( M \) is mixed geodesic, we derive

\[
g(A^a_i X, Z) = -g(\nabla_X Z, E_i) = \tilde{g}(E_i, \nabla_X Z)
\]

\[= \tilde{g}(E_i, V_X Z) = 0, \tag{71}\]

which is a contradiction. Thus \( A^a_i X \neq \Gamma(\tilde{\phi}\tilde{D}_2 \perp \tilde{\phi}\tilde{D}_2 \perp D_0) \); that is, \( A^a_i X = 0 \). The converse is obvious.

**Corollary 17.** Let \((M, g, S(TM), S(TM^+))\) be a proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold \((\tilde{M}, \tilde{g})\) is called \( D \)-totally geodesic if its second fundamental form \( h \) satisfies \( h(X, Y) = 0 \), \( \forall X, Y \in \Gamma(D) \).
Since $M$ is a screen QGCR-null submanifold, we have $\overline{g}(X,\xi) = 0$, for all $X \in \Gamma(D)$. Applying $\overline{\nabla}_Y$ to $\overline{g}(X,\xi) = 0$ we get

$$
\eta\left(\overline{\nabla}_Y X\right) = -\overline{g}\left(X,\overline{\nabla}_Y \xi\right) = \overline{g}\left(X,\overline{\nabla}_Y Y\right). \tag{72}
$$

Interchanging $X$ and $Y$ in (72) and then adding the resulting equation to (72) gives

$$
\eta\left(\overline{\nabla}_X Y\right) + \eta\left(\overline{\nabla}_Y X\right) = \overline{g}\left(Y,\overline{\nabla}_X X\right) + \overline{g}\left(X,\overline{\nabla}_Y Y\right) = 0. \tag{73}
$$

**Theorem 19.** Let $(M,g,S(TM),S(TM^\perp))$ be a proper ascreen QGCR-null submanifold of an indefinite nearly cosymplectic manifold $(\tilde{M},\tilde{g})$. Then, $M$ is $D$-totally geodesic if and only if $\overline{\partial}_h(x,\overline{\phi}E)$ and $\overline{\partial}_h(x,\overline{\phi}W)$, respectively, have no components along $\text{itr}(TM)$ and $S(TM^\perp)$, while both $\overline{\nabla}_X \overline{\phi}E$ and $\overline{\nabla}_X \overline{\phi}W \notin \Gamma(D_0)$ for all $X, Y \in \Gamma(D), E \in \Gamma(\text{Rad}(TM))$, and $W \in \Gamma(\delta)$.

**Proof.** By the definition of ascreen QGCR-null submanifold, $M$ is $D$ geodesic if and only if $\overline{g}(h(X,Y),E) = \overline{g}(h(X,Y),W) = 0$, for all $X, Y \in \Gamma(D), W_a \in \Gamma(S(TM^\perp))$, and $E, W \in \Gamma(\text{Rad}(TM))$.

Using (7) and (18), we derive

$$
\overline{g}(h(X,Y),E) = \overline{g}\left(\overline{\nabla}_X Y,\overline{\phi}E\right) = \overline{g}\left(\overline{\phi} \overline{\nabla}_X Y,\overline{\phi} E\right) + \overline{g}\left(Y,\overline{\nabla}_X X\right) \overline{g}(E,\xi), \tag{74}
$$

from which when we apply (20) we get

$$
\overline{g}(h(X,Y),E) = \overline{g}\left(\overline{\phi} \overline{\nabla}_X Y,\overline{\phi} E\right) + \overline{g}\left(Y,\overline{\nabla}_X X\right) \overline{g}(E,\xi). \tag{75}
$$

Interchanging $X$ and $Y$ in (75) and considering the fact that $h$ is symmetric we get

$$
\overline{g}(h(X,Y),E) = \overline{g}\left(\overline{\phi} \overline{\nabla}_Y X,\overline{\phi} E\right) + \overline{g}\left(X,\overline{\nabla}_Y Y\right) \overline{g}(E,\xi). \tag{76}
$$

Summing (75) and (76) and then applying (73), we have

$$
2\overline{g}(h(X,Y),E) = \overline{g}\left(\overline{\phi} \overline{\nabla}_X Y,\overline{\phi} E\right) + \overline{g}\left(\overline{\phi} \overline{\nabla}_Y X,\overline{\phi} E\right) + \overline{g}\left(Y,\overline{\nabla}_X X\right) \overline{g}(E,\xi). \tag{77}
$$

Now, applying the nearly cosymplectic condition in (19) to (77) leads to

$$
2\overline{g}(h(X,Y),E) = \overline{g}\left(\overline{\phi} \overline{\nabla}_X Y,\overline{\phi} E\right) + \overline{g}\left(\overline{\phi} \overline{\nabla}_Y X,\overline{\phi} E\right) + \overline{g}\left(h(X,Y),E\right). \tag{78}
$$

From (78) and (7) we derive

$$
2\overline{g}(h(X,Y),E) = \overline{g}\left(\overline{\nabla}_X \overline{\phi} Y,\overline{\phi} E\right) + \overline{g}\left(\overline{\nabla}_Y \overline{\phi} X,\overline{\phi} E\right) - \overline{g}\left(\overline{\phi} Y, h(X,\overline{\phi} E)\right) \tag{79}
$$

$$
- \overline{g}\left(\overline{\phi} X, h(Y,\overline{\phi} E)\right).
$$


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