Research Article

Mass Renormalization in the Nelson Model

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The asymptotic behavior of the effective mass $m_{\text{eff}}(\Lambda)$ of the so-called Nelson model in quantum field theory is considered, where Λ is an ultraviolet cutoff parameter of the model. Let m be the bare mass of the model. It is shown that for sufficiently small coupling constant $|\alpha|$ of the model, $m_{\text{eff}}(\Lambda)/m$ can be expanded as $m_{\text{eff}}(\Lambda)/m = 1 + \sum_{n=1}^{\infty} a_n(\Lambda)\alpha^{2n}$. A physical folklore is that $a_n(\Lambda) = O([\log \Lambda]^{(n-1)})$ as $\Lambda \rightarrow \infty$. It is rigorously shown that $0 < \lim_{\Lambda \rightarrow \infty} a_1(\Lambda) < C$, $C_1 \leq \lim_{\Lambda \rightarrow \infty} a_2(\Lambda)/\log \Lambda \leq C_2$ with some constants C , C_1 , and C_2 .

1. Introduction and Main Results

The model considered in this paper is the so-called Nelson model [1], which describes a nonrelativistic nucleon with bare mass $m > 0$ interacting with a quantized scalar field with mass $\nu > 0$. The nucleon is governed by a Schrödinger operator. Let us first define the Nelson Hamiltonian. We use relativistic unit and employ the total momentum representation. Then the Hilbert space of states is the boson Fock space over $L^2(\mathbb{R}^3)$ which is given by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^3) \right], \quad (1)$$

where $\bigotimes_s^n$ denotes the n -fold symmetric tensor product and $\bigotimes_s^0 L^2(\mathbb{R}^3) = \mathbb{C}$. Then $\Phi \in \mathcal{F}$ can be written as $\Phi = \{\Phi^{(0)}, \Phi^{(1)}, \Phi^{(2)}, \dots\}$, where $\Phi^{(n)} \in \bigotimes_s^n L^2(\mathbb{R}^3)$. The Fock vacuum $\Omega \in \mathcal{F}$ is defined by $\Omega = \{1, 0, 0, \dots\}$. Let $a(f)$, $f \in L^2(\mathbb{R}^3)$, be the annihilation operator and $a(f)^*$, $f \in L^2(\mathbb{R}^3)$, the creation operator on $\mathcal{F}$, which are defined by

$$\begin{aligned} D(a(f)^*) \\ = \left\{ \Psi \in \mathcal{F} \mid \sum_{n=0}^{\infty} (n+1) \|S_{n+1}(f \otimes \Psi^{(n)})\|_{\bigotimes^n L^2(\mathbb{R}^3)}^2 < \infty \right\}, \end{aligned}$$

$$(a(f)^* \Psi)^{(0)} = 0,$$

$$(a(f)^* \Psi)^{(n+1)} = \sqrt{n+1} S_{n+1}(f \otimes \Psi^{(n)}), \quad (2)$$

and $a(f) = (a(f)^*)^*$, where S_n is the symmetrizer, $D(X)$ the domain of operator X , and $\|\cdot\|_{\mathcal{H}}$ the norm on $\mathcal{H}$. They satisfy canonical commutation relations as follows:

$$\begin{aligned} [a(f), a(g)^*] &= (f, g), \\ [a(f), a(g)] &= 0, \\ [a(f)^*, a(g)^*] &= 0 \end{aligned} \quad (3)$$

on a suitable dense domain, where $[X, Y] = XY - YX$ and $(\cdot, \cdot)$ is the inner product on $\mathcal{H}$ (linear in the second variable). Let T be a self-adjoint operator on $L^2(\mathbb{R}^3)$. Then we define the self-adjoint operator $d\Gamma(T)$ on $\mathcal{F}$ by $d\Gamma(T) = \bigoplus_{n=0}^{\infty} T^{(n)}$, where

$$\begin{aligned} T^{(n)} &= \left(\sum_{j=1}^n I \otimes \dots \otimes I \otimes \overset{j\text{th}}{\tilde{T}} \otimes I \otimes \dots \otimes I \right) \left[\bigotimes_s^n D(T) \right] \\ &\quad (n \geq 1) \end{aligned} \quad (4)$$

with $T^{(0)} = 0$. Here, for a closable operator T , $\bar{T}$ denotes the closure of T . The operator $d\Gamma(T)$ is called the second quantization of T . The free energy of the scalar field is given by $H_f = d\Gamma(\omega)$, where $\omega(k) = \sqrt{|k|^2 + \nu^2}$ ($k = (k_1, k_2, k_3) \in \mathbb{R}^3$, $\nu > 0$) is considered as a multiplication operator on $L^2(\mathbb{R}^3)$. Similarly the momentum of the scalar field is given by $P_{f\mu} = d\Gamma(k_\mu)$ ($\mu = 1, 2, 3$). The coupling of the nucleon and a scalar field is mediated through the Segal field operator $\Phi_s(g)$ defined by

$$\Phi_s(g) = \frac{1}{\sqrt{2}} (a(g) + a(g)^*), \quad (5)$$

where g is a cutoff function given by $g(k) = \hat{\phi}(k)/\sqrt{\omega(k)}$. Here $\hat{\phi}$ is the form factor with infrared cutoff $\kappa > 0$ and ultraviolet cutoff $\Lambda > 0$, which are defined by

$$\hat{\phi}(k) = \begin{cases} 0 & |k| < \kappa, \\ (2\pi)^{-3/2} & \kappa \leq |k| \leq \Lambda, \\ 0 & |k| > \Lambda. \end{cases} \quad (6)$$

The Nelson Hamiltonian with total momentum $p \in \mathbb{R}^3$ is given by a self-adjoint operator on $\mathcal{F}$ as follows:

$$H(p) = \frac{1}{2m} (p - P_f)^2 + H_f + \alpha \Phi_s(g), \quad (7)$$

where $\alpha \in \mathbb{R}$ is a coupling constant. Let $E(p, \alpha)$ be the energy-momentum relation (the infimum of the spectrum $\sigma(H(p))$) defined by

$$E(p, \alpha) = \inf \sigma(H(p)). \quad (8)$$

Then the effective mass $m_{\text{eff}} = m_{\text{eff}}(\Lambda)$ is defined by

$$\frac{1}{m_{\text{eff}}} = \frac{1}{3} \Delta_p E(p, \alpha) \upharpoonright_{p=0}. \quad (9)$$

Here Δ_p denotes the three-dimensional Laplacian in the variable p . We are concerned with the asymptotic behavior of m_{eff} as the ultraviolet cutoff goes to infinity. It is however a subtle problem. Removal of the ultraviolet cutoff Λ through mass renormalization means finding sequences $\{m\}$ and $\{\Lambda\}$ such that $m \rightarrow 0$, $\Lambda \rightarrow \infty$, and m_{eff} converges. Since we can see that m_{eff}/m is a function of Λ/m , to achieve this, we want to find constants $0 < \gamma < 1$ and $0 < b_0 < \infty$ such that

$$\lim_{\Lambda \rightarrow \infty} \frac{m_{\text{eff}}/m}{(\Lambda/m)^\gamma} = b_0. \quad (10)$$

If we succeed in finding constants γ and b_0 such as in (10), scaling the bare mass m as

$$m = \frac{1}{\Lambda^{\gamma/(1-\gamma)}} M, \quad (11)$$

where $M = (m^*/b_0)^{1/(1-\gamma)}$ with an arbitrary positive constant m^* , we have

$$\lim_{\Lambda \rightarrow \infty} m_{\text{eff}}(\Lambda) = m^*. \quad (12)$$

The mass renormalization is, however, a subtle problem, and unfortunately, we cannot yet find constants γ and b_0 such as in (10). For that reason we turn to perturbative renormalization, by which we try to guess the proper value of γ . Main results obtained in this paper are summarized as follows.

Theorem 1. *Let $\kappa > 0$. Then m_{eff} is an analytic function of α^2 and can be expanded in the following power series for sufficiently small $|\alpha|$:*

$$\frac{m_{\text{eff}}}{m} = 1 + \sum_{n=1}^{\infty} a_n(\Lambda) \alpha^{2n}. \quad (13)$$

Theorem 2. *There exists a strictly positive constant C such that*

$$\lim_{\Lambda \rightarrow \infty} a_1(\Lambda) = C. \quad (14)$$

Theorem 3. *There exist some constants C_1 and C_2 such that*

$$C_1 \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda)}{\log \Lambda} \leq C_2. \quad (15)$$

From Theorems 2 and 3, if $D = \lim_{\Lambda \rightarrow \infty} a_2(\Lambda)/\log \Lambda > 0$, it is suggested that $\gamma = D\alpha^2/C$. So, the mass of the Nelson model is renormalizable for sufficiently small $|\alpha|$.

The effective mass and energy-momentum relation have been studied mainly in nonrelativistic electrodynamics. Spohn [2] investigates the upper and lower bound of the effective mass of the polaron model from a functional integral point of view. Hiroshima and Spohn [3] study a perturbative mass renormalization including fourth order in the coupling constant in the case of a spinless electron. Hiroshima and Ito [4, 5] study it in the case of an electron with spin 1/2. Bach et al. [6] show that the energy-momentum relation is equal to the infimum of the essential spectrum of the Hamiltonian for $\kappa \geq 0$. Fröhlich and Pizzo [7] investigate energy-momentum relation when infrared cutoff goes to 0.

2. Analytic Properties

In order to investigate the effective mass in a perturbation theory we have to check the analytic properties of $E(p, \alpha)$.

2.1. Analytic Family in the Sense of Kato

Lemma 4. *$H(p)$ is an analytic family in the sense of Kato.*

Proof. We prove $H(p)$ is an analytic family of type (A). We see that

$$H(p) = H_0 + \sum_{\mu=1}^3 p_\mu \frac{1}{2m} (p_\mu - 2P_{f\mu}) + \alpha H_I, \quad (16)$$

where $H_0 = (1/2m)P_f^2 + H_f$ and $H_I = \Phi_s(g)$. Hence all we have to do is to prove the following facts.

$$(a) \ D(H_0) \subset \bigcap_{\mu=1}^3 D(P_{f\mu}) \cap D(H_I).$$

- (b) There exist real constants a_μ, b_μ ($\mu = 1, 2, 3$), c , and d such that for any $\Psi \in D(H_0)$

$$\left\| \frac{1}{2m} (p_\mu - 2P_{f\mu}) \Psi \right\|_{\mathcal{F}} \leq a_\mu \|H_0 \Psi\|_{\mathcal{F}} + b_\mu \|\Psi\|_{\mathcal{F}} \quad (\mu = 1, 2, 3), \quad (17)$$

$$\|H_I \Psi\|_{\mathcal{F}} \leq c \|H_0 \Psi\|_{\mathcal{F}} + d \|\Psi\|_{\mathcal{F}}.$$

We prove (a) at first. Since $\bigcap_{\mu=1}^3 D(P_{f\mu}^2) \subset D(P_{f\mu})$, we have $D(H_0) = \bigcap_{\mu=1}^3 D(P_{f\mu}^2) \cap D(H_f) \subset \bigcap_{\mu=1}^3 D(P_{f\mu})$. Additionally, since $\|\omega^{-1/2} g\|_{L^2(\mathbb{R}^3)} < \infty$, we have $g \in D(\omega^{-1/2})$. Furthermore, since ω is a nonnegative and injective self-adjoint operator on $L^2(\mathbb{R}^3)$, it follows that $D(d\Gamma(\omega)^{1/2}) \subset D(a(g)) \cap D(a(g)^*) = D(H_I)$. Hence we have $D(H_0) \subset D(d\Gamma(\omega)) \subset D(d\Gamma(\omega)^{1/2})$. Together with them, (a) is proven. Next we prove (b). Let Ψ be an arbitrary vector in $D(H_0)$. Then we have

$$\left\| \frac{1}{2m} (p_\mu - 2P_{f\mu}) \Psi \right\|_{\mathcal{F}} \leq \frac{|p_\mu|}{2m} \|\Psi\|_{\mathcal{F}} + \frac{1}{m} \|P_{f\mu} \Psi\|_{\mathcal{F}}. \quad (18)$$

Since $\|P_{f\mu} \Psi\|_{\mathcal{F}}^2 \leq 2m \|H_0^{1/2} \Psi\|_{\mathcal{F}}^2$, we have $\|H_0^{1/2} \Psi\|_{\mathcal{F}}^2 \leq \|(H_0 + 1)\Psi\|_{\mathcal{F}}^2$. Hence

$$\begin{aligned} & \left\| \frac{1}{2m} (p_\mu - 2P_{f\mu}) \Psi \right\|_{\mathcal{F}} \\ & \leq \sqrt{\frac{2}{m}} \|H_0 \Psi\|_{\mathcal{F}} + \left(\frac{|p_\mu|}{2m} + \sqrt{\frac{2}{m}} \right) \|\Psi\|_{\mathcal{F}}. \end{aligned} \quad (19)$$

Since $D(H_0) \subset D(d\Gamma(\omega)^{1/2})$,

$$\begin{aligned} \|a(g) \Psi\|_{\mathcal{F}} & \leq \|\omega^{-1/2} g\|_{L^2(\mathbb{R}^3)} \|H_f^{1/2} \Psi\|_{\mathcal{F}}, \\ \|a(g)^* \Psi\|_{\mathcal{F}} & \leq \|\omega^{-1/2} g\|_{L^2(\mathbb{R}^3)} \|H_f^{1/2} \Psi\|_{\mathcal{F}} \\ & \quad + \|g\|_{L^2(\mathbb{R}^3)} \|\Psi\|_{\mathcal{F}} \end{aligned} \quad (20)$$

hold. Hence

$$\begin{aligned} \|H_I \Psi\|_{\mathcal{F}} & \leq \sqrt{2} \|\omega^{-1/2} g\|_{L^2(\mathbb{R}^3)} \|H_f^{1/2} \Psi\|_{\mathcal{F}} \\ & \quad + \frac{1}{\sqrt{2}} \|g\|_{L^2(\mathbb{R}^3)} \|\Psi\|_{\mathcal{F}}. \end{aligned} \quad (21)$$

From triangle inequality, we have $\|H_f^{1/2} \Psi\|_{\mathcal{F}} \leq \|H_f \Psi\|_{\mathcal{F}} + \|\Psi\|_{\mathcal{F}}$. In addition,

$$\begin{aligned} \|H_0 \Psi\|_{\mathcal{F}}^2 - \|H_f \Psi\|_{\mathcal{F}}^2 & = \left\| \frac{1}{2m} P_f^2 \Psi \right\|_{\mathcal{F}}^2 \\ & \quad + \frac{1}{m} \Re(P_f^2 \Psi, H_f \Psi). \end{aligned} \quad (22)$$

Since P_f^2 and H_f are strongly commutative and nonnegative self-adjoint operators on $\mathcal{F}$, $(P_f^2 \Psi, H_f \Psi) \geq 0$ holds. Hence $\|H_f \Psi\|_{\mathcal{F}} \leq \|H_0 \Psi\|_{\mathcal{F}}$. Then we have

$$\begin{aligned} & \|H_I \Psi\|_{\mathcal{F}} \\ & \leq \sqrt{2} \|\omega^{-1/2} g\|_{L^2(\mathbb{R}^3)} \|H_0 \Psi\|_{\mathcal{F}} \\ & \quad + \left(\sqrt{2} \|\omega^{-1/2} g\|_{L^2(\mathbb{R}^3)} + \frac{1}{\sqrt{2}} \|g\|_{L^2(\mathbb{R}^3)} \right) \|\Psi\|_{\mathcal{F}}. \end{aligned} \quad (23)$$

From (19) and (23), (b) is proven. Hence $H(p)$ is an analytic family of type (A). Since every analytic family of type (A) is an analytic family of in the sense of Kato, it is an analytic family in the sense of Kato. $\square$

We denote the ground state of $H(p)$ by $\psi_g(p)$.

Lemma 5. (1) $E(p, \alpha)$ is analytic in p and α if $|p|$ and $|\alpha|$ are sufficiently small. (2) $\psi_g(p)$ is strongly analytic in p and α if $|p|$ and $|\alpha|$ are sufficiently small.

Proof. From [8, Theorem XII.9], (1) follows, and from [8, Theorem XII.8], (2) follows. $\square$

2.2. Formula. In this section we expand m/m_{eff} with respect to α .

Lemma 6. The ratio m/m_{eff} can be expressed as

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1}^3 \frac{(P_{f\mu} \psi_g(0), \psi'_{g_\mu}(0))}{(\psi_g(0), \psi_g(0))}, \quad (24)$$

where $\psi'_{g_\mu}(0) = s - \partial p_\mu \psi_g(p) \upharpoonright_{p=0}$.

Proof. Since $E(p, \alpha)$ is symmetry, $E(p, \alpha) = E(-p, \alpha)$, we have

$$\partial p_\mu E(p, \alpha) \upharpoonright_{p=0} = 0, \quad \mu = 1, 2, 3. \quad (25)$$

Since $H(p) \psi_g(p) = E(p, \alpha) \psi_g(p)$, for any $\Psi \in D(H(p))$,

$$(H(p) \Psi, \psi_g(p)) = E(p, \alpha) (\Psi, \psi_g(p)) \quad (26)$$

holds. Taking a derivative with respect to p_μ on both sides above, we have

$$\begin{aligned} & (H'_\mu(p) \Psi, \psi_g(p)) \\ & \quad + (H(p) \Psi, \psi'_{g_\mu}(p)) = E'_\mu(p, \alpha) (\Psi, \psi_g(p)) \\ & \quad + E(p, \alpha) (\Psi, \psi'_{g_\mu}(p)), \\ & (H''_\mu(p) \Psi, \psi_g(p)) + 2 (H'_\mu(p) \Psi, \psi'_{g_\mu}(p)) \\ & \quad + (H(p) \Psi, \psi''_{g_\mu}(p)) = E''_\mu(p, \alpha) (\Psi, \psi_g(p)) \\ & \quad + 2E'_\mu(p, \alpha) (\Psi, \psi'_{g_\mu}(p)) + E(p, \alpha) (\Psi, \psi''_{g_\mu}(p)). \end{aligned} \quad (27)$$

Here ' denotes the derivative or strong derivative with respect to p_μ , and $H'_\mu(p) = (1/m)(p_\mu - P_{f\mu})$, $H''_\mu(p) = 1/m$. Setting $\Psi = \psi_g(0)$ and $p = 0$, we have

$$E''_\mu(0, \alpha) = \frac{1}{m} \frac{(\psi_g(0), \psi_g(0)) - 2(P_{f\mu}\psi_g(0), \psi'_{g\mu}(0))}{(\psi_g(0), \psi_g(0))}. \quad (28)$$

This expression and the definition of the effective mass prove the lemma. $\square$

2.3. Perturbative Expansions. We define operators A^+ and A^- by $A^+ = (1/\sqrt{2})a(g)^*$ and $A^- = (1/\sqrt{2})a(g)$. Then $H_I = A^+ + A^-$. Moreover, let $\mathcal{F}^{(n)} = \bigotimes_s^n L^2(\mathbb{R}^3)$ and

$$\psi_g(0) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \varphi_n. \quad (29)$$

Since $E(p, \alpha)$ is symmetry $E(p, -\alpha) = E(p, \alpha)$, we have

$$E(0, \alpha) = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} E_{2n}. \quad (30)$$

Since $\ker H_0 \neq \{0\}$, H_0 is not injective. However, we define the operator $1/H_0$ (for notational simplicity we write $1/H_0$ for H_0^{-1} in what follows) on $\mathcal{F}$ as follows.

$$\begin{aligned} D\left(\frac{1}{H_0}\right) &= \left\{ \Psi = \bigoplus_{n=0}^{\infty} \Psi^{(n)} \in \mathcal{F} \mid \sum_{n=1}^{\infty} \|\beta^n \Psi^{(n)}\|^2 < \infty \right\}, \\ \left(\frac{1}{H_0} \Psi\right)^{(0)} &= 0, \\ \left(\frac{1}{H_0} \Psi\right)^{(n)}(k_1, \dots, k_n) &= \beta^n(k_1, \dots, k_n) \Psi^{(n)}(k_1, \dots, k_n) \quad (n \geq 1). \end{aligned} \quad (31)$$

Here

$$\begin{aligned} \beta^n &= \beta^n(k_1, \dots, k_n) \\ &= \frac{1}{(1/2m) |k_1 + \dots + k_n|^2 + \sum_{i=1}^n \omega(k_i)}. \end{aligned} \quad (32)$$

We define the subspace $\mathcal{F}_{\text{fin}}$ of $\mathcal{F}$ as $\mathcal{F}_{\text{fin}} = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} \mid \Psi^{(l)} = 0 \text{ for } l \geq q \text{ with some } q\}$.

Lemma 7. It holds that $\mathcal{F}_{\text{fin}} \subset D(1/H_0)$.

Proof. Let $\Psi \in \mathcal{F}_{\text{fin}}$. Then $\|(1/H_0)\Psi\|^2 = \sum_{n=1}^{\infty} \|((1/H_0)\Psi)^{(n)}\|^2 < \infty$. Hence the lemma follows. $\square$

Lemma 8. Let $\psi_g(0) = \sum_{n=0}^{\infty} (\alpha^n/n!) \varphi_n$. Then $\varphi_0 = \Omega$, $\varphi_1 = -(1/H_0)H_I\Omega$, and the recurrence formulas

$$\varphi_{2l} = \frac{1}{H_0} \left\{ -2lH_I\varphi_{2l-1} + \sum_{j=1}^l \binom{2l}{2j} E_{2j}\varphi_{2l-2j} \right\} \quad (l \geq 1), \quad (33)$$

$$\begin{aligned} \varphi_{2l+1} &= \frac{1}{H_0} \left\{ -(2l+1)H_I\varphi_{2l} + \sum_{j=1}^l \binom{2l+1}{2j} E_{2j}\varphi_{2l+1-2j} \right\} \quad (l \geq 0) \end{aligned} \quad (34)$$

follow, with

$$\begin{aligned} \varphi_{2l} &\in \mathcal{F}^{(2)} \oplus \mathcal{F}^{(4)} \oplus \dots \oplus \mathcal{F}^{(2l)} \quad (l \geq 1), \\ \varphi_{2l+1} &\in \mathcal{F}^{(1)} \oplus \mathcal{F}^{(3)} \oplus \dots \oplus \mathcal{F}^{(2l+1)} \quad (l \geq 0), \end{aligned} \quad (35)$$

and E_{2l} is given by

$$E_{2l} = 2l(\Omega, H_I\varphi_{2l-1}) \quad (l \geq 1). \quad (36)$$

Proof. We have $E(0, 0) = E_0$ by substituting $\alpha = 0$ in (30). Since $E(0, 0)$ is the ground state energy of H_0 , $E(0, 0) = 0$. Hence $E_0 = 0$. Since φ_0 is the ground state of H_0 , φ_0 can be Ω . We can find that $(\varphi_n, \Omega) = \delta_{0n}$ for $n = 0, 1, \dots$ holds in the same way as [3]. From now we set $H = H(p)$, $\psi_g = \psi_g(0)$, $E = E(p, \alpha)$, and ' means (strong) derivative with respect to α .

$$(H\Psi, \psi_g) = E(\Psi, \psi_g) \quad (37)$$

holds for $\Psi \in D(H)$. Differentiating (37) with respect to α , we have

$$(H_I\Psi, \psi_g) + (H\Psi, \psi'_g) = E'(\Psi, \psi_g) + E(\Psi, \psi'_g). \quad (38)$$

Hence $\psi'_g \in D(H)$ and we have

$$H_I\psi_g + H\psi'_g = E'\psi_g + E\psi'_g. \quad (39)$$

Substituting $p = 0$ and $\alpha = 0$ into (39) and taking into account $(\varphi_n, \Omega) = \delta_{0n}$, we have $\varphi_1 = -(1/H_0)H_I\Omega$. Differentiating (37) n times with respect to α , we also have

$$\begin{aligned} &(H\Psi, \psi_g^{(n)}) + n(\Psi, H_I\psi_g^{(n-1)}) \\ &= \sum_{j=1}^n \binom{n}{j} E^{(j)}(\Psi, \psi_g^{(n-j)}). \end{aligned} \quad (40)$$

By the induction on n , we have $\psi_g^{(n)} \in D(H)$ and

$$H_I\psi_g^{(n)} + nH_I\psi_g^{(n-1)} = \sum_{j=1}^n \binom{n}{j} E^{(j)}\psi_g^{(n-j)}. \quad (41)$$

Substituting $p = 0$ and $\alpha = 0$ into both sides above, we have

$$H_0 \varphi_{2l} + 2l H_1 \varphi_{2l-1} = \sum_{j=1}^l \binom{2l}{2j} E_{2j} \varphi_{2l-2j} \quad (l \geq 1), \quad (42)$$

$$H_0 \varphi_{2l+1} + (2l+1) H_1 \varphi_{2l} = \sum_{j=1}^l \binom{2l+1}{2j} E_{2j} \varphi_{2l+1-2j} \quad (l \geq 0).$$

From now on, we shall prove

$$\begin{aligned} \varphi_n^{(i)} &= 0, \quad (i > n, i = 0), \\ \varphi_{2l}^{(2i+1)} &= 0, \quad (l \geq 1, 0 \leq i \leq l-1), \\ \text{supp}_{k \in \mathbb{R}^{3 \cdot 2l}} \varphi_{2l}^{(2i)}(k) &= S_{2i} \text{ or } \emptyset, \quad (l \geq 1, 1 \leq i \leq l), \end{aligned} \quad (43)$$

$$\begin{aligned} \varphi_{2l+1}^{(2i)} &= 0, \quad (l \geq 0, 0 \leq i \leq l), \\ \text{supp}_{k \in \mathbb{R}^{3 \cdot (2l+1)}} \varphi_{2l+1}^{(2i+1)}(k) &= S_{2i+1} \text{ or } \emptyset, \quad (l \geq 0, 0 \leq i \leq l), \end{aligned}$$

where we set $\varphi_n = \{\varphi_n^{(i)}\}_{i=0}^\infty$ by induction for $n \geq 1$, and

$$S_i = \{(k_1, \dots, k_i) \in \mathbb{R}^{3i} \mid \kappa \leq |k_1| \leq \Lambda, \dots, \kappa \leq |k_i| \leq \Lambda\}. \quad (44)$$

Since $\varphi_1 = -(1/H_0)H_1\Omega \in \mathcal{F}^{(1)}$, $\varphi_1^{(i)} = 0$, $i > 1$, $i = 0$. Moreover, since

$$\varphi_1^{(1)}(k_1) = -\frac{1}{\sqrt{2}(2\pi)^{3/2}\sqrt{\omega(k_1)}E(k_1)}\chi_{[\kappa, \Lambda]}(|k_1|), \quad (45)$$

we have $\text{supp}_{k_1 \in \mathbb{R}^3} \varphi_1^{(1)}(k_1) = S_1$, where $E(k) = |k|^2/2m + \omega(k)$. Assume that the assumption of the induction holds when $n \leq 2l+1$, ($l \geq 0$). Then

$$\begin{aligned} H_0 \varphi_{2l+2} + (2l+2) H_1 \varphi_{2l+1} \\ = \sum_{j=1}^{l+1} \binom{2l+2}{2j} E_{2j} \varphi_{2l+2-2j}. \end{aligned} \quad (46)$$

It is derived that $\varphi_{2l+2}^{(i)} = 0$, $i > 2l+2$, $i = 0$, by $(\varphi_n, \Omega) = \delta_{0n}$ and (46). By the assumption of the induction, $(H_1 \varphi_{2l+1})^{(2i+1)} = 0$, $0 \leq i \leq l$, holds. When $1 \leq q \leq l$, it holds that

$$\begin{aligned} (H_0 \varphi_{2l+2})^{(2q)} &= -2(l+1)(H_1 \varphi_{2l+1})^{(2q)} \\ &+ \sum_{j=1}^{l+1} \binom{2l+2}{2j} E_{2j} \varphi_{2l+2-2j}^{(2q)} = -\sqrt{2}(l+1) \left\{ \frac{1}{\sqrt{2q}} \right. \\ &\cdot \sum_{i=1}^{2q} \frac{1}{(2\pi)^{3/2}} \frac{\chi_{[\kappa, \Lambda]}(|k_i|)}{\sqrt{2}\sqrt{\omega(k_i)}} \varphi_{2l+1}^{(2q-1)}(k_1, \dots, \widehat{k_i}, \dots, k_{2q}) \end{aligned}$$

$$\begin{aligned} &+ \sqrt{2q+1} \int \frac{1}{(2\pi)^{3/2}} \frac{\chi_{[\kappa, \Lambda]}(|k|)}{\sqrt{2}\sqrt{\omega(k)}} \varphi_{2l+1}^{(2q+1)}(k, k_1, \dots, \\ &k_{2q}) dk \Big\} + \sum_{j=1}^{l+1} \binom{2l+2}{2j} E_{2j} \varphi_{2l+1-2j}^{(2q)}, \end{aligned} \quad (47)$$

where $\widehat{k_i}$ means that k_i is omitted. By the assumption of the induction, the supports of the functions

$$\frac{1}{(2\pi)^{3/2}} \frac{\chi_{[\kappa, \Lambda]}(|k_i|)}{\sqrt{2}\sqrt{\omega(k_i)}} \varphi_{2l+1}^{(2q-1)}(k_1, \dots, \widehat{k_i}, \dots, k_{2q}), \quad (48)$$

$$\int \frac{1}{(2\pi)^{3/2}} \frac{\chi_{[\kappa, \Lambda]}(|k|)}{\sqrt{2}\sqrt{\omega(k)}} \varphi_{2l+1}^{(2q+1)}(k, k_1, \dots, k_{2q}) dk$$

and $\varphi_{2l+1-2j}^{(2q)}$ are S_{2q} or $\emptyset$. Furthermore,

$$\begin{aligned} (H_0 \varphi_{2l+2})^{(2l+2)}(k_1, \dots, k_{2l+2}) &= -2(l+1) \\ &\cdot (A^+ \varphi_{2l+1})^{(2l+2)}(k_1, \dots, k_{2l+2}) \\ &= -\sqrt{2}(l+1) \sum_{i=1}^{2l+2} \frac{1}{(2\pi)^{3/2}} \frac{\chi_{[\kappa, \Lambda]}(|k_i|)}{\sqrt{2}\sqrt{\omega(k_i)}} \\ &\cdot \varphi_{2l+1}^{(2l+1)}(k_1, \dots, \widehat{k_i}, \dots, k_{2l+2}) \end{aligned} \quad (49)$$

holds. By the assumption of the induction, the support of the right hand side is S_{2l+2} or $\emptyset$. Hence we have $\text{supp}_{k \in \mathbb{R}^{3 \cdot 2l}} \varphi_{2l+2}^{(2i)}(k) = S_{2i}$ or $\emptyset$, $1 \leq i \leq l+1$. We can prove $\varphi_{2l+3}^{(i)} = 0$, $i > 2l+3$, $i = 0$, $\varphi_{2l+3}^{(2i)} = 0$, $1 \leq i \leq l+1$, and $\text{supp}_{k \in \mathbb{R}^{3 \cdot (2l+1)}} \varphi_{2l+3}^{(2i+1)}(k) = S_{2i+1}$ or $\emptyset$, $0 \leq i \leq l+1$, in a similar way. From the discussion so far, we have

$$\begin{aligned} -2l H_1 \varphi_{2l-1} + \sum_{j=1}^l \binom{2l}{2j} E_{2j} \varphi_{2l-2j} &\in \mathcal{F}_{\text{fin}} \quad (l \geq 1), \\ -(2l+1) H_1 \varphi_{2l} + \sum_{j=1}^{l+1} \binom{2l+1}{2j} E_{2j} \varphi_{2l+1-2j} &\in \mathcal{F}_{\text{fin}} \quad (l \geq 0). \end{aligned} \quad (50)$$

Hence we have

$$\begin{aligned} \varphi_{2l} &= \frac{1}{H_0} \left\{ -2l H_1 \varphi_{2l-1} + \sum_{j=1}^l \binom{2l}{2j} E_{2j} \varphi_{2l-2j} \right\} + b_{2l} \Omega \\ &\quad (l \geq 1), \\ \varphi_{2l+1} &= \frac{1}{H_0} \left\{ -(2l+1) H_1 \varphi_l + \sum_{j=1}^{l+1} \binom{2l+1}{2j} E_{2j} \varphi_{2l+1-2j} \right\} \\ &\quad + b_{2l+1} \Omega \quad (l \geq 0), \end{aligned} \quad (51)$$

where b_{2l} and b_{2l+1} are some constants. Since $(\varphi_{2l}, \Omega) = 0$, $l \geq 1$, and $(\varphi_{2l+1}, \Omega) = 0$, $l \geq 0$, $b_{2l} = b_{2l+1} = 0$. Hence (33) and (34) are proven. By the discussion so far, (35) are also proven. We can derive (36) by (33) and $(\varphi_n, \Omega) = \delta_{0n}$. $\square$

3. Main Theorems

For notational simplicity we set $\hat{\phi}_j = \hat{\phi}(k_j)$ and $\omega_j = \omega(k_j)$ for $k_j \in \mathbb{R}^3$, $j = 1, 2$. Let

$$\begin{aligned} E_j &= \frac{|k_j|^2}{2m} + \omega_j, \quad j = 1, 2, \\ E_{12} &= \frac{|k_1 + k_2|^2}{2m} + \omega_1 + \omega_2, \\ \omega(r) &= \sqrt{r^2 + v^2}, \\ F(r) &= \frac{r^2}{2m} + \omega(r). \end{aligned} \quad (52)$$

Theorem 9. Let $\kappa > 0$. Then m_{eff} is an analytic function of α^2 and can be expanded in the following power series for sufficiently small $|\alpha|$:

$$\frac{m_{\text{eff}}}{m} = 1 + \sum_{n=1}^{\infty} a_n(\Lambda) \alpha^{2n}. \quad (53)$$

Proof. By the power series (29), we have

$$\begin{aligned} (\psi_g, \psi_g) &= \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \varphi_n, \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} \varphi_m \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{n+m}}{n!m!} (\varphi_n, \varphi_m). \end{aligned} \quad (54)$$

By Lemma 8, $(\varphi_n, \varphi_m) \neq 0$ if and only if both n and m are even or odd. Then we have

$$(\psi_g, \psi_g) = 1 + \sum_{n=1}^{\infty} b_n(\Lambda) \alpha^{2n}. \quad (55)$$

From the fact that both m_{eff}^{-1} and (ψ_g, ψ_g) are analytic functions of α^2 and Lemma 6, we have the following power series:

$$-\frac{2}{3} \sum_{\mu=1}^3 (P_{f\mu} \psi_g, \psi'_{g\mu}(0)) = \sum_{n=0}^{\infty} c_n(\Lambda) \alpha^{2n}. \quad (56)$$

Since $\psi'_{g\mu}(0)$ is an analytic function of α , we can write

$$\psi'_{g\mu}(0) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Phi_n^{\mu}. \quad (57)$$

We note that

$$\begin{aligned} c_0(\Lambda) &= -\frac{2}{3} \sum_{\mu=1}^3 (P_{f\mu} \varphi_0, \Phi_0^{\mu}) \\ &= -\frac{2}{3} \sum_{\mu=1}^3 (d\Gamma(k_{\mu}) \varphi_0, \Phi_0^{\mu}) = 0. \end{aligned} \quad (58)$$

Hence if $|\alpha|$ is sufficiently small, then we have the following power series:

$$\begin{aligned} \frac{m_{\text{eff}}}{m} &= \frac{(\psi_g, \psi_g)}{(\psi_g, \psi_g) - (2/3) \sum_{\mu=1}^3 (P_{f\mu} \psi_g(0), \psi'_{g\mu}(0))} \\ &= \frac{1 + \sum_{n=1}^{\infty} b_n(\Lambda) \alpha^{2n}}{1 + \sum_{n=1}^{\infty} (b_n(\Lambda) + c_n(\Lambda)) \alpha^{2n}} \\ &= \left(1 + \sum_{n=1}^{\infty} b_n(\Lambda) \alpha^{2n} \right) \\ &\quad \cdot \sum_{n=0}^{\infty} \left(- \sum_{l=1}^{\infty} (b_l(\Lambda) + c_l(\Lambda)) \alpha^{2l} \right)^n. \end{aligned} \quad (59)$$

This proves the theorem. $\square$

Theorem 10. There exists strictly positive constant C such that $\lim_{\Lambda \rightarrow \infty} a_1(\Lambda) = C$.

Proof. From (59), we have

$$\begin{aligned} \frac{m_{\text{eff}}}{m} &= \{1 + b_1(\Lambda) \alpha^2 + O(\alpha^4)\} \\ &\quad \cdot [1 - \{b_1(\Lambda) + c_1(\Lambda)\} \alpha^2 + O(\alpha^4)] = 1 - c_1(\Lambda) \\ &\quad \cdot \alpha^2 + O(\alpha^4). \end{aligned} \quad (60)$$

Therefore $a_1(\Lambda) = -c_1(\Lambda)$. Since

$$\begin{aligned} \sum_{n=1}^{\infty} c_n(\Lambda) \alpha^{2n} &= -\frac{2}{3} \sum_{\mu=1}^3 (P_{f\mu} \psi_g, \psi'_{g\mu}(0)) \\ &= -\frac{2}{3} \sum_{\mu=1}^3 \left(P_{f\mu} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \varphi_n, \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Phi_n^{\mu} \right), \end{aligned} \quad (61)$$

we have

$$\begin{aligned} c_1(\Lambda) &= -\frac{2}{3} \sum_{\mu=1}^3 \left\{ \frac{1}{0!2!} (P_{f\mu} \varphi_0, \Phi_2^{\mu}) + \frac{1}{1!1!} (P_{f\mu} \varphi_1, \Phi_1^{\mu}) \right. \\ &\quad \left. + \frac{1}{2!0!} (P_{f\mu} \varphi_2, \Phi_0^{\mu}) \right\} = -\frac{2}{3} \sum_{\mu=1}^3 \left\{ (P_{f\mu} \varphi_1, \Phi_1^{\mu}) \right. \\ &\quad \left. + \frac{1}{2} (P_{f\mu} \varphi_2, \Phi_0^{\mu}) \right\}. \end{aligned} \quad (62)$$

Substituting $p = 0$ into (27) and using (25), we have

$$\begin{aligned} &-\frac{1}{m} (P_{f\mu} \Psi, \psi_g) + ((H_0 + \alpha H_I) \Psi, \psi'_{g\mu}(0)) \\ &= E(0, \alpha) (\Psi, \psi'_{g\mu}(0)). \end{aligned} \quad (63)$$

In addition, by setting $\alpha = 0$, we have $-(P_{f\mu}/m) \varphi_0 + H_0 \Phi_0^{\mu} = 0$. Since $P_{f\mu} \varphi_0 = d\Gamma(k_{\mu}) \Omega = 0$, $H_0 \Phi_0^{\mu} = 0$ holds. Hence we have

$$\Phi_0^{\mu} = c_0 \Omega, \quad (c_0 \text{ is some constant}). \quad (64)$$

Differentiating both sides of (63) with respect to α , we have

$$\begin{aligned} & -\frac{1}{m} \left(P_{f\mu} \Psi, s - \frac{d}{d\alpha} \psi_g \right) + \left(H(0) \Psi, s - \frac{d}{d\alpha} \psi'_{g\mu}(0) \right) \\ & + \left(H_I \Psi, \psi'_{g\mu}(0) \right) \\ & = E(0, \alpha) \left(\Psi, s - \frac{d}{d\alpha} \psi'_{g\mu}(0) \right) \\ & + \frac{d}{d\alpha} E(0, \alpha) \left(\Psi, \psi'_{g\mu}(0) \right). \end{aligned} \quad (65)$$

Substituting $\alpha = 0$ into both sides, we have

$$(H_0 \Psi, \Phi_1^\mu) = \frac{1}{m} (\Psi, P_{f\mu} \varphi_1) - (\Psi, H_I \Phi_0^\mu). \quad (66)$$

Therefore $\Phi_1^\mu \in D(H_0)$ and

$$\begin{aligned} H_0 \Phi_1^\mu &= \frac{1}{m} P_{f\mu} \varphi_1 - H_I \Phi_0^\mu \\ &= -\frac{1}{m} P_{f\mu} \frac{1}{H_0} H_I \Omega - c_0 H_I \Omega. \end{aligned} \quad (67)$$

Since $-(1/m)P_{f\mu}(1/H_0)H_I\Omega - c_0H_I\Omega \in \mathcal{F}_{\text{fin}}$, we have

$$\Phi_1^\mu = -\frac{1}{m} \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega - c_0 \frac{1}{H_0} H_I \Omega + c_1 \Omega, \quad (68)$$

where c_1 is some constant. By $a_1(\Lambda) = -c_1(\Lambda)$, (62), (64), and (68), we have

$$\begin{aligned} a_1(\Lambda) &= \frac{2}{3} \sum_{\mu=1}^3 (P_{f\mu} \varphi_1, \Phi_1^\mu) \\ &= \frac{2}{3m} \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\ &\quad - \frac{2c_0}{3} \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} H_I \Omega \right) \\ &\quad - \frac{2c_1}{3} \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \Omega \right). \end{aligned} \quad (69)$$

It is also seen that

$$\begin{aligned} \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} H_I \Omega \right) &= \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \Omega \right) \\ &= 0. \end{aligned} \quad (70)$$

Thus we have

$$\begin{aligned} a_1(\Lambda) &= \frac{2}{3m} \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\ &= \frac{2}{3m} \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} A^+ \Omega, \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} A^+ \Omega \right) \\ &= \frac{2}{3m} \int \frac{|\hat{\phi}(k)|^2}{\omega(k)} \frac{|k|^2}{E(k)^3} dk. \end{aligned} \quad (71)$$

Changing variables into polar coordinate, we have

$$\begin{aligned} & \frac{2}{3m} \int \frac{|\hat{\phi}(k)|^2}{\omega(k)} \frac{|k|^2}{E(k)^3} dk \\ &= \frac{8\pi}{3m(2\pi)^3} \int_{\kappa}^{\Lambda} \frac{r^4}{\omega(r)F(r)^3} dr. \end{aligned} \quad (72)$$

Since $r^4/\omega(r)F(r)^3 = O(r^{-3})$ ($r \rightarrow \infty$), the improper integral $\int_{\kappa}^{\infty} (r^4/\omega(r)F(r)^3) dr$ converges. It is trivial to see that $\lim_{\Lambda \rightarrow \infty} a_1(\Lambda) > 0$. Thus the theorem follows. $\square$

Lemma 11. It follows that $\Phi_n^\mu \in \mathcal{F}_{\text{fin}}$ for $n \in \mathbb{N} \cup \{0\}$.

Proof. By (64), we have $\Phi_0^\mu \in \mathcal{F}_{\text{fin}}$. Assume that $\Phi_n^\mu \in \mathcal{F}_{\text{fin}}$ holds when $n \leq k-1$. Differentiating both sides of (63) k times with respect to α and substituting $\alpha = 0$, we have

$$\begin{aligned} & -\frac{1}{m} (P_{f\mu} \Psi, \varphi_k) + (H_0 \Psi, \Phi_k^\mu) + k (H_I \Psi, \Phi_{k-1}^\mu) \\ &= \sum_{j=1}^k \binom{k}{j} E_j (\Psi, \Phi_{k-j}^\mu); \end{aligned} \quad (73)$$

however, $E_j = 0$ when j is odd. Since $\Phi_{k-1}^\mu \in \mathcal{F}_{\text{fin}}$, $\Phi_{k-1}^\mu \in D(H_I)$ and

$$\begin{aligned} (H_0 \Psi, \Phi_k^\mu) &= \frac{1}{m} (\Psi, P_{f\mu} \varphi_k) - k (\Psi, H_I \Phi_{k-1}^\mu) \\ &\quad + \left(\Psi, \sum_{j=1}^k \binom{k}{j} E_j \Phi_{k-j}^\mu \right). \end{aligned} \quad (74)$$

Thus $\Phi_k^\mu \in D(H_0)$ and

$$H_0 \Phi_k^\mu = \frac{1}{m} P_{f\mu} \varphi_k - k H_I \Phi_{k-1}^\mu + \sum_{j=1}^k \binom{k}{j} E_j \Phi_{k-j}^\mu. \quad (75)$$

Since $\varphi_k \in \mathcal{F}_{\text{fin}}$, $H_I \Phi_{k-1}^\mu \in \mathcal{F}_{\text{fin}}$, and $\Phi_{k-j}^\mu \in \mathcal{F}_{\text{fin}}$ ($j = 1, \dots, k$), by the assumption of induction, $H_0 \Phi_k^\mu \in \mathcal{F}_{\text{fin}}$. Hence $\Phi_n^\mu \in \mathcal{F}_{\text{fin}}$ holds when $n = k$. $\square$

Lemma 12. It holds that $\Phi_0^\mu = c_0 \Omega$, $\Phi_1^\mu = -(1/m)(1/H_0)P_{f\mu}(1/H_0)H_I\Omega - c_0(1/H_0)H_I\Omega + c_1\Omega$, and the recurrence formulas

$$\begin{aligned} \Phi_{2l}^\mu &= \frac{1}{H_0} \left\{ \frac{1}{m} P_{f\mu} \varphi_{2l} - 2l H_I \Phi_{2l-1}^\mu \right. \\ &\quad \left. + \sum_{j=1}^l \binom{2l}{2j} E_{2j} \Phi_{2l-2j}^\mu \right\} + c_{2l} \Omega \\ &\quad (l \geq 1, \quad c_{2l} \text{ is some constant}). \end{aligned}$$

$$\begin{aligned} \Phi_{2l+1}^\mu &= \frac{1}{H_0} \left\{ \frac{1}{m} P_{f\mu} \varphi_{2l+1} - (2l+1) H_I \Phi_{2l}^\mu \right. \\ &\quad \left. + \sum_{j=1}^l \binom{2l+1}{2j} E_{2j} \Phi_{2l+1-2j}^\mu \right\} + c_{2l+1} \Omega, \\ &\quad (l \geq 0, \quad c_{2l+1} \text{ is some constant.}) \end{aligned} \quad (76)$$

Proof. The first and second expressions are proven in Theorem 10. From (75), it follows that

$$\begin{aligned} H_0 \Phi_{2l}^\mu &= \frac{1}{m} P_{f\mu} \varphi_{2l} - 2l H_I \Phi_{2l-1}^\mu \\ &\quad + \sum_{j=1}^l \binom{2l}{2j} E_{2j} \Phi_{2l-2j}^\mu \quad (l \geq 1) \\ H_0 \Phi_{2l+1}^\mu &= \frac{1}{m} P_{f\mu} \varphi_{2l+1} - (2l+1) H_I \Phi_{2l}^\mu \\ &\quad + \sum_{j=1}^l \binom{2l+1}{2j} E_{2j} \Phi_{2l+1-2j}^\mu \quad (l \geq 0). \end{aligned} \quad (77)$$

These prove the lemma. $\square$

Lemma 13. *It is proven that $a_2(\Lambda)$ can be expanded as*

$$\begin{aligned} a_2(\Lambda) &= \frac{2}{3m} \sum_{j=1}^8 I_j(\Lambda) + \frac{E_2(\Lambda)}{m} I_9(\Lambda) - a_1(\Lambda) I_{10}(\Lambda) \\ &\quad + a_1(\Lambda)^2, \end{aligned} \quad (78)$$

where I_j are given by

$$\begin{aligned} I_1(\Lambda) &= \frac{1}{4} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^3} + \frac{|k_2|^2}{E_2^3} \right) \\ &\quad \cdot \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{1}{E_{12}} dk_1 dk_2, \\ I_2(\Lambda) &= \frac{1}{8} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^4} + \frac{|k_2|^2}{E_2^4} \right) \frac{1}{E_{12}} dk_1 dk_2, \\ I_3(\Lambda) &= \frac{1}{8} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1^2} + \frac{1}{E_2^2} \right) \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \\ &\quad \cdot \frac{(k_1, k_2)}{E_{12}^2} dk_1 dk_2, \\ I_4(\Lambda) &= \frac{1}{4} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^2} + \frac{|k_2|^2}{E_2^2} \right) \\ &\quad \cdot \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{1}{E_{12}^2} dk_1 dk_2, \end{aligned}$$

$$I_5(\Lambda) = \frac{1}{4} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2 E_1^2 E_2^2} \frac{(k_1, k_2)}{E_{12}} dk_1 dk_2,$$

$$\begin{aligned} I_6(\Lambda) &= \frac{1}{8} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^2 \\ &\quad \cdot \frac{|k_1|^2 + |k_2|^2}{E_{12}^3} dk_1 dk_2, \end{aligned}$$

$$\begin{aligned} I_7(\Lambda) &= \frac{1}{4} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^2 \\ &\quad \cdot \frac{(k_1, k_2)}{E_{12}^3} dk_1 dk_2, \end{aligned}$$

$$I_8(\Lambda) = \frac{1}{4} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{(k_1, k_2)}{E_{12}^4} dk_1 dk_2,$$

$$I_9(\Lambda) = \frac{1}{2} \int \frac{|\widehat{\phi}(k)|^2 |k|^2}{\omega(k) E(k)^4} dk,$$

$$I_{10}(\Lambda) = \frac{1}{2} \int \frac{|\widehat{\phi}(k)|^2}{\omega(k) E(k)^2} dk.$$

(79)

The proof of Lemma 13 is given in the next section. The asymptotic behaviors of terms $I_j(\Lambda)$ as $\Lambda \rightarrow \infty$ is given in the lemma below. Only two terms $I_1(\Lambda)$ and $I_2(\Lambda)$ logarithmically diverge, and other terms converge as $\Lambda \rightarrow \infty$.

Lemma 14. (1)–(3) follow the following:

- (1) There exist some constants C_3 and C_4 such that $C_3 \leq \lim_{\Lambda \rightarrow \infty} (I_1(\Lambda) / \log \Lambda) \leq C_4$.
- (2) There exist some constants C_5 and C_6 such that $C_5 \leq \lim_{\Lambda \rightarrow \infty} (I_2(\Lambda) / \log \Lambda) \leq C_6$.
- (3) For $j = 3, 4, 5, 6, 7, 8$ $\lim_{\Lambda \rightarrow \infty} |I_j(\Lambda)| < \infty$.

The proof of Lemma 14 is technical and also given in the next section.

Lemma 15. *It holds that*

$$\lim_{\Lambda \rightarrow \infty} \frac{E_2(\Lambda)}{\log \Lambda} = -\frac{m}{\pi^2}. \quad (80)$$

Proof. From (36), we have $E_2(\Lambda) = -(1/2\pi^2) \int_{\mathbb{R}^3} (r^2/\omega(r)F(r))dr$ and $\lim_{r \rightarrow \infty} ((r^2/\omega(r)F(r))/(1/r)) = 2m$. It implies (80). $\square$

Now we are in the position to state the main theorem in this paper.

Theorem 16. *There exist some constants C_1 and C_2 such that*

$$C_1 \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda)}{\log \Lambda} \leq C_2. \quad (81)$$

Proof. We have $I_9(\Lambda) = (1/4\pi^2) \int_{\kappa}^{\Lambda} (r^4/\omega(r)F(r)^4)dr$. Since $r^4/\omega(r)F(r)^4 = O(r^{-5})$ ($r \rightarrow \infty$), we have

$$\lim_{\Lambda \rightarrow \infty} |I_9(\Lambda)| < \infty. \quad (82)$$

We also have $I_{10}(\Lambda) = (1/4\pi^2) \int_{\kappa}^{\Lambda} (r^2/\omega(r)F(r)^2)dr$. Then $r^2/\omega(r)F(r)^2 = O(r^{-3})$ ($r \rightarrow \infty$) and we also have

$$\lim_{\Lambda \rightarrow \infty} |I_{10}(\Lambda)| < \infty. \quad (83)$$

By (82) and (83), Theorem 10, and Lemmas 13, 14, and 15 we can conclude the theorem. $\square$

4. Proof of Lemmas 13 and 14

In this section we prove Lemmas 13 and 14.

4.1. Proof of Lemma 13. From (59) and $a_1(\Lambda) = -c_1(\Lambda)$, we have $a_2(\Lambda) = -c_2(\Lambda) - b_1(\Lambda)a_1(\Lambda) + a_1(\Lambda)^2$. Here

$$b_1(\Lambda) = (\varphi_1, \varphi_1) = \frac{1}{2} \int \frac{|\widehat{\phi}(k)|^2}{\omega(k)E(k)^2} dk, \quad (84)$$

$$\begin{aligned} c_2(\Lambda) = & -\frac{2}{3} \left\{ \frac{1}{0!4!} \sum_{\mu=1}^3 (P_{f\mu}\varphi_0, \Phi_4^\mu) \right. \\ & + \frac{1}{1!3!} \sum_{\mu=1}^3 (P_{f\mu}\varphi_1, \Phi_3^\mu) + \frac{1}{2!2!} \sum_{\mu=1}^3 (P_{f\mu}\varphi_2, \Phi_2^\mu) \\ & \left. + \frac{1}{3!1!} \sum_{\mu=1}^3 (P_{f\mu}\varphi_3, \Phi_1^\mu) + \frac{1}{4!0!} \sum_{\mu=1}^3 (P_{f\mu}\varphi_4, \Phi_0^\mu) \right\} \quad (85) \\ = & -\frac{1}{9} \sum_{\mu=1}^3 (P_{f\mu}\varphi_1, \Phi_3^\mu) - \frac{1}{6} \sum_{\mu=1}^3 (P_{f\mu}\varphi_2, \Phi_2^\mu) - \frac{1}{9} \\ & \cdot \sum_{\mu=1}^3 (P_{f\mu}\varphi_3, \Phi_1^\mu). \end{aligned}$$

Using recurrence formulas (33), (34), and (76), we have

$$\begin{aligned} \varphi_2 = & 2 \left(\frac{1}{H_0} H_I \right)^2 \Omega, \\ \varphi_3 = & -6 \left(\frac{1}{H_0} H_I \right)^3 \Omega - 3E_2 \left(\frac{1}{H_0} \right)^2 H_I \Omega, \\ \Phi_2^\mu = & \frac{2}{m} \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega \\ & + \frac{2}{m} \frac{1}{H_0} H_I \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \\ & + 2c_0 \left(\frac{1}{H_0} H_I \right)^2 \Omega - 2c_1 \frac{1}{H_0} H_I \Omega + c_2 \Omega, \end{aligned}$$

$$\begin{aligned} \Phi_3^\mu = & -\frac{6}{m} \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega \\ & - \frac{6}{m} \frac{1}{H_0} H_I \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega \\ & - \frac{6}{m} \left(\frac{1}{H_0} H_I \right)^2 \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \\ & - \frac{3}{m} E_2 \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega \\ & - \frac{3}{m} E_2 \left(\frac{1}{H_0} \right)^2 P_{f\mu} \frac{1}{H_0} H_I \Omega \\ & - 6c_0 \left(\frac{1}{H_0} H_I \right)^3 \Omega + 6c_1 \left(\frac{1}{H_0} H_I \right)^2 \Omega \\ & - 3c_2 \frac{1}{H_0} H_I \Omega - 3c_0 E_2 \left(\frac{1}{H_0} \right)^2 H_I \Omega + c_3 \Omega. \end{aligned} \quad (86)$$

Substituting them into (85), we have

$$\begin{aligned} c_2(\Lambda) = & -\frac{2}{3m} \left\{ \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega \right) \right. \\ & + \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} H_I \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) \\ & + \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} H_I \right)^2 \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\ & + \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) \\ & + \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \frac{1}{H_0} H_I \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\ & + \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega, \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \left. \right\} \\ & - \frac{E_2}{3m} \left\{ \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega \right) \right. \\ & + \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} \right)^2 P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\ & + \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega, \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \left. \right\} - \frac{2c_0}{3} \\ & \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} H_I \right)^3 \Omega \right) + \frac{2c_1}{3} \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) - \frac{c_2}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} H_I \Omega \right) - \frac{c_0 E_2}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} \right)^2 H_I \Omega \right) + \frac{c_3}{9} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \Omega \right) - \frac{2c_0}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) + \frac{2c_1}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \frac{1}{H_0} H_I \Omega \right) - \frac{c_2}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \Omega \right) - \frac{2c_0}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega, \frac{1}{H_0} H_I \Omega \right) + \frac{2c_1}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega, \Omega \right) - \frac{c_0 E_2}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega, \frac{1}{H_0} H_I \Omega \right) + \frac{c_1 E_2}{3} \\
& \cdot \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega, \Omega \right) = \sum_{j=1}^{21} (j). \tag{87}
\end{aligned}$$

$$\begin{aligned}
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \frac{1}{H_0} H_I \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega, \frac{1}{H_0} H_I \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega, \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} \right)^3 \Omega, \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega, \frac{1}{H_0} H_I \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega, \Omega \right) = 0. \tag{88}
\end{aligned}$$

We estimate 21 terms (1)–(21) above. We can however directly see that $0 = (10) = \dots = (21)$ as follows:

$$\begin{aligned}
& \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} H_I \right)^3 \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} H_I \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} \right)^2 H_I \Omega \right) \\
& = \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \Omega \right)
\end{aligned}$$

We can compute remaining terms (1)–(9) as

$$\begin{aligned}
(1) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega \right) \\
& = \sum_{\mu=1}^3 \left(A^+ \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} A^+ \Omega, \left(\frac{1}{H_0} A^+ \right)^2 \Omega \right) \\
& = \frac{1}{8} \iint \frac{|\hat{\phi}_1|^2 |\hat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^3} + \frac{|k_2|^2}{E_2^3} \right) \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \\
& \quad \cdot \frac{1}{E_{12}} dk_1 dk_2, \\
(2) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} H_I \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) \\
& = \sum_{\mu=1}^3 \left(A^+ \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} A^+ \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} A^+ \right)^2 \Omega \right) \\
& = \frac{1}{8} \iint \frac{|\hat{\phi}_1|^2 |\hat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^2} + \frac{|k_2|^2}{E_2^2} \right) \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \\
& \quad \cdot \frac{1}{E_{12}^2} dk_1 dk_2 + \frac{1}{8} \iint \frac{|\hat{\phi}_1|^2 |\hat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1^2} + \frac{1}{E_2^2} \right) \\
& \quad \cdot \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{(k_1, k_2)}{E_{12}^2} dk_1 dk_2,
\end{aligned}$$

$$\begin{aligned}
(3) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} H_I \right)^2 \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\
&= \sum_{\mu=1}^3 \left(A^+ \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} A^+ \Omega, \frac{1}{H_0} A^+ \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} A^+ \Omega \right) \\
&= \frac{1}{8} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^4} + \frac{|k_2|^2}{E_2^4} \right) \frac{1}{E_{12}} dk_1 dk_2 \\
&\quad + \frac{1}{4} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2 E_1^2 E_2^2} \frac{(k_1, k_2)}{E_{12}} dk_1 dk_2, \\
(4) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega \right) \\
&= \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} A^+ \right)^2 \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} A^+ \right)^2 \Omega \right) \\
&= \frac{1}{8} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^2 \\
&\quad \cdot \frac{|k_1|^2 + |k_2|^2}{E_{12}^3} dk_1 dk_2 + \frac{1}{4} \\
&\quad \cdot \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^2 \frac{(k_1, k_2)}{E_{12}^3} dk_1 dk_2, \\
(5) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^2 \Omega, \frac{1}{H_0} H_I \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\
&= \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} A^+ \right)^2 \Omega, \frac{1}{H_0} A^+ \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} A^+ \Omega \right) \\
&= \frac{1}{8} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^2} + \frac{|k_2|^2}{E_2^2} \right) \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \\
&\quad \cdot \frac{1}{E_{12}^2} dk_1 dk_2 + \frac{1}{4} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \\
&\quad \cdot \frac{(k_1, k_2)}{E_{12}^4} dk_1 dk_2, \\
(6) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} H_I \right)^3 \Omega, \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\
&= \sum_{\mu=1}^3 \left(\left(\frac{1}{H_0} A^+ \right)^2 \Omega, A^+ \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} A^+ \Omega \right) \\
&= \frac{1}{8} \iint \frac{|\widehat{\phi}_1|^2 |\widehat{\phi}_2|^2}{\omega_1 \omega_2} \left(\frac{|k_1|^2}{E_1^3} + \frac{|k_2|^2}{E_2^3} \right) \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \\
&\quad \cdot \frac{1}{E_{12}} dk_1 dk_2,
\end{aligned}$$

$$\begin{aligned}
(7) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \frac{1}{H_0} P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega \right) \\
&= \frac{1}{2} \int \frac{|\widehat{\phi}(k)|^2 |k|^2}{\omega(k) E(k)^4} dk, \\
(8) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \frac{1}{H_0} H_I \Omega, \left(\frac{1}{H_0} \right)^2 P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\
&= \frac{1}{2} \int \frac{|\widehat{\phi}(k)|^2 |k|^2}{\omega(k) E(k)^4} dk, \\
(9) \quad & \sum_{\mu=1}^3 \left(P_{f\mu} \left(\frac{1}{H_0} \right)^2 H_I \Omega, \frac{1}{H_0} P_{f\mu} \frac{1}{H_0} H_I \Omega \right) \\
&= \frac{1}{2} \int \frac{|\widehat{\phi}(k)|^2 |k|^2}{\omega(k) E(k)^4} dk.
\end{aligned} \tag{89}$$

Thus the lemma follows.

4.2. Proof of Lemma 14

Proof of $C_3 \leq \lim_{\Lambda \rightarrow \infty} (I_1(\Lambda)/\log \Lambda)$ and $C_5 \leq \lim_{\Lambda \rightarrow \infty} (I_2(\Lambda)/\log \Lambda)$. Changing variables to polar coordinates, we have

$$\begin{aligned}
I_1(\Lambda) &= \frac{2\pi^2}{(2\pi)^6} \\
&\cdot \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} \frac{r_1^2 r_2^2}{\omega(r_1) \omega(r_2)} \left(\frac{r_1^2}{F(r_1)^3} + \frac{r_2^2}{F(r_2)^3} \right) \\
&\cdot \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \frac{1}{L(r_1, r_2, z)} dz dr_1 dr_2,
\end{aligned} \tag{90}$$

where

$$L(r_1, r_2, z) = \frac{r_1^2 + r_2^2 + 2r_1 r_2 z}{2m} + \omega(r_1) + \omega(r_2). \tag{91}$$

We define $h(r_1, r_2)$, $h_1(r_1, r_2)$, $h_2(r_1, r_2)$, $S(\Lambda)$, $S_1(\Lambda)$ and $S_2(\Lambda)$ as

$$\begin{aligned}
h(r_1, r_2) &= \frac{r_1^2 r_2^2}{\omega(r_1) \omega(r_2)} \left(\frac{r_1^2}{F(r_1)^3} + \frac{r_2^2}{F(r_2)^3} \right) \\
&\cdot \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \frac{1}{L(r_1, r_2, 1)}, \\
h_1(r_1, r_2) &= \frac{r_1^2 r_2^4}{\omega(r_1) \omega(r_2) F(r_2)^4 L(r_1, r_2, 1)}, \\
h_2(r_1, r_2) &= h(r_1, r_2) - h_1(r_1, r_2),
\end{aligned}$$

$$\begin{aligned}
S(\Lambda) &= \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2) dr_1 dr_2, \\
S_1(\Lambda) &= \int_{\kappa}^{\kappa+1} \int_{\kappa+1+\nu+m}^{\Lambda} h_1(r_1, r_2) dr_2 dr_1, \\
S_2(\Lambda) &= \int_{\kappa}^{\kappa+1} \int_{\kappa}^{\kappa+1+\nu+m} h_1(r_1, r_2) dr_2 dr_1 \\
&\quad + \int_{\kappa+1}^{\Lambda} \int_{\kappa}^{\Lambda} h_1(r_1, r_2) dr_1 dr_2 \\
&\quad + \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h_2(r_1, r_2) dr_1 dr_2.
\end{aligned} \tag{92}$$

Then

$$\frac{4\pi^2}{(2\pi)^6} S(\Lambda) \leq I_1(\Lambda). \tag{93}$$

In addition, $S(\Lambda) = S_1(\Lambda) + S_2(\Lambda)$ follows. Since $h_1(r_1, r_2) > 0$ and $h_2(r_1, r_2) > 0$, $S_2(\Lambda) > 0$. Hence

$$S(\Lambda) > S_1(\Lambda). \tag{94}$$

Let r_2 satisfy $\kappa \leq r_2 \leq \kappa + 1$. Suppose that $\kappa + 1 + \nu + m \leq r_1 \leq \Lambda$. Since $\nu < r_1$, $r_1^2 + \nu^2 < 2r_1^2$ holds. Therefore we have $\omega(r_1) < \sqrt{2}r_1$. Since $r_2 < r_1$, we have $r_1 r_2 < r_1^2$ and $r_2^2 < r_1^2$. Thus $L(r_1, r_2, 1) < 2(1/m + \sqrt{2})r_1^2$. So,

$$\begin{aligned}
&\int_{\kappa+1+\nu+m}^{\Lambda} \frac{r_1^2}{\omega(r_1) L(r_1, r_2, 1)} dr_1 \\
&> \frac{1}{2\sqrt{2}(1/m + \sqrt{2})} \int_{\kappa+1+\nu+m}^{\Lambda} r_1^{-1} dr_1 \\
&= \frac{1}{2\sqrt{2}(1/m + 2)} (\log \Lambda - \log(\kappa + 1 + \nu + m))
\end{aligned} \tag{95}$$

follows. When $\kappa \leq r_2 \leq \kappa + 1$, we have

$$\begin{aligned}
\omega(r_2) &\leq \sqrt{(\kappa + 1)^2 + \nu^2}, \\
F(r_2) &\leq \frac{(\kappa + 1)^2}{2m} + \sqrt{(\kappa + 1)^2 + \nu^2}.
\end{aligned} \tag{96}$$

Then

$$\begin{aligned}
S_1(\Lambda) &= \int_{\kappa}^{\kappa+1} \frac{r_2^4}{\omega(r_2) F(r_2)^4} dr_2 \\
&\quad \cdot \int_{\kappa+1+\nu+m}^{\Lambda} \frac{r_1^2}{\omega(r_1) L(r_1, r_2, 1)} dr_1 \\
&> \frac{K}{2\sqrt{2}(1/m + \sqrt{2})} (\log \Lambda - \log(\kappa + 1 + \nu + m)) \\
&\quad \cdot \int_{\kappa}^{\kappa+1} r_2^4 dr_2 = \frac{K \{(\kappa + 1)^5 - \kappa^5\}}{10\sqrt{2}(1/m + \sqrt{2})} (\log \Lambda \\
&\quad - \log(\kappa + 1 + \nu + m)),
\end{aligned} \tag{97}$$

where

$$\begin{aligned}
&K \\
&= \frac{1}{\sqrt{(\kappa + 1)^2 + \nu^2} \left((\kappa + 1)^2 / 2m + \sqrt{(\kappa + 1)^2 + \nu^2} \right)^4}.
\end{aligned} \tag{98}$$

From (93), (94), and (97), $C_3 \leq \lim_{\Lambda \rightarrow \infty} (I_1(\Lambda) / \log \Lambda)$ follows.

The proof of $C_5 \leq \lim_{\Lambda \rightarrow \infty} (I_2(\Lambda) / \log \Lambda)$ is similar to that of $C_3 \leq \lim_{\Lambda \rightarrow \infty} (I_1(\Lambda) / \log \Lambda)$. Then we omit it. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_1(\Lambda) / \log \Lambda) \leq C_4$. We redefine $h(r_1, r_2)$, $h_1(r_1, r_2)$, $h_2(r_1, r_2)$, $S_1(\Lambda)$, $S_2(\Lambda)$, and $S(\Lambda)$ as

$$\begin{aligned}
h(r_1, r_2) &= \frac{r_1^2 r_2^2}{\omega(r_1) \omega(r_2)} \left(\frac{r_1^2}{F(r_1)^3} + \frac{r_2^2}{F(r_2)^3} \right) \\
&\quad \cdot \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \frac{1}{L(r_1, r_2, -1)}, \\
h_1(r_1, r_2) &= \frac{r_1^2 r_2^4}{\omega(r_1) \omega(r_2) F(r_2)^4 L(r_1, r_2, -1)}, \\
h_2(r_1, r_2) &= \frac{r_1^2 r_2^4}{\omega(r_1) \omega(r_2) F(r_1)^3 F(r_2)^3 L(r_1, r_2, -1)},
\end{aligned} \tag{99}$$

$$S_1(\Lambda) = \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h_1(r_1, r_2) dr_1 dr_2,$$

$$S_2(\Lambda) = \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h_2(r_1, r_2) dr_1 dr_2,$$

$$S(\Lambda) = \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2) dr_1 dr_2.$$

Then we have

$$I_1(\Lambda) \leq \frac{4\pi^2}{(2\pi)^6} S(\Lambda). \tag{100}$$

Since $h(r_1, r_2) = h_1(r_1, r_2) + h_1(r_2, r_1) + h_2(r_1, r_2) + h_2(r_2, r_1)$, we have

$$S(\Lambda) = 2(S_1(\Lambda) + S_2(\Lambda)). \tag{101}$$

Let B be $B = ((\kappa + 1)^3 - \kappa^3) / 6\nu^2$. Since $\nu < \omega(r_1)$ and $2\nu < L(r_1, r_2, -1)$,

$$\int_{\kappa}^{\kappa+1} \frac{r_1^2}{\omega(r_1) L(r_1, r_2, -1)} dr_1 < B \tag{102}$$

follows. Let Y be $Y = 2r_2 + \kappa + 1$. Since $r_1 < \omega(r_1)$ and $r_1 < L(r_1, r_2, -1)$,

$$\int_{\kappa+1}^Y \frac{r_1^2}{\omega(r_1) L(r_1, r_2, -1)} dr_1 < 2r_2 \tag{103}$$

holds. When $Y \leq r_1$, since $2r_2 < r_1$, we have $r_1 - r_2 > r_1/2$. Then $L(r_1, r_2, -1) > (r_1 - r_2)^2/2m > r_1^2/8m$. So,

$$\int_Y^\Lambda \frac{r_1^2}{\omega(r_1)L(r_1, r_2, -1)} dr_1 < 8m \int_Y^\Lambda \frac{dr_1}{r_1} < 8m \log \Lambda \quad (104)$$

follows. From (102), (103), and (104), we have

$$\int_\kappa^\Lambda \frac{r_1^2}{\omega(r_1)L(r_1, r_2, -1)} dr_1 < B + 2r_2 + 8m \log \Lambda. \quad (105)$$

Using this, we see that $S_1(\Lambda) < \int_\kappa^\Lambda (B + 2r_2 + 8m \log \Lambda)(r_2^4/\omega(r_2)F(r_2)^4) dr_2$. Since $r_2 < \omega(r_2)$ and $r_2^2/2m < F(r_2)$, we have

$$\begin{aligned} S_1(\Lambda) &< \int_\kappa^\Lambda (2r_2 + B + 8m \log \Lambda) \frac{16m^4}{r_2^5} dr_2 \\ &= \frac{32m^4}{3} \left(\frac{1}{\kappa^3} - \frac{1}{\Lambda^3} \right) \\ &\quad + 4m^4 (B + 8m \log \Lambda) \left(\frac{1}{\kappa^4} - \frac{1}{\Lambda^4} \right) \\ &< \frac{32m^5}{\kappa^4} \log \Lambda + \frac{32m^4}{3\kappa^3} + \frac{4m^4 B}{\kappa^4}. \end{aligned} \quad (106)$$

Since $r_1 < L(r_1, r_2, -1)$,

$$\begin{aligned} \int_\kappa^\Lambda \frac{r_1^2}{\omega(r_1)F(r_1)L(r_1, r_2, -1)} dr_1 &< 2m \int_\kappa^\Lambda \frac{dr_1}{r_1^2} \\ &< \frac{2m}{\kappa}. \end{aligned} \quad (107)$$

Then we have

$$\begin{aligned} S_2(\Lambda) &= \int_\kappa^\Lambda \frac{r_2^4}{\omega(r_2)F(r_2)^3} dr_2 \\ &\cdot \int_\kappa^\Lambda \frac{r_1^2}{\omega(r_1)F(r_1)L(r_1, r_2, -1)} dr_1 < \frac{2m}{\kappa} \\ &\cdot \int_\kappa^\Lambda \frac{r_2^4}{\omega(r_2)F(r_2)^3} dr_2 < \frac{16m^4}{\kappa} \int_\kappa^\Lambda \frac{dr_2}{r_2^3} < \frac{8m^4}{\kappa^3}. \end{aligned} \quad (108)$$

From (101), (106), and (108), it follows that

$$S(\Lambda) < \frac{64m^5}{\kappa^4} \log \Lambda + \frac{112m^4}{3\kappa^3} + \frac{8m^2 B}{\kappa^4}. \quad (109)$$

From (100) and (109), the lemma follows. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_2(\Lambda)/\log \Lambda) \leq C_6$. We redefine $h(r_1, r_2)$, $h_1(r_1, r_2)$, $S(\Lambda)$, and $S_1(\Lambda)$ as

$$\begin{aligned} h(r_1, r_2) &= \frac{r_1^2 r_2^2}{\omega(r_1)\omega(r_2)} \left(\frac{r_1^2}{F(r_1)^4} + \frac{r_2^2}{F(r_2)^4} \right) \frac{1}{L(r_1, r_2, -1)}, \\ h_1(r_1, r_2) &= \frac{r_1^2 r_2^4}{\omega(r_1)\omega(r_2)F(r_2)^4 L(r_1, r_2, -1)}, \end{aligned} \quad (110)$$

$$S(\Lambda) = \int_\kappa^\Lambda \int_\kappa^\Lambda h(r_1, r_2) dr_1 dr_2,$$

$$S_1(\Lambda) = \int_\kappa^\Lambda \int_\kappa^\Lambda h_1(r_1, r_2) dr_1 dr_2.$$

We have

$$\begin{aligned} I_2(\Lambda) &= \frac{\pi^2}{(2\pi)^6} \\ &\cdot \int_{-1}^1 \int_\kappa^\Lambda \int_\kappa^\Lambda \frac{r_1^2 r_2^2}{\omega(r_1)\omega(r_2)} \left(\frac{r_1^2}{F(r_1)^4} + \frac{r_2^2}{F(r_2)^4} \right) \\ &\cdot \frac{1}{L(r_1, r_2, z)} dz dr_1 dr_2. \end{aligned} \quad (111)$$

Then

$$I_2(\Lambda) \leq \frac{2\pi^2}{(2\pi)^6} S(\Lambda) \quad (112)$$

holds. Since $h(r_1, r_2) = h_1(r_1, r_2) + h_1(r_2, r_1)$, we have

$$S(\Lambda) = 2S_1(\Lambda). \quad (113)$$

We have

$$\int_\kappa^\Lambda \frac{r_1^2}{\omega(r_1)L(r_1, r_2, -1)} dr_1 < B + 2r_2 + 8m \log \Lambda \quad (114)$$

in the same way as the proof of $\lim_{\Lambda \rightarrow \infty} (I_1(\Lambda)/\log \Lambda) \leq C_4$. Since $r_2 < \omega(r_2)$ and $r_2^2/2m < F(r_2)$, we have

$$\begin{aligned} S_1(\Lambda) &< \int_\kappa^\Lambda (B + 2r_2 + 8m \log \Lambda) \frac{r_2^4}{\omega(r_2)F(r_2)^4} dr_2 \\ &< 32m^4 \int_\kappa^\Lambda \frac{dr_2}{r_2^4} \end{aligned} \quad (115)$$

$$+ 16m^4 (B + 8m \log \Lambda) \int_\kappa^\Lambda \frac{dr_2}{r_2^5}$$

$$< \frac{32m^5}{\kappa^4} \log \Lambda + \frac{32m^4}{3\kappa^3} + \frac{4m^4 B}{\kappa^4}.$$

From (112), (113), and (115), the lemma follows. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_3(\Lambda) / \log \Lambda) = 0$. We define $h(r_1, r_2, z)$ as

$$h(r_1, r_2, z) = \frac{zr_1^3 r_2^3}{\omega(r_1)\omega(r_2)} \left(\frac{1}{F(r_1)^2} + \frac{1}{F(r_2)^2} \right) \cdot \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \frac{1}{L(r_1, r_2, z)^2} \quad (116)$$

and redefine $S(\Lambda)$ as

$$S(\Lambda) = \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_2 dr_1. \quad (117)$$

Then we have $I_3(\Lambda) = (\pi^2 / (2\pi)^6) S(\Lambda)$. We divide $S(\Lambda)$ in the following way.

$$\begin{aligned} S(\Lambda) &= \int_{-1}^0 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_2 dr_1 \\ &\quad + \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_2 dr_1 \\ &= - \int_1^0 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, -z) dz dr_2 dr_1 \\ &\quad + \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_2 dr_1 \\ &= \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} (h(r_1, r_2, z) + h(r_1, r_2, -z)) dz dr_2 dr_1 \\ &= \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} g(r_1, r_2, z) dz dr_2 dr_1, \end{aligned} \quad (118)$$

where

$$\begin{aligned} g(r_1, r_2, z) &= h(r_1, r_2, z) + h(r_1, r_2, -z) \\ &= - \frac{2z^2 r_1^4 r_2^4}{m\omega(r_1)\omega(r_2)} \left(\frac{1}{F(r_1)^2} + \frac{1}{F(r_2)^2} \right) \\ &\quad \cdot \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \\ &\quad \cdot \frac{((r_1^2 + r_2^2)/m + 2\omega(r_1) + 2\omega(r_2))}{L(r_1, r_2, z)^2 L(r_1, r_2, -z)^2}. \end{aligned} \quad (119)$$

Since $g(r_1, r_2, z) \leq 0$, $S(\Lambda)$ is decreasing in Λ .

$$\begin{aligned} S(\Lambda) &= \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} g(r_1, r_2, z) dz dr_2 dr_1 \\ &\quad + \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_1}^{\Lambda} g(r_1, r_2, z) dz dr_1 dr_2 \\ &= 2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} g(r_1, r_2, z) dz dr_2 dr_1 \\ &= 2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_2}^{2r_2} g(r_1, r_2, z) dz dr_2 dr_1 \\ &\quad + 2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{2r_2}^{\Lambda} g(r_1, r_2, z) dz dr_2 dr_1. \end{aligned} \quad (120)$$

Since $\kappa \leq r_1$, we have $1 \leq r_1/\kappa$. Hence $r_1^2 + \nu^2 \leq ((\kappa^2 + \nu^2)/\kappa^2)r_1^2$. Therefore we have $\omega(r_1) \leq (\sqrt{\kappa^2 + \nu^2}/\kappa)r_1$, and similarly $\omega(r_2) \leq (\sqrt{\kappa^2 + \nu^2}/\kappa)r_2$. When $0 \leq z \leq 1$, we have

$$L(r_1, r_2, z) > \frac{r_1^2}{2m}. \quad (121)$$

Then

$$\begin{aligned} L(r_1, r_2, -z) &= \frac{(r_1 - r_2)^2 + 2r_1 r_2 (1 - z)}{2m} + \omega(r_1) \\ &\quad + \omega(r_2) > \omega(r_1) > r_1. \end{aligned} \quad (122)$$

When $r_2 \leq r_1$, we have $r_2^2/m \leq r_1^2/m$. Then it holds that

$$\begin{aligned} 2\omega(r_1) &\leq \frac{2r_1^2}{\kappa^2} \sqrt{\kappa^2 + \nu^2}, \\ 2\omega(r_2) &\leq \frac{2r_2}{\kappa} \sqrt{\kappa^2 + \nu^2} \leq \frac{2r_1}{\kappa^2} \sqrt{\kappa^2 + \nu^2}. \end{aligned} \quad (123)$$

Thus we have

$$\begin{aligned} \frac{r_1^2}{m} + \frac{r_2^2}{m} + 2\omega(r_1) + 2\omega(r_2) \\ \leq 2 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) r_1^2. \end{aligned} \quad (124)$$

When $r_2 \leq r_1 \leq 2r_2$, we have

$$\begin{aligned} \frac{1}{F(r_1)} + \frac{1}{F(r_2)} &< \frac{10m}{r_1^2}, \\ \frac{1}{F(r_1)^2} + \frac{1}{F(r_2)^2} &< \frac{68m^2}{r_1^4}. \end{aligned} \quad (125)$$

From (119), (121), (122), (124), and (125), it follows that

$$\begin{aligned} -g(r_1, r_2, z) &\leq \frac{2}{m} \frac{z^2 r_1^4 r_2^4}{r_1 r_2} \frac{68m^2}{r_1^4} \frac{10m}{r_1^2} \\ &\cdot 2 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) r_1^2 \left(\frac{2m}{r_1^2} \right)^2 \frac{1}{r_1^2} \\ &= 68 \cdot 10 \cdot 16m^4 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) \frac{z^2 r_2^3}{r_1^7}. \end{aligned} \quad (126)$$

Hence

$$\begin{aligned} &-2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_2}^{2r_2} g(r_1, r_2, z) dz dr_2 dr_1 \\ &\leq 1190 \frac{m^4}{\kappa^2} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right). \end{aligned} \quad (127)$$

When $2r_2 \leq r_1$, we have

$$\begin{aligned} \frac{1}{F(r_1)} + \frac{1}{F(r_2)} &< \frac{5m}{2r_2^2}, \\ \frac{1}{F(r_1)^2} + \frac{1}{F(r_2)^2} &< \frac{17m^2}{4r_2^4}. \end{aligned} \quad (128)$$

Since $r_2 \leq r_1/2$, we have $r_1/2 \leq r_1 - r_2$. Then we have

$$\begin{aligned} L(r_1, r_2, \pm z) &= \frac{(r_1 - r_2)^2 + 2r_1 r_2 (1 \pm z)}{2m} + \omega(r_1) \\ &+ \omega(r_2) > \frac{(r_1 - r_2)^2}{2m} \geq \frac{r_1^2}{8m}. \end{aligned} \quad (129)$$

From (119), (124), (128), and (129), it follows that

$$\begin{aligned} -g(r_1, r_2, z) &\leq \frac{2}{m} \frac{z^2 r_1^4 r_2^4}{r_1 r_2} \frac{17m^2}{4r_2^4} \frac{5m}{2r_2^2} \\ &\cdot 2 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) r_1^2 \left(\frac{8m}{r_1^2} \right)^4 = 2^{11} \cdot 5 \\ &\cdot 17m^6 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) \frac{z^2}{r_1^3 r_2^3}. \end{aligned} \quad (130)$$

Hence

$$\begin{aligned} &-2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{2r_2}^{\Lambda} g(r_1, r_2, z) dz dr_2 dr_1 \\ &\leq 2^7 \cdot 5 \cdot 17 \frac{m^6}{\kappa^4} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right). \end{aligned} \quad (131)$$

Then by (120), (127), and (131), we have

$$\begin{aligned} -S(\Lambda) &\leq \frac{1190m^4}{\kappa^2} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) \\ &+ 2^7 \cdot 5 \cdot 17 \frac{m^6}{\kappa^4} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right). \end{aligned} \quad (132)$$

Since $S(\Lambda)$ is decreasing and bounded below, it converges as $\Lambda \rightarrow \infty$. This fact proves the lemma. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_4(\Lambda)/\log \Lambda) = 0$. We redefine $h(r_1, r_2, z)$ as

$$\begin{aligned} h(r_1, r_2, z) &= \frac{r_1^2 r_2^2}{\omega(r_1) \omega(r_2)} \left(\frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} \right) \\ &\cdot \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \frac{1}{L(r_1, r_2, z)^2}. \end{aligned} \quad (133)$$

Then we have $I_4(\Lambda) = (2\pi^2/(2\pi)^6) \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_1 dr_2$. We define $J(\Lambda)$ as

$$J(\Lambda) = \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_1 dr_2. \quad (134)$$

Step 1. We define $S(\Lambda, z)$ as

$$S(\Lambda, z) = \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dr_1 dr_2. \quad (135)$$

Our first task is to prove that $\lim_{\Lambda \rightarrow \infty} S(\Lambda, z)$ exists for all $z \in I = [-1, 1]$. Since $h(r_1, r_2, z) > 0$, $S(\Lambda, z)$ is increasing in Λ . Let

$$\begin{aligned} S_1(\Lambda, z) &= 2 \int_{\kappa}^{\Lambda} \int_{r_2}^{2r_2} h(r_1, r_2, z) dr_2 dr_1, \\ S_2(\Lambda, z) &= 2 \int_{\kappa}^{\Lambda} \int_{2r_2}^{\Lambda} h(r_1, r_2, z) dr_2 dr_1. \end{aligned} \quad (136)$$

Then

$$\begin{aligned} S(\Lambda, z) &= \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} h(r_1, r_2, z) dr_2 dr_1 \\ &+ \int_{\kappa}^{\Lambda} \int_{r_1}^{\Lambda} h(r_1, r_2, z) dr_1 dr_2 \\ &= 2 \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} h(r_1, r_2, z) dr_2 dr_1 \\ &= 2 \int_{\kappa}^{\Lambda} \int_{r_2}^{2r_2} h(r_1, r_2, z) dr_2 dr_1 \\ &+ 2 \int_{\kappa}^{\Lambda} \int_{2r_2}^{\Lambda} h(r_1, r_2, z) dr_2 dr_1 \\ &= S_1(\Lambda, z) + S_2(\Lambda, z) \end{aligned} \quad (137)$$

holds. We have

$$\begin{aligned} \frac{1}{F(r_1)} + \frac{1}{F(r_2)} &< 2m \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right), \\ \frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} &< 4m^2 \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right), \\ r_1 &< L(r_1, r_2, z). \end{aligned} \quad (138)$$

$$r_1 < L(r_1, r_2, z). \quad (139)$$

Let $r_2 \leq r_1 \leq 2r_2$. Since $1/r_2^2 \leq 4/r_1^2$, it holds that

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} \leq \frac{5}{r_1^2}. \quad (140)$$

Then from (138) and (140), it follows that

$$\begin{aligned} \frac{1}{F(r_1)} + \frac{1}{F(r_2)} &< \frac{10m}{r_1^2}, \\ \frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} &< \frac{20m^2}{r_1^2}. \end{aligned} \quad (141)$$

Hence from (139) and (141), it follows that $h(r_1, r_2, z) < 200m^3 r_2 / r_1^5$. Therefore we have

$$\begin{aligned} S_1(\Lambda, z) &< 400m^3 \int_{\kappa}^{\Lambda} r_2 dr_2 \int_{r_2}^{2r_2} \frac{dr_1}{r_1^5} \\ &= \frac{375m^3}{8} \left(\frac{1}{\kappa^2} - \frac{1}{\Lambda^2} \right) < \frac{375m^3}{8\kappa^2}. \end{aligned} \quad (142)$$

Let $2r_2 \leq r_1$. Since $r_1/2 \leq r_1 - r_2$, we have

$$\frac{r_1^2}{8m} \leq \frac{(r_1 - r_2)^2}{2m} < L(r_1, r_2, z). \quad (143)$$

Since $r_2^2 \leq r_1^2$, we have

$$\begin{aligned} \frac{1}{F(r_1)} + \frac{1}{F(r_2)} &< \frac{4m}{r_2^2}, \\ \frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} &< \frac{8m^2}{r_2^2}. \end{aligned} \quad (144)$$

Hence from (143) and (144), it follows that $h(r_1, r_2, z) < 2048m^5 / r_1^3 r_2^3$. Therefore we have

$$\begin{aligned} S_2(\Lambda, z) &< 4096m^5 \int_{\kappa}^{\Lambda} \frac{dr_2}{r_2^3} \int_{2r_2}^{\Lambda} \frac{dr_1}{r_1^3} \\ &< 512m^5 \int_{\kappa}^{\Lambda} \frac{dr_2}{r_2^5} < \frac{128m^5}{\kappa^4}. \end{aligned} \quad (145)$$

From (137), (142), and (145), it follows that

$$S(\Lambda, z) < \frac{375m^3}{8\kappa^2} + \frac{128m^5}{\kappa^4}. \quad (146)$$

Since $S(\Lambda, z)$ is increasing in Λ and bounded above for all $z \in I$, it converges as Λ goes to infinity.

Step 2. Our second task is to prove that $J(\Lambda)$ converges when Λ goes to infinity. Let $M(r_1, r_2)$ be

$$\begin{aligned} M(r_1, r_2) &= \frac{r_1^2 r_2^2}{\omega(r_1) \omega(r_2)} \left(\frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} \right) \\ &\cdot \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \frac{1}{L(r_1, r_2, -1)^2}. \end{aligned} \quad (147)$$

$|h(r_1, r_2, z)| \leq M(r_1, r_2)$ holds for all $(r_1, r_2, z) \in [\kappa, \Lambda]^2 \times I$, and by Step 1 there exists

$$\lim_{\Lambda \rightarrow \infty} \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} M(r_1, r_2) dr_1 dr_2. \quad (148)$$

Since

$$\begin{aligned} M_{\Lambda} &= \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} M(r_1, r_2) dr_1 dr_2 \longrightarrow \\ M_{\infty} &= \lim_{\Lambda \rightarrow \infty} \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} M(r_1, r_2) dr_1 dr_2, \end{aligned} \quad (149)$$

from Cauchy convergence condition, for any $\epsilon > 0$, there exists $\Lambda_0 \in [\kappa, \infty)$ such that if $\Lambda_0 < \Lambda_1 \leq \Lambda_2$, $|M_{\Lambda_2} - M_{\Lambda_1}| < \epsilon$. Then for $\Lambda_0 < \Lambda_1 \leq \Lambda_2$ and all $z \in I$,

$$\begin{aligned} |S(\Lambda_2, z) - S(\Lambda_1, z)| &= \left| \int_{\kappa}^{\Lambda_1} \int_{\Lambda_1}^{\Lambda_2} h(r_1, r_2, z) dr_1 dr_2 \right. \\ &+ \int_{\kappa}^{\Lambda_1} \int_{\Lambda_1}^{\Lambda_2} h(r_1, r_2, z) dr_2 dr_1 \\ &+ \left. \int_{\Lambda_1}^{\Lambda_2} \int_{\Lambda_1}^{\Lambda_2} h(r_1, r_2, z) dr_1 dr_2 \right| \\ &\leq \int_{\kappa}^{\Lambda_1} \int_{\Lambda_1}^{\Lambda_2} M(r_1, r_2) dr_1 dr_2 \\ &+ \int_{\Lambda_1}^{\Lambda_2} \int_{\Lambda_1}^{\Lambda_2} M(r_1, r_2) dr_2 dr_1 \\ &+ \int_{\Lambda_1}^{\Lambda_2} \int_{\Lambda_1}^{\Lambda_2} M(r_1, r_2) dr_1 dr_2 = |M_{\Lambda_2} - M_{\Lambda_1}| < \epsilon. \end{aligned} \quad (150)$$

Therefore $\sup_{z \in I} |S(\Lambda_2, z) - S(\Lambda_1, z)| \leq |M_{\Lambda_2} - M_{\Lambda_1}| < \epsilon$ holds. Since family of functions $(S(\Lambda, \cdot))_{\Lambda \in [\kappa, \infty)}$ on I satisfies uniform Cauchy conditions, it converges uniformly on I . Since $[\kappa, \Lambda]^2$ is a Jordan measurable bounded closed set of $\mathbb{R}^2$, the function $S(\Lambda, z)$ is continuous on I . Hence

$$S(\infty, z) = \lim_{\Lambda \rightarrow \infty} \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dr_1 dr_2 \quad (151)$$

is continuous on I . Since both $S(\Lambda, z)$ and $S(\infty, z)$ are integrable on Jordan measurable set I , by uniform convergence theorem, we have

$$\lim_{\Lambda \rightarrow \infty} \int_{-1}^1 S(\Lambda, z) dz = \int_{-1}^1 S(\infty, z) dz. \quad (152)$$

It implies that $J(\Lambda)$ converges as $\Lambda \rightarrow \infty$. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_5(\Lambda)/\log \Lambda) = 0$. We redefine $h(r_1, r_2, z)$, $g(r_1, r_2, z)$, and $S(\Lambda)$ as

$$\begin{aligned} h(r_1, r_2, z) &= \frac{z r_1^3 r_2^3}{\omega(r_1) \omega(r_2) F(r_1)^2 F(r_2)^2 L(r_1, r_2, z)}, \\ g(r_1, r_2, z) &= h(r_1, r_2, z) + h(r_1, r_2, -z), \\ S(\Lambda) &= \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_1 dr_2. \end{aligned} \quad (153)$$

Then $I_5(\Lambda) = (2\pi^2/(2\pi)^6)S(\Lambda)$. We have

$$S(\Lambda) = \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} g(r_1, r_2, z) dz dr_2 dr_1 \quad (154)$$

in the same way as (118). Since

$$\begin{aligned} g(r_1, r_2, z) &= -\frac{2z^2 r_1^4 r_2^4}{m^2 \omega(r_1) \omega(r_2) F(r_1)^2 F(r_2)^2 L(r_1, r_2, z) L(r_1, r_2, -z)} \\ &\leq 0, \end{aligned} \quad (155)$$

$S(\Lambda)$ is decreasing in Λ . Since $r_1 < L(r_1, r_2, z)$, we have

$$\frac{r_1^4}{\omega(r_1) F(r_1)^2 L(r_1, r_2, z)} < \frac{4m^2}{r_1^2}. \quad (156)$$

Similarly, we have

$$\frac{r_2^4}{\omega(r_2) F(r_2)^2 L(r_1, r_2, -z)} < \frac{4m^2}{r_2^2}. \quad (157)$$

Hence

$$\begin{aligned} -S(\Lambda) &= \frac{2}{m^2} \int_0^1 z^2 dz \\ &\cdot \int_{\kappa}^{\Lambda} \frac{r_2^4}{\omega(r_2) F(r_2)^2 L(r_1, r_2, -z)} dr_2 \\ &\cdot \int_{\kappa}^{\Lambda} \frac{r_1^4}{\omega(r_1) F(r_1)^2 L(r_1, r_2, z)} dr_1 < \frac{2}{3m^2} \\ &\cdot \int_{\kappa}^{\Lambda} \frac{4m^2}{r_2^2} dr_2 \int_{\kappa}^{\Lambda} \frac{4m^2}{r_1^2} dr_1 < \frac{32m^2}{3\kappa^2}. \end{aligned} \quad (158)$$

Since $S(\Lambda)$ is decreasing and bounded below, it converges as $\Lambda \rightarrow \infty$. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_6(\Lambda)/\log \Lambda) = 0$. We redefine $h(r_1, r_2, z)$, $J(\Lambda)$, $S(\Lambda, z)$, $S_1(\Lambda, z)$, and $S_2(\Lambda, z)$ as

$$\begin{aligned} h(r_1, r_2, z) &= \frac{r_1^2 r_2^2}{\omega(r_1) \omega(r_2)} \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right)^2 \frac{r_1^2 + r_2^2}{L(r_1, r_2, z)^3}, \\ J(\Lambda) &= \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_2 dr_1, \\ S(\Lambda, z) &= \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dr_1 dr_2, \\ S_1(\Lambda, z) &= 2 \int_{\kappa}^{\Lambda} \int_{r_2}^{2r_2} h(r_1, r_2, z) dr_2 dr_1, \\ S_2(\Lambda, z) &= 2 \int_{\kappa}^{\Lambda} \int_{2r_2}^{\Lambda} h(r_1, r_2, z) dr_2 dr_1. \end{aligned} \quad (159)$$

We have $I_6(\Lambda) = (\pi^2/(2\pi)^6)J(\Lambda)$.

Step 1. Our first task is to prove that $\lim_{\Lambda \rightarrow \infty} S(\Lambda, z)$ exists for all $z \in I$. Since $h(r_1, r_2, z) > 0$, $S(\Lambda, z)$ is increasing in Λ . We have

$$S(\Lambda, z) = S_1(\Lambda, z) + S_2(\Lambda, z) \quad (160)$$

in the same way as (137). When $r_2 \leq r_1$, it holds that

$$\begin{aligned} r_1^2 + r_2^2 &\leq 2r_1^2, \\ \frac{1}{F(r_1)} + \frac{1}{F(r_2)} &< \frac{4m}{r_2^2}. \end{aligned} \quad (161)$$

When $r_2 \leq r_1 \leq 2r_2$, it also holds that

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{10m}{r_1^2}. \quad (162)$$

Then we have

$$\begin{aligned} \int_{r_2}^{2r_2} h(r_1, r_2, z) dr_1 &< \int_{r_2}^{2r_2} \frac{r_1^2 r_2^2}{r_1 r_2} \left(\frac{10m}{r_1^2} \right)^2 \frac{2r_1^2}{r_1^3} dr_1 \\ &= \frac{175m^2}{3r_2^2}. \end{aligned} \quad (163)$$

Hence

$$S_1(\Lambda, z) < \frac{350m^2}{3} \int_{\kappa}^{\Lambda} \frac{dr_2}{r_2^2} < \frac{350m^2}{3\kappa}. \quad (164)$$

Let $2r_2 \leq r_1$. Since $r_1/2 \leq r_1 - r_2$, we have

$$\frac{r_1^2}{8m} \leq \frac{(r_1 - r_2)^2}{2m} < L(r_1, r_2, z). \quad (165)$$

Then

$$\begin{aligned} & \int_{2r_2}^{\Lambda} h(r_1, r_2) dr_1 \\ & < \int_{2r_2}^{\Lambda} \frac{r_1^2 r_2^2}{r_1 r_2} \left(\frac{4m}{r_2^2} \right)^2 \left(\frac{8m}{r_1^2} \right)^3 2r_1^2 dr_1 \quad (166) \\ & = \frac{16384m^5}{r_2^3} \int_{2r_2}^{\Lambda} \frac{dr_1}{r_1^3} < \frac{2048m^5}{r_2^5}. \end{aligned}$$

Therefore

$$S_2(\Lambda, z) < 4096m^5 \int_{\kappa}^{\Lambda} \frac{dr_2}{r_2^5} < \frac{1024m^5}{\kappa^4}. \quad (167)$$

From (160), (164), and (167), it follows that $S(\Lambda, z) < 350m^2/3\kappa + 1024m^5/\kappa^4$. Since $S(\Lambda, z)$ is increasing in Λ and bounded above, it converges as Λ goes to infinity.

Step 2. Our second task is to prove $J(\Lambda)$ converges as Λ goes to infinity. This step is the same as that of $\lim_{\Lambda \rightarrow \infty} (I_4(\Lambda)/\log \Lambda) = 0$. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_7(\Lambda)/\log \Lambda) = 0$. We redefine $h(r_1, r_2, z)$, $g(r_1, r_2, z)$, and $S(\Lambda)$ as

$$\begin{aligned} & h(r_1, r_2, z) \\ & = \frac{zr_1^3 r_2^3}{\omega(r_1)\omega(r_2)} \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right)^2 \frac{1}{L(r_1, r_2, z)^3}, \quad (168) \\ & g(r_1, r_2, z) = h(r_1, r_2, z) + h(r_1, r_2, -z), \end{aligned}$$

$$S(\Lambda) = \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_1 dr_2.$$

Then we have $I_7(\Lambda) = (2\pi^2/(2\pi)^6)S(\Lambda)$, and

$$S(\Lambda) = \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} g(r_1, r_2, z) dz dr_1 dr_2 \quad (169)$$

in the same way as $\lim_{\Lambda \rightarrow \infty} (I_3(\Lambda)/\log \Lambda) = 0$. We define $g_1(r_1, r_2, z)$ and $g_2(r_1, r_2, z)$ as

$$\begin{aligned} & g_1(r_1, r_2, z) = -\frac{6z^2 r_1^4 r_2^4}{m\omega(r_1)\omega(r_2)} \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right)^2 \\ & \cdot \frac{\left((r_1^2 + r_2^2)/2m + \omega(r_1) + \omega(r_2) \right)^2}{L(r_1, r_2, z)^3 L(r_1, r_2, -z)^3}, \\ & g_2(r_1, r_2, z) \quad (170) \\ & = -\frac{2z^4 r_1^6 r_2^6}{m^3 \omega(r_1)\omega(r_2)} \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right)^2 \\ & \cdot \frac{1}{L(r_1, r_2, z)^3 L(r_1, r_2, -z)^3}. \end{aligned}$$

Then we have $g(r_1, r_2, z) = g_1(r_1, r_2, z) + g_2(r_1, r_2, z)$. We redefine $S_1(\Lambda)$ and $S_2(\Lambda)$ by

$$S_1(\Lambda) = \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} g_1(r_1, r_2, z) dz dr_1 dr_2, \quad (171)$$

$$S_2(\Lambda) = \int_0^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} g_2(r_1, r_2, z) dz dr_1 dr_2.$$

Then

$$S(\Lambda) = S_1(\Lambda) + S_2(\Lambda). \quad (172)$$

Since $g_1(r_1, r_2, z) \leq 0$, $S_1(\Lambda)$ is decreasing in Λ . We divide $S_1(\Lambda)$ in the following way:

$$\begin{aligned} S_1(\Lambda) & = \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_1}^{\Lambda} g_1(r_1, r_2, z) dz dr_1 dr_2 \\ & + \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} g_1(r_1, r_2, z) dz dr_2 dr_1 \\ & = \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_1}^{\Lambda} g_1(r_1, r_2, z) dz dr_1 dr_2 \quad (173) \\ & + \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_1}^{\Lambda} g_1(r_2, r_1, z) dz dr_1 dr_2 \\ & = 2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_1}^{\Lambda} g_1(r_1, r_2, z) dz dr_1 dr_2. \end{aligned}$$

Let $\kappa \leq r_1 \leq r_2$. Then we have

$$\frac{r_1^2 + r_2^2}{2m} + \omega(r_1) + \omega(r_2) \leq \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) r_2^2 \quad (174)$$

in the same way as (124). Let $r_1 \leq r_2 \leq 2r_1$. Then we also have

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{10m}{r_2^2}. \quad (175)$$

Therefore

$$\begin{aligned} & -g_1(r_1, r_2, z) \leq \frac{6}{m} \frac{z^2 r_1^4 r_2^4}{r_1 r_2} \left(\frac{10m}{r_2^2} \right)^2 \\ & \cdot \left\{ \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) r_2^2 \right\}^2 \left(\frac{2m}{r_2^2} \right)^3 \frac{1}{r_2^3} \\ & = 4800m^4 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \frac{z^2 r_1^3}{r_2^6}, \\ & -\int_{r_1}^{2r_1} g_1(r_1, r_2, z) dr_2 \quad (176) \\ & \leq 4800m^4 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 z^2 r_1^3 \int_{r_1}^{2r_1} \frac{dr_2}{r_2^6} \\ & = 930m^4 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \frac{z^2}{r_1^2}. \end{aligned}$$

Hence

$$\begin{aligned}
 & -2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_1}^{2r_1} g_1(r_1, r_2, z) dz dr_1 dr_2 \\
 & \leq 1860m^4 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \int_0^1 z^2 dz \int_{\kappa}^{\Lambda} \frac{dr_2}{r_2^2} \quad (177) \\
 & < \frac{620m^4}{\kappa} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2.
 \end{aligned}$$

Let $2r_1 \leq r_2$. Then we have

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{5m}{2r_1^2}. \quad (178)$$

In addition, since $r_2/2 \leq r_2 - r_1$, we can see that

$$\frac{r_2^2}{8m} \leq \frac{(r_2 - r_1)^2}{2m} < L(r_1, r_2, -z). \quad (179)$$

Therefore

$$\begin{aligned}
 -g_1(r_1, r_2, z) & \leq \frac{6}{m} \frac{z^2 r_1^4 r_2^4}{r_1 r_2} \left(\frac{5m^2}{2r_1^2} \right)^2 \\
 & \cdot \left\{ \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right) r_2^2 \right\}^2 \left(\frac{2m}{r_2^2} \right)^3 \left(\frac{8m}{r_2^2} \right)^3 \quad (180) \\
 & = 153600m^9 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \frac{z^2}{r_1 r_2^5}.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 & - \int_{2r_1}^{\Lambda} g_1(r_1, r_2, z) dr_2 \\
 & \leq \frac{153600m^9 z^2}{r_1} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \int_{2r_1}^{\Lambda} \frac{dr_2}{r_2^5} \quad (181) \\
 & \leq 2400m^9 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \frac{z^2}{r_1^5}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 & -2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{2r_1}^{\Lambda} g_1(r_1, r_2, z) dz dr_1 dr_2 \\
 & \leq 4800m^9 \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \int_0^1 z^2 dz \int_{\kappa}^{\Lambda} \frac{dr_1}{r_1^5} \quad (182) \\
 & < \frac{400m^9}{\kappa^4} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 -S_1(\Lambda) & < \frac{620m^4}{\kappa} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2 \\
 & + \frac{400m^9}{\kappa^4} \left(\frac{1}{m} + \frac{2\sqrt{\kappa^2 + \nu^2}}{\kappa^2} \right)^2. \quad (183)
 \end{aligned}$$

Since $S_1(\Lambda)$ is decreasing and bounded below, it converges as $\Lambda \rightarrow \infty$. Since $g_2(r_1, r_2, z) \leq 0$, $S_2(\Lambda)$ is also decreasing in Λ . Let $r_1 \leq r_2$. Then

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{4m}{r_1^2}. \quad (184)$$

Therefore

$$\begin{aligned}
 -g_2(r_1, r_2, z) & \leq \frac{2}{m^3} \frac{z^4 r_1^6 r_2^6}{r_1 r_2} \left(\frac{4m}{r_1^2} \right)^2 \frac{8m^3}{r_2^6} \frac{1}{r_2^3} \\
 & = \frac{256m^2 z^4 r_1}{r_2^4}. \quad (185)
 \end{aligned}$$

Then

$$\begin{aligned}
 - \int_{r_1}^{\Lambda} g_2(r_1, r_2, z) dr_2 & \leq 256m^2 z^4 r_1 \int_{r_1}^{\Lambda} \frac{dr_2}{r_2^4} \\
 & \leq \frac{256m^2 z^4}{3r_1^2}. \quad (186)
 \end{aligned}$$

Hence

$$-S_2(\Lambda) \leq \frac{512m^2}{3} \int_0^1 z^4 dz \int_{\kappa}^{\Lambda} \frac{dr_1}{r_1^2} < \frac{512m^2}{15\kappa}. \quad (187)$$

Since $S_2(\Lambda)$ is decreasing in Λ and bounded below, it converges. Since both $S_1(\Lambda)$ and $S_2(\Lambda)$ converge, $S(\Lambda)$ converges. $\square$

Proof of $\lim_{\Lambda \rightarrow \infty} (I_8(\Lambda)/\log \Lambda) = 0$. We redefine $h(r_1, r_2, z)$, $g(r_1, r_2, z)$, $g_1(r_1, r_2, z)$, $g_2(r_1, r_2, z)$, $S(\Lambda)$, $S_1(\Lambda)$, and $S_2(\Lambda)$ as

$$\begin{aligned}
 h(r_1, r_2, z) & = \frac{z r_1^3 r_2^3}{\omega(r_1) \omega(r_2)} \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \\
 & \cdot \frac{1}{L(r_1, r_2, z)^4}, \quad (188)
 \end{aligned}$$

$$g(r_1, r_2, z) = h(r_1, r_2, z) + h(r_1, r_2, -z),$$

$$\begin{aligned}
 g_1(r_1, r_2, z) & = -\frac{2z^2 r_1^4 r_2^4}{m\omega(r_1)\omega(r_2)} \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \\
 & \cdot \frac{1}{L(r_1, r_2, z)L(r_1, r_2, -z)^4}, \quad (189)
 \end{aligned}$$

$$\begin{aligned}
 g_2(r_1, r_2, z) & = -\frac{2z^2 r_1^4 r_2^4}{m\omega(r_1)\omega(r_2)} \left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)} \right) \\
 & \cdot G(r_1, r_2, z), \quad (190)
 \end{aligned}$$

$$S(\Lambda) = \int_{-1}^1 \int_{\kappa}^{\Lambda} \int_{\kappa}^{\Lambda} h(r_1, r_2, z) dz dr_2 dr_1,$$

$$S_1(\Lambda) = \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} \int_0^{1-1/r_1^{1/4}r_2^{1/2}} g_1(r_1, r_2, z) dr_2 dr_1 dz, \quad (191)$$

$$S_2(\Lambda) = \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} \int_{1-1/r_1^{1/4}r_2^{1/2}}^1 g_1(r_1, r_2, z) dr_2 dr_1 dz,$$

where

$$G(r_1, r_2, z) = \frac{1}{L(r_1, r_2, z)^2 L(r_1, r_2, -z)^3} + \frac{1}{L(r_1, r_2, z)^3 L(r_1, r_2, -z)^2} + \frac{1}{L(r_1, r_2, z)^4 L(r_1, r_2, -z)}. \quad (192)$$

Furthermore, we define $S_3(\Lambda)$ as

$$S_3(\Lambda) = \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} g_2(r_1, r_2, z) dz dr_2 dr_1. \quad (193)$$

Then we have $I_8(\Lambda) = (2\pi^2/(2\pi)^6)S(\Lambda)$, and

$$S(\Lambda) = 2 \int_0^1 \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} g(r_1, r_2, z) dz dr_2 dr_1 \quad (194)$$

in the same way as the proof of $\lim_{\Lambda \rightarrow \infty} (I_3(\Lambda)/\log \Lambda) = 0$. Since $g(r_1, r_2, z) = g_1(r_1, r_2, z) + g_2(r_1, r_2, z)$, it holds that

$$S(\Lambda) = 2S_1(\Lambda) + 2S_2(\Lambda) + 2S_3(\Lambda). \quad (195)$$

Since $g_1(r_1, r_2, z) \leq 0$ and $g_2(r_1, r_2, z) \leq 0$, $S_i(\Lambda)$ ($i = 1, 2, 3$) are decreasing in Λ . Let $r_2 \leq r_1$. Then

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{4m}{r_2^2}. \quad (196)$$

Let $0 \leq z \leq 1 - 1/r_1^{1/4} r_2^{1/2}$. Then we have

$$\frac{1}{(1-z)^4} \leq r_1 r_2^2. \quad (197)$$

We have

$$L(r_1, r_2, -z) = \frac{(r_1 - r_2)^2 + 2r_1 r_2 (1-z)}{2m} + \omega(r_1) + \omega(r_2) > \frac{r_1 r_2 (1-z)}{m}. \quad (198)$$

Using (197) and (198), we have

$$\frac{1}{L(r_1, r_2, -z)^4} < \frac{m^4}{r_1^4 r_2^4 (1-z)^4} \leq \frac{m^4}{r_1^3 r_2^2}. \quad (199)$$

From (189), (196), and (199), it follows that

$$-g_1(r_1, r_2, z) \leq \frac{2}{m} \frac{z^2 r_1^4 r_2^4}{r_1 r_2} \frac{4m}{r_2^2} \frac{2m}{r_1^2} \frac{m^4}{r_1^3 r_2^2} = \frac{16m^5}{r_1^2 r_2}. \quad (200)$$

Hence we have

$$\begin{aligned} -S_1(\Lambda) &\leq 16m^5 \int_{\kappa}^{\Lambda} \frac{dr_2}{r_2} \int_{r_2}^{\Lambda} \frac{1}{r_1^2} \left(1 - \frac{1}{r_1^{1/4} r_2^{1/2}}\right) dr_1 \\ &< 16m^5 \int_{\kappa}^{\Lambda} \frac{dr_2}{r_2} \int_{r_2}^{\Lambda} \frac{dr_1}{r_1^2} < \frac{16m^5}{\kappa}. \end{aligned} \quad (201)$$

Since $S_1(\Lambda)$ is decreasing in Λ and bounded below, it converges. When $r_2 \leq r_1$ and $1 - 1/r_1^{1/4} r_2^{1/2} \leq z \leq 1$, from (189) and (196), it holds that

$$-g_1(r_1, r_2, z) < \frac{2}{m} \frac{r_1^4 r_2^4}{r_1 r_2} \frac{4m}{r_2^2} \frac{2m}{r_1^2} \frac{1}{r_1^4} = \frac{16mr_2}{r_1^3}. \quad (202)$$

Hence we have

$$\begin{aligned} -S_2(\Lambda) &< 16m \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} \int_{1-1/r_1^{1/4} r_2^{1/2}}^1 \frac{r_2}{r_1^3} dr_2 dr_1 dz \\ &= 16m \int_{\kappa}^{\Lambda} \int_{r_2}^{\Lambda} \frac{r_2}{r_1^3} r_1^{-1/4} r_2^{-1/2} dr_2 dr_1 \\ &= 16m \int_{\kappa}^{\Lambda} r_2^{1/2} dr_2 \int_{r_2}^{\Lambda} r_1^{-13/4} dr_1 \\ &= \frac{64m}{9} \int_{\kappa}^{\Lambda} r_2^{1/2} (r_2^{-9/4} - \Lambda^{-9/4}) dr_2 \\ &< \frac{64m}{9} \int_{\kappa}^{\Lambda} r_2^{-7/4} dr_2 \\ &= \frac{256m}{27} (\kappa^{-3/4} - \Lambda^{-3/4}) < \frac{256m}{27\kappa^{3/4}}. \end{aligned} \quad (203)$$

Since $S_2(\Lambda)$ is decreasing in Λ and bounded below, it converges. We have

$$G(r_1, r_2, z) < \frac{4m^2}{r_1^7} + \frac{8m^3}{r_1^8} + \frac{16m^4}{r_1^9}. \quad (204)$$

From (190), (196), and (204), we have

$$\begin{aligned} -g_2(r_1, r_2, z) &\leq \frac{2}{m} \frac{z^2 r_1^4 r_2^4}{r_1 r_2} \frac{4m}{r_2^2} \left(\frac{4m^2}{r_1^7} + \frac{8m^3}{r_1^8} + \frac{16m^4}{r_1^9} \right) \\ &< 8r_1^3 r_2 \left(\frac{4m^2}{r_1^7} + \frac{8m^3}{r_1^8} + \frac{16m^4}{r_1^9} \right) \\ &= \frac{32m^2 r_2}{r_1^4} + \frac{64m^3 r_2}{r_1^5} + \frac{128m^4 r_2}{r_1^6}. \end{aligned} \quad (205)$$

Hence

$$\begin{aligned} -S_3(\Lambda) &< 32m^2 \int_{\kappa}^{\Lambda} r_2 dr_2 \int_{r_2}^{\Lambda} \frac{dr_1}{r_1^4} \\ &\quad + 64m^3 \int_{\kappa}^{\Lambda} r_2 dr_2 \int_{r_2}^{\Lambda} \frac{dr_1}{r_1^5} \\ &\quad + 128m^4 \int_{\kappa}^{\Lambda} r_2 dr_2 \int_{r_2}^{\Lambda} \frac{dr_1}{r_1^6} \\ &< \frac{32m^2}{3} \int_{\kappa}^{\Lambda} \frac{1}{r_2^2} dr_2 + 16m^3 \int_{\kappa}^{\Lambda} \frac{1}{r_2^3} dr_2 \end{aligned}$$

$$\begin{aligned}
& + \frac{128m^4}{5} \int_{\kappa}^{\Lambda} \frac{1}{r_2^4} dr_2 \\
& < \frac{32m^2}{3\kappa} + \frac{8m^3}{\kappa^2} + \frac{128m^4}{15\kappa^3}.
\end{aligned} \quad (206)$$

Since $S_3(\Lambda)$ is decreasing in Λ and bounded below, it converges. Since $S_i(\Lambda)$ ($i = 1, 2, 3$) converge, $I_8(\Lambda)$ converges by (195). $\square$

5. Concluding Remarks

(1) The Nelson model is defined as the self-adjoint operator

$$H_V = \left(-\frac{1}{2}\Delta + V\right) \otimes 1 + 1 \otimes H_f + \alpha \int_{\mathbb{R}^3}^{\oplus} \phi(x) dx, \quad (207)$$

acting in the Hilbert space $L^2(\mathbb{R}) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx$. Here $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is an external potential and

$$\begin{aligned}
& \phi(x) \\
& = \frac{1}{\sqrt{2}} \int \left\{ \frac{a^\dagger(k) e^{-ikx} \hat{\phi}(k)}{\sqrt{\omega(k)}} + \frac{a(k) e^{ikx} \hat{\phi}(-k)}{\sqrt{\omega(k)}} \right\} dk.
\end{aligned} \quad (208)$$

In the case of $V = 0$, $H_{V=0}$ is translation invariant and the relationship between H_V and $H(p)$ is given by

$$H_{V=0} = \int_{\mathbb{R}^3}^{\oplus} H(p) dp. \quad (209)$$

Furthermore the ground state energy of $H(p = 0)$ coincides with that of $H_{V=0}$.

(2) We show that $m_{\text{eff}}(\Lambda)/m = 1 + \sum_{n=1}^{\infty} a_n(\Lambda) \alpha^{2n}$ and $\lim_{\Lambda \rightarrow \infty} a_2(\Lambda) = \pm\infty$. It is also expected that $\lim_{\Lambda \rightarrow \infty} a_n(\Lambda)$ diverges and the signatures are alternatively changed. Hence $\lim_{\Lambda \rightarrow \infty} m_{\text{eff}}(\Lambda)/m$ may converge but it is not trivial to see it directly.

(3) The relativistic Nelson model is defined by replacing $-(1/2)\Delta + V$ with the semirelativistic Schrödinger operator $\sqrt{-\Delta + 1} + V$ in (207); that is,

$$H_V^{\text{rel}} = \left(\sqrt{-\Delta + 1} + V\right) \otimes 1 + 1 \otimes H_f + \int_{\mathbb{R}^3}^{\oplus} \phi(x) dx. \quad (210)$$

Then it follows that

$$H_{V=0}^{\text{rel}} = \int_{\mathbb{R}^3}^{\oplus} H^{\text{rel}}(p) dp, \quad (211)$$

where $H^{\text{rel}}(p) = \sqrt{(p - P_f)^2 + 1} + H_f + \phi(0)$. Then the effective mass $m_{\text{eff}}(\Lambda)$ of $H^{\text{rel}}(p)$ is defined in the same way as that of $H(p)$. We are also interested in seeing the asymptotic behavior of $m_{\text{eff}}(\Lambda)$ as $\Lambda \rightarrow \infty$. However $\sqrt{(p - P_f)^2 + 1}$ is a nonlocal operator and then estimates are rather complicated.

Another interesting nonlocal model is the so-called semirelativistic Pauli-Fierz model defined by

$$\begin{aligned}
H_V^{\text{PF}} &= \sqrt{\left(-i\nabla \otimes -\alpha \int_{\mathbb{R}^3}^{\oplus} A(x) dx\right)^2 + 1} + V \otimes 1 + 1 \\
&\otimes H_f,
\end{aligned} \quad (212)$$

where $A(x)$ is a quantized radiation field. See [9] for the detail. Then it follows that

$$H_{V=0}^{\text{PF}} = \int_{\mathbb{R}^3}^{\oplus} H^{\text{PF}}(p) dp, \quad (213)$$

where $H^{\text{PF}}(p) = \sqrt{(p - P_f - \alpha A(0))^2 + 1} + H_f$. It is also interesting to investigate the asymptotic behavior of the effective mass of the semirelativistic Pauli-Fierz model.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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