Research Article

Star-Supermagic Decompositions of the Complete Bipartite Graph Minus a One-Factor

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Let G be a graph and let H be a subgraph of G . Assume that G has an H -decomposition $T = \{H_1, H_2, \dots, H_t\}$ such that $H_i \cong H$ for all $1 \leq i \leq t$. An H -supermagic decomposition of G is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that $\sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e)$ is a constant k for each H_i in the decomposition T and $f(V(G)) = \{1, 2, \dots, |V(G)|\}$. If G admits an H -supermagic decomposition, then G is called H -supermagic decomposable. In this paper, we give necessary and sufficient conditions for the existence of $K_{1,n-1}$ -supermagic decomposition of the complete bipartite graph $K_{n,n}$ minus a one-factor.

1. Introduction

All graphs in this paper are finite and simple. We use terminologies of graphs from West's textbook [1]. If m and n are integers with $m \leq n$, we denote $\{m, m+1, \dots, n\}$ by $[m, n]$.

Let G be a graph. A *covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of G belongs to at least one of the subgraphs H_i . If $H_i \cong H$, for all $i \in [1, t]$, then the covering is called an H -covering of G .

Assume that G has an H -covering. An H -magic labeling of G is a bijection $f : V(G) \cup E(G) \rightarrow [1, |V(G)| + |E(G)|]$ such that $\sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e)$ is a constant k for each H_i in the H -covering, and k is called a *magic constant*. An H -magic labeling f is called an H -supermagic labeling if $f(V(G)) = [1, |V(G)|]$. A graph G is said to be H -supermagic if G admits an H -magic labeling.

The H -supermagic labeling was first introduced by Gutiérrez and Lladó [2] in 2005. They proved that many classes of graphs are H -supermagic, such as the complete bipartite graphs $K_{m,n}$ and the star $K_{1,n}$ are $K_{1,l}$ -supermagic for some positive integer l . In 2007, Lladó and Moragas [3] studied some windmills, wheels, and thetas which are

H -supermagic. For example, they showed that the wheel W_n for odd n is C_3 -supermagic.

In 2010, Ngurah et al. [4] proved that the fan $P_n \vee K_1$, the triangle ladder TL_n , and $K_{1,n} \vee K_1$ are C_3 -supermagic. Furthermore, they showed that the book B_n , the ladder L_n , and the grid $P_m \times P_n$ are C_4 -supermagic.

An H -decomposition of a graph G is an H -covering of G which forms a partition of the edge set of G . If G has an H -decomposition, then G is called H -decomposable. Assume that G is H -decomposable. If G admits an H -supermagic labeling, then G is said to be H -supermagic decomposable or G has an H -supermagic decomposition.

The concept of an H -supermagic decomposition of G arises from the combination between graph labelings and graph decompositions which was introduced by Liang [5] in 2012. He found the conditions for the existence of C_{2k} -supermagic decomposition of the complete n -partite graph as well as of multiple copies of it. Moreover, in 2015, Marimuthu and Kumar [6] showed that the complete bipartite graph $K_{n,n}$ is $K_{1,n}$ -supermagic decomposable. For more information about graph labelings, please see Gallian's survey [7].

Motivated by the results of these authors, we are going to find necessary and sufficient conditions for the existence of an H -supermagic decomposition of the complete bipartite graph $K_{n,n}$ minus a one-factor, where $H \cong K_{1,n-1}$. Hereafter, if $H \cong K_{1,n}$, for some positive integer n , then H is called an n -star subgraph. The notation of an n -star-subgraph was introduced by Akiyama and Kano [8].

2. Main Result

In this section, we consider the graph $K_{n,n} - I$, where I is a one-factor of $K_{n,n}$ and $n \geq 2$. Note that $K_{n,n} - I$ is $(n-1)$ -star-decomposable for all $n \geq 2$. Since $|V(K_{n,n} - I)| = 2n$ and $|E(K_{n,n} - I)| = n^2 - n$, we have $|V(K_{n,n} - I)| + |E(K_{n,n} - I)| = n^2 + n$.

Theorem 1. *Let n be a positive integer with $n \equiv 2 \pmod{4}$. Then $K_{n,n} - I$ is not an $(n-1)$ -star-supermagic decomposable graph.*

Proof. Suppose that $K_{n,n} - I$ is $(n-1)$ -star-supermagic decomposable. Let f be an $(n-1)$ -star-supermagic labeling of $K_{n,n} - I$ with the magic constant k . Let $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$ be vertex-bipartitions of $K_{n,n} - I$. Let $\{H_1, H_2, \dots, H_n\}$ be an $(n-1)$ -star-decomposition of $K_{n,n} - I$, where H_i contains one vertex of U and $n-1$ vertices of V for all $i \in [1, 2n]$.

Consider the labeling f ; assume that all vertices of U are labeled by $\{a_1, a_2, \dots, a_n\}$ and all vertices of V are labeled by $\{a_{n+1}, a_{n+2}, \dots, a_{2n}\}$. Note that $a_i \in [1, 2n]$ for all $i \in [1, 2n]$. Then

$$\begin{aligned} \sum_{i=1}^n \sum_{v \in V(H_i)} f(v) + \sum_{i=1}^n \sum_{e \in E(H_i)} f(e) &= [(a_1 + a_2 + \dots + a_n) \\ &+ (n-1)(a_{n+1} + a_{n+2} + \dots + a_{2n})] + [(2n+1) \\ &+ (2n+2) + \dots + (n^2 + n)] = [(a_1 + a_2 + \dots + a_{2n}) \\ &+ (n-2)(a_{n+1} + a_{n+2} + \dots + a_{2n})] + \sum_{i=2n+1}^{n^2+n} i = \sum_{i=1}^{2n} i \\ &+ (n-2)(a_{n+1} + a_{n+2} + \dots + a_{2n}) + \sum_{i=2n+1}^{n^2+n} i = \sum_{i=1}^{n^2+n} i \\ &+ (n-2)(a_{n+1} + a_{n+2} + \dots + a_{2n}) \\ &= \frac{(n^2 + n)(n^2 + n + 1)}{2} + (n-2)(a_{n+1} + a_{n+2} + \dots \\ &+ a_{2n}). \end{aligned} \quad (1)$$

Since $n \equiv 2 \pmod{4}$, we have $2 \mid n$ but $4 \nmid n$; that is, $n = 2(2k+1)$ for some integer k . Note that $(n^2 + n)(n^2 + n + 1)/2$ is odd and $(n-2)(a_{n+1} + a_{n+2} + \dots + a_{2n})$ is even. Thus, $\sum_{i=1}^n \sum_{v \in V(H_i)} f(v) + \sum_{i=1}^n \sum_{e \in E(H_i)} f(e)$ is odd. We know that $\sum_{i=1}^n \sum_{v \in V(H_i)} f(v) + \sum_{i=1}^n \sum_{e \in E(H_i)} f(e) = kn$ because the magic constant is k . Since n is even, this is impossible and

results in a contradiction to our assumption. Hence, $K_{n,n} - I$ is not an $(n-1)$ -star-supermagic decomposable graph. $\square$

The necessary condition for the existence of an $(n-1)$ -star-supermagic decomposition of $K_{n,n} - I$ is settled directly from Theorem 1.

Corollary 2. *If $K_{n,n} - I$ is $(n-1)$ -star-supermagic decomposable, then n is odd or $n \equiv 0 \pmod{4}$.*

Next we establish a structural lemma which will be used in the proof of Theorem 4.

Lemma 3. *For any positive integer m , the set $[1, 12m]$ can be partitioned into 3 elements subsets which satisfy two conditions as follows:*

- (i) *There are $3m$ subsets of $[1, 12m]$ such that the summation of all elements in each subset is $18m + 1$.*
- (ii) *There are m subsets of $[1, 12m]$ such that the summation of all elements in each subset is $18m + 3$.*

Proof. For $m \in \mathbb{N}$, let $A = [1, 4m]$, let $B = [4m + 1, 8m]$, and let $C = [8m + 1, 12m]$.

Define $f : A \rightarrow B$ by

$$f(i) = \begin{cases} 8m - 2i + 1, & \text{if } i \in [1, 2m], \\ 6m, & \text{if } i = 2m + 1, \\ 12m - 2i + 4, & \text{if } i \in [2m + 2, 3m + 1], \\ 12m - 2i + 2, & \text{if } i \in [3m + 2, 4m]. \end{cases} \quad (2)$$

Let $s_i = i + f(i)$ for all $i \in [1, 4m]$. Then the set

$$\begin{aligned} S &= \{s_i \mid i \in [1, 4m]\} \\ &= [6m + 1, 9m] \cup [9m + 3, 10m + 2]. \end{aligned} \quad (3)$$

Now, let $s'_1 = 6m + 1$, let $s'_2 = 6m + 2, \dots$, and let $s'_{3m} = 9m$ and let $s'_{3m+1} = 9m + 3$, let $s'_{3m+2} = 9m + 4, \dots$, and let $s'_{4m} = 10m + 2$. Thus, $S = \{s'_i \mid i \in [1, 4m]\}$.

Let $c_i = 12m - (i - 1)$ for all $i \in [1, 4m]$. Then $C = [8m + 1, 12m] = \{c_i \mid i \in [1, 4m]\}$.

Define $g : S \rightarrow C$ by $g(s'_i) = c_i$ for all $i \in [1, 4m]$. For each $i \in [1, 4m]$, there exists an element $s_j \in S$ such that $s_j = s'_i$, so $g(s_j) = g(s'_i) = c_i$. Then we have

$$j + f(j) + c_i = \begin{cases} 18m + 1, & \text{if } i \in [1, 3m], \\ 18m + 3, & \text{if } i \in [3m + 1, 4m]. \end{cases} \quad (4)$$

Hence, there are $3m$ subsets of $[1, 12m]$ such that the summation of all elements in each subset is $18m + 1$, and there are m subsets of $[1, 12m]$ such that the summation of all elements in each subset is $18m + 3$. $\square$

Theorem 4. *Let n be a positive integer with $n \equiv 0 \pmod{4}$. Then $K_{n,n} - I$ is an $(n-1)$ -star-supermagic decomposable graph with the magic constant $(2n^3 + 8n^2 - 3n)/4$.*

Proof. Let $n = 4m$ for some positive integer m , and let $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$ be vertex-bipartition of $K_{n,n} - I$. Let $\{G_1, G_2, \dots, G_n\}$ be an $(n-1)$ -star-decomposition of $K_{n,n} - I$, where $G_i \cong K_{1,n-1}$ with $V(G_i) = \{u_i, v_1, v_2, \dots, v_n\} \setminus \{v_i\}$ and $E(G_i) = \{u_i v_1, u_i v_2, \dots, u_i v_n\} \setminus \{u_i v_i\}$ for all $i \in [1, n]$.

Define a total labeling $f : V(K_{n,n} - I) \cup E(K_{n,n} - I) \rightarrow [1, n^2 + n]$ by

$$f(u_i) = \begin{cases} 2i, & \text{if } i \in [1, 3m], \\ 2i - 1, & \text{if } i \in [3m + 1, 4m], \end{cases} \quad (5)$$

$$f(v_i) = \begin{cases} 2i - 1, & \text{if } i \in [1, 3m], \\ 2i, & \text{if } i \in [3m + 1, 4m], \end{cases}$$

for $i \in [1, n]$, and

$$f(u_i v_j) = \begin{cases} 2kn + i, & \text{if } i < j, j = 2k, \\ 2kn + 1 - i, & \text{if } i < j, j = 2k - 1, \\ 2kn + i, & \text{if } i > j, j = 2k - 1, j \neq n - 3, \\ 2kn + 2n + 1 - i, & \text{if } i > j, j = 2k, j \neq n - 3, \end{cases} \quad (6)$$

for $i, j \in [1, n - 3]$, where $i \neq j$.

Note that

$$\sum_{v \in V(G_i)} f(v) = \begin{cases} n^2 + \frac{n}{4} + 1, & \text{if } i \in [1, 3m], \\ n^2 + \frac{n}{4} - 1, & \text{if } i \in [3m + 1, 4m]. \end{cases} \quad (7)$$

Now, $n - 4$ edges of each G_i have been labeled with the partial magic sum $(n^3 - 4n^2 + n - 4)/2$, and their edges are labeled with values from $[2n + 1, n^2 - 2n]$. There are still three edges within each G_i which need to be labeled, and their label will come from $[n^2 - 2n + 1, n^2 + n]$. By applying Lemma 3 to the labeling f , we have the following properties:

- (i) There are $3m$ subsets of $[n^2 - 2n + 1, n^2 + n]$ such that the summation of all elements in each subset is $3n^2 - 3n/2 + 1$.
- (ii) There are m subsets of $[n^2 - 2n + 1, n^2 + n]$ such that the summation of all elements in each subset is $3n^2 - 3n/2 + 3$.

Thus, G_i has the summation of its three remaining edges as $3n^2 - 3n/2 + 1$ for $i \in [1, 3m]$ and $3n^2 - 3n/2 + 3$ for $i \in [3m + 1, 4m]$.

Hence, for $i \in [1, 3m]$,

$$\begin{aligned} \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e) &= \left(n^2 + \frac{n}{4} + 1\right) + \left(\frac{n^3 - 4n^2 + n - 4}{2}\right) \\ &+ \left(3n^2 - \frac{3n}{2} + 1\right) = \frac{2n^3 + 8n^2 - 3n}{4}, \end{aligned} \quad (8)$$

and, for $i \in [3m + 1, 4m]$,

$$\begin{aligned} \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e) &= \left(n^2 + \frac{n}{4} - 1\right) + \left(\frac{n^3 - 4n^2 + n - 4}{2}\right) \\ &+ \left(3n^2 - \frac{3n}{2} + 3\right) = \frac{2n^3 + 8n^2 - 3n}{4}. \end{aligned} \quad (9)$$

Therefore $K_{n,n} - I$ is an $(n-1)$ -star-supermagic decomposable graph, where $n \equiv 2 \pmod{4}$ with the magic constant $(2n^3 + 8n^2 - 3n)/4$. $\square$

Example 5. Consider the graph $K_{8,8} - I$. Let $U = \{u_1, u_2, \dots, u_8\}$ and $V = \{v_1, v_2, \dots, v_8\}$ be vertex-bipartition of $K_{8,8} - I$. We have that $K_{8,8} - I$ is 7-star-supermagic decomposable by using the labeling f in Table 1 with the magic constant 378.

Theorem 6. Let n be a positive integer, where n is odd and $n \geq 3$. Then $K_{n,n} - I$ is an $(n-1)$ -star-supermagic decomposable graph with the magic constant $(n^3 + 5n^2 - 3n - 1)/2$.

Proof. Let $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$ be vertex-bipartition of $K_{n,n} - I$. Let $\{G_1, G_2, \dots, G_n\}$ be an $(n-1)$ -star-decomposition of $K_{n,n} - I$, where $G_i \cong K_{1,n-1}$ with $V(G_i) = \{u_i, v_1, v_2, \dots, v_n\} \setminus \{v_i\}$ and $E(G_i) = \{u_i v_1, u_i v_2, \dots, u_i v_n\} \setminus \{u_i v_i\}$ for all $i \in [1, n]$.

Define a total labeling $f : V(K_{n,n} - I) \cup E(K_{n,n} - I) \rightarrow [1, n^2 + n]$ by

$$\begin{aligned} f(u_i) &= i, \\ f(v_i) &= n + i, \end{aligned} \quad (10)$$

for $i \in [1, n]$, and

$$f(u_i v_j) = \begin{cases} 2kn + i, & \text{if } i < j, j = 2k, \\ 2kn + 1 - i, & \text{if } i < j, j = 2k - 1, \\ 2kn + i, & \text{if } i > j, j = 2k - 1, \\ 2kn + 2n + 1 - i, & \text{if } i > j, j = 2k, \end{cases} \quad (11)$$

for $i, j \in [1, n]$, where $i \neq j$.

To get the magic constant from the labeling, we compute

$$\begin{aligned} \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e) &= \left(f(u_i) + \sum_{\substack{j=1 \\ j \neq i}}^n f(u_j)\right) + \sum_{j=1}^n f(u_i v_j) \\ &= \left(\frac{3n^2 - n}{2}\right) + \frac{n^3 + 2n^2 - 2n - 1}{2} \\ &= \frac{n^3 + 5n^2 - 3n - 1}{2}, \end{aligned} \quad (12)$$

TABLE 1: The 7-star-supermagic labeling f of $K_{8,8} - I$.

(a)								
i	1	2	3	4	5	6	7	8
$f(u_i)$	2	4	6	8	10	12	13	15
$f(v_i)$	1	3	5	7	9	11	14	16
(b)								
$f(u_i v_j)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
u_1	—	17	32	33	48	52	57	72
u_2	18	—	31	34	47	51	59	71
u_3	19	30	—	35	46	50	61	70
u_4	20	29	36	—	45	49	63	69
u_5	21	28	37	44	—	53	60	68
u_6	22	27	38	43	56	—	58	67
u_7	23	26	39	42	55	62	—	66
u_8	24	25	40	41	54	64	65	—

for all $i \in [1, n]$. Hence $K_{n,n} - I$ is an $(n - 1)$ -star-supermagic decomposable graph for all odd n . $\square$

The following corollary is a direct consequence of Corollary 2 and Theorems 4–6.

Corollary 7. *Let n be a positive integer, where $n \geq 2$. Then $K_{n,n} - I$ has an $(n - 1)$ -star-supermagic decomposition if and only if n is odd or $n \equiv 0 \pmod{4}$.*

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] D. B. West, *Introduction to Graph Theory*, Prentice Hall, New Jersey, NJ, USA, 2001.
- [2] A. Gutiérrez and A. Lladó, "Magic coverings," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 55, pp. 43–56, 2005.
- [3] A. Lladó and J. Moragas, "Cycle-magic graphs," *Discrete Mathematics*, vol. 307, no. 23, pp. 2925–2933, 2007.
- [4] A. A. G. Ngurah, A. N. M. Salman, and L. Susilowati, "H-supermagic labelings of graphs," *Discrete Mathematics*, vol. 310, no. 8, pp. 1293–1300, 2010.
- [5] Z. Liang, "Cycle-supermagic decompositions of complete multipartite graphs," *Discrete Mathematics*, vol. 312, no. 22, pp. 3342–3348, 2012.
- [6] G. Marimuthu and S. S. Kumar, "H-V-super magic decomposition of complete bipartite graphs," *Electronic Notes in Discrete Mathematics*, vol. 48, pp. 231–233, 2015.
- [7] J. A. Gallian, "A dynamic survey of graph labeling," *The Electronic Journal of Combinatorics*, vol. 16, p. S6, 2013.
- [8] J. Akiyama and M. Kano, "Path factors of a graph," in *Graphs and applications (Boulder, Colo., 1982)*, Wiley-Intersci. Publ., pp. 1–21, Wiley, New York, 1985.

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