Research Article

New Results on the (Super) Edge-Magic Deficiency of Chain Graphs

Ngurah Anak Agung Gede¹ and Adiwijaya²

¹Department of Civil Engineering, Universitas Mercade Malang, Jl. Taman Agung No. 1, Malang 65146, Indonesia
²School of Computing, Telkom University, Jl. Telekomunikasi No. 1, Bandung 40257, Indonesia

Correspondence should be addressed to Adiwijaya; kang.adiwijaya@gmail.com

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1. Introduction

Let $G$ be a finite and simple graph, where $V(G)$ and $E(G)$ are its vertex set and edge set, respectively. Let $n = |V(G)|$ and $e = |E(G)|$ be the number of the vertices and edges, respectively. In [1], Kotzig and Rosa introduced the concepts of edge-magic labeling and edge-magic graph as follows: an edge-magic labeling of a graph $G$ is a bijection $f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, n + e\}$ such that $f(x) + f(xy) + f(y)$ is a constant for every edge $xy \in E(G)$. An edge-magic labeling $f$ of a graph $G$ with $f(V(G)) = \{1, 2, 3, \ldots, v\}$ is called a super edge-magic labeling. Furthermore, the edge-magic deficiency of a graph $G$, $\mu(G)$, is defined as the smallest nonnegative integer $n$ such that $G \cup nK_1$ has an edge-magic labeling. Similarly, the super edge-magic deficiency of a graph $G$, $\mu_s(G)$, is either the smallest nonnegative integer $n$ such that $G \cup nK_1$ has a super edge-magic labeling or $\infty$ if there exists no such integer $n$. In this paper, we investigate the (super) edge-magic deficiency of chain graphs. Referring to these, we propose some open problems.

1. Chain Graphs

A chain graph is a graph with blocks $B_1, B_2, \ldots, B_k$ such that, for every $i$, $B_i$ and $B_{i+1}$ have a common vertex in such a way that the block-cut-vertex graph is a path. We will denote the chain graph with $k$ blocks $B_1, B_2, \ldots, B_k$ by $C[B_1, B_2, \ldots, B_k]$. If $B_1 = \cdots = B_k = B$, we will write $C[B_1, B_2, \ldots, B_k]$ as $C[B^{(t)}]$. If, for every $i$, $B_i = H$ for a given graph $H$, then $C[B_1, B_2, \ldots, B_k]$ is denoted by $kH$-path. Suppose that $c_1, c_2, \ldots, c_{k-1}$ are the consecutive cut vertices of $C[B_1, B_2, \ldots, B_k]$. The string of $C[B_1, B_2, \ldots, B_k]$ is $(k-2)$-tuple $(d_1, d_2, \ldots, d_{k-2})$, where $d_i$ is the distance between $c_i$ and $c_{i+1}$, $1 \leq i \leq k-2$. We will write $(d_1, d_2, \ldots, d_{k-2})$ as $(d^{(p)}, d_{t+1}, \ldots, d_{k-2})$, if $d_1 = \cdots = d_t = d$.

For any integer $m \geq 2$, let $L_m = P_m \times P_2$. Let $TL_m$ and $DL_m$ be the graphs obtained from the ladder $L_m$ by adding a single diagonal and two diagonals in each rectangle of $L_m$, respectively. Thus, $|V(TL_m)| = |V(DL_m)| = 2m$, $|E(TL_m)| = 4m - 3$, and $|E(DL_m)| = 5m - 4$. $TL_m$ and $DL_m$ are called triangle ladder and diagonal ladder, respectively.

Recently, the author studied the (super) edge-magic deficiency of $kDL_m$-path, $C[K_1^{(k)}]$, $DL_m$, $K_1^{(n)}$, and $K_4$-path with some strings. Other results on the (super) edge-magic
deficiency of chain graphs can be seen in [4]. The latest developments in this area can be found in the survey of graph labelings by Gallian [5]. In this paper, we further investigate the (super) edge-magic deficiency of chain graphs whose blocks are combination of TL_m and DL_m and K_4 and TL_m, as well as the combination of C_4 and L_m. Additionally, we propose some open problems related to the (super) edge-magic deficiency of these graphs. To present our results, we use the following lemmas.

Lemma 1 (see [6]). A graph G is a super edge-magic graph if and only if there exists a bijective function f : V(G) → \{1,2,...,v\} such that the set S = {f(x) + f(y) : xy ∈ E(G)} consists of e consecutive integers.

Lemma 2 (see [2]). If G is a super edge-magic graph, then e ≤ 2v − 3.

2. Main Results

For k ≥ 3, let G = C[B_1,B_2,...,B_k], where B_i = TL_m when j is odd and B_j = DL_m when j is even. Thus G is a chain graph with |V(G)| = (2m−1)k+1 and |E(G)| = (1/2)(k+1)(4m−3)+ (1/2)(k−1)(5m−4) when k is odd, or |E(G)| = (k/2)(4m−3)+(k/2)(5m−4) when k is even. By Lemma 2, it can be checked that G is not super edge-magic when m ≥ 3 and k is even and when m ≥ 4 and k is odd. As we can see later, when m = 3 and k is odd, G is super edge-magic. Next, we investigate the super edge-magic deficiency of G. Our first result gives its lower bound. This result is a direct consequence of Lemma 2, so we state the result without proof.

Lemma 3. Let k ≥ 3 be an integer. For any integer m ≥ 3,

\[ μ_s(G) \geq \begin{cases} \frac{1}{4}k(m-3) + 1, & \text{if } k \text{ is even,} \\ \frac{1}{4}(k(m-3)-(m-1)) + 1, & \text{if } k \text{ is odd.} \end{cases} \]  

Notice that the lower bound presented in Lemma 3 is sharp. We found that when m is odd, the chain graph G with particular string has the super edge-magic deficiency equal to its lower bound as we state in Theorem 4. First, we define vertex and edge sets of B_i as follows.

V(B_i) = \{u^i_j, v^i_j : 1 ≤ i ≤ m\}, for 1 ≤ j ≤ k, E(B_j) = \{u^i_jv^{i+1}_j, v^i_jv^{i+1}_j : 1 ≤ i ≤ m-1\} ∪ {e}; where e is either u^i_jv^{i+1}_j or v^i_jv^{i+1}_j, 1 ≤ i ≤ m−1, for 1 ≤ j ≤ k, when j is odd, and E(B_j) = \{u^i_jv^{i+1}_j, v^i_jv^{i+1}_j, v^i_jv^{i+1}_j, v^i_jv^{i+1}_j : 1 ≤ i ≤ m−1\} ∪ {u^i_jv^{i+1}_j : 1 ≤ i ≤ m}, for 1 ≤ j ≤ k, when j is even.

Theorem 4. Let k ≥ 3 be an integer and G = C[B_1,B_2,...,B_k] with string (m−1,d_1,m−1,d_2,m−1,...,d_{(1/2)(k−3)},m−1) when k is odd or (m−1,d_1,m−1,d_2,...,m−1,d_{(1/2)(k−2)}) when k is even, where d_1,d_2,...,d_{(1/2)(k−2)} ∈ \{m−1,m\}. For any odd integer m ≥ 3,

\[ μ_s(G) = \begin{cases} \frac{1}{4}k(m-3) + 1, & \text{if } k \text{ is even,} \\ \frac{1}{4}(k-1)(m-3), & \text{if } k \text{ is odd.} \end{cases} \]  

Proof. First, we define G as a graph with vertex set V(G) = \bigcup_{j=1}^{k} V(B_j), where u^m_j = v^{i+1}_j, 1 ≤ j ≤ k − 1, and edge set E(G) = \bigcup_{j=1}^{k} E(B_j). Under this definition, u^m_j = v^{i+1}_j, 1 ≤ j ≤ k − 1, are the cut vertices of G.

Next, for 1 ≤ i ≤ m and 1 ≤ j ≤ k, define the labeling f : V(G)∪αK_1 → {1,2,3,...,2m−1}k+1+α, where α = (1/4)(k(m−3)+1) when k is even or α = (1/4)(k−1)(m−3) when k is odd, as follows:

\[ f(x) = \begin{cases} \frac{1}{4}((j-1)(9m-7) + 2i - 1), & \text{if } x = u^i_j \text{, } j \text{ is odd,} \\ \frac{1}{4}((j-1)(9m-7) + 2i), & \text{if } x = v^{i+1}_j \text{, } j \text{ is odd,} \\ \frac{3}{4}((j-1)(9m-7) + 2i), & \text{if } x = v^{i+1}_j \text{, } j \text{ is even,} \\ \frac{3}{4}((j-1)(9m-7) + 2i-1), & \text{if } x = u^i_j \text{, } j \text{ is even,} \end{cases} \]  

where β = (1/4)(j−2)(9m−7)+2m.

Under the vertex labeling f, it can be checked that no labels are repeated, f(u^m_j) = f(v^{i+1}_j), 1 ≤ j ≤ k − 1, \{f(x) + f(y) : xy ∈ E(G)\} is a set of |E(G)| consecutive integers, and the largest vertex label used is (1/4)(k−2)(9m−7)+(1/2)(9m−3) when k is even or (1/4)(k−1)(9m−7)+2m when k is odd. Also, it can be checked that f(u^i_j) + f(v^{i+1}_j) = f(v^{i+1}_j) + f(u^{i+1}_j) = f(v^{i+1}_j) + f(u^{i+1}_j) when j is odd.

Next, label the isolated vertices in the following way.

Case k Is Odd. In this case, we denote the isolated vertices with \{z^i_{2j-1} : 1 ≤ l ≤ (1/2)(m-3), 1 ≤ j ≤ (1/2)(k-1)\} and set f(z^i_{2j-1}) = f(v^{m}_j) + 5l.

Case k Is Even. In this case, we denote the isolated vertices with \{z^i_{2j-1} : 1 ≤ l ≤ (1/2)(m-3), 1 ≤ j ≤ k/2\} and set f(z^i_{2j-1}) = f(v^{m}_j) + 5l + f(z_0) = f(v^{m}_k) + 1.

By Lemma 1, f can be extended to a super edge-magic labeling of G∪αK_1 with the magic constant (k/4)(27m−21)+5 when k is even or (1/4)(k−1)(27m−21)+6m when k is odd. Based on these facts and Lemma 3, we have the desired result.

An example of the labeling defined in the proof of Theorem 4 is shown in Figure 1(a).

Notice that when m = 3 and k is odd, \mu_s(G) = 0. In other words, the chain graph G with string (2,d_1,2,d_2,2,...,d_{(1/2)(k−3)},2), where d_1 ∈ {2,3}, is super edge-magic.
when \( m = 3 \) and \( k \) is odd. Based on this fact and previous results, we propose the following open problems.

**Open Problem 1.** Let \( k \geq 3 \) be an integer. For \( m = 2 \), decide if there exists a super edge-magic labeling of \( G \). Further, for any even integer \( m \geq 2 \), find the super edge-magic deficiency of \( G \).

Next, we investigate the super edge-magic deficiency of the chain graph \( H = C[K_4^{(p)}], TL_m, K_4^{(q)}] \) with string \((1^{(p-1)}, d, 1^{(q-1)})\), where \( d \in \{m-1, m\} \). \( H \) is a graph of order \( 3(p+q) + 2m \) and size \( 6(p+q) + 4m - 3 \). We define the vertex and edge sets of \( H \) as follows: \( V(H) = \{a_i, b_i; 1 \leq i \leq p\} \cup \{c_i; 1 \leq i \leq p + 1\} \cup \{u_j, v_j; 1 \leq j \leq m\} \cup \{x_j, y_j; 1 \leq j \leq t \leq q\} \cup \{z_j; 1 \leq t \leq q+1\} \), where \( c_{p+1} = u_1 \) and \( v_m = z_1 \), and \( E(H) = \{a_i b_i, a_i c_i, a_i c_{i+1}, b_i c_i, b_i c_{i+1}, c_i c_{i+1}; 1 \leq i \leq p\} \cup \{u_j v_j; 1 \leq j \leq m\} \cup \{u_j u_{j+1}, v_j v_{j+1}; 1 \leq j \leq m-1\} \cup \{e_i; e_i \text{ is either } u_j v_{j+1} \text{ or } v_j u_{j+1}, 1 \leq j \leq m-1\} \cup \{x_j y_j, x_j x_{j+1}, x_j z_{j+1}, y_j z_j, y_j z_{j+1}, z_j z_{j+1}; 1 \leq t \leq q\} \). Hence, the cut vertices of \( H \) are \( c_i, 2 \leq i \leq p + 1 \), and \( z_j, 1 \leq t \leq q \). Notice that \( H \) has string \((1^{(p-1)}, m-1, 1^{(q-1)})\), if at least one of \( e_j \) is \( u_j v_{j+1} \), and its string is \((1^{(p-1)}, m, 1^{(q-1)})\), if \( e_j = v_j u_{j+1} \) for every \( 1 \leq j \leq m-1 \).

**Theorem 5.** For any integers \( p, q \geq 1 \) and \( m \geq 2 \), \( \mu_s(H) = 0 \).

**Proof.** Define a bijective function \( g : V(H) \to \{1, 2, 3, \ldots, 3(p+q) + 2m\} \) as follows:

\[
g(x) = \begin{cases} 
3i - 2, & \text{if } x = a_i, 1 \leq i \leq p, \\
3i, & \text{if } x = b_i, 1 \leq i \leq p, \\
3i - 1, & \text{if } x = c_i, 1 \leq i \leq p + 1, \\
3p + 2j, & \text{if } x = u_j, 1 \leq j \leq m, \\
3p + 2j - 1, & \text{if } x = v_j, 1 \leq j \leq m, \\
3p + 2m + 3t - 2, & \text{if } x = x_j, 1 \leq t \leq q, \\
3p + 2m + 3t, & \text{if } x = y_j, 1 \leq t \leq q, \\
3p + 2m + 3t - 4, & \text{if } x = z_j, 1 \leq t \leq q + 1.
\end{cases}
\]

Under the labeling \( g \), it can be checked that \( g(c_{p+1}) = g(u_1) \) and \( g(v_m) = g(z_1) \). Also, it can be checked that \( g(u_j) + g(v_{j+1}) = g(u_{j+1}) + g(v_j), 1 \leq j \leq m - 1, \) and \( |g(x) + g(y)|, xy \in E(H) = \{3, 4, 5, \ldots, 6(p+q) + 4m - 1\} \). By Lemma 1, \( g \) can be extended to a super edge-magic labeling of \( H \) with the magic constant \( 9(p+q) + 6m \). Hence, \( \mu_s(H) = 0 \).

**Open Problem 2.** For any integers \( p, q \geq 1 \) and \( m \geq 2 \), find the super edge-magic deficiency of \( C[K_4^{(p)}], TL_m, K_4^{(q)}] \) with string \((1^{(p-1)}, d, 1^{(q-1)})\), where \( d \in \{1, 2, 3, \ldots, m - 2\} \).
Proof. Let $V(L_m) = \{u_i, v_i : 1 \leq i \leq m\}$ and $E(G) = \{u_iu_{i+1}, v_i v_{i+1} : 1 \leq i \leq m-1\} \cup \{u_i v_{i+1} : 1 \leq i \leq m\}$ be the vertex set and edge set, respectively, of $L_m$. It is easy to verify that the labeling $h : V(L_m) \cup E(L_m) \to \{1, 2, 3, \ldots, 5m-2\}$ is a bijection and, for every $xy \in E(L_m)$, $h(x)+h(xy)+h(y) = 6m$.

Thus, $\mu(L_m) = 0$ for every $m \geq 2$.

Theorem 6. For any integer $m \geq 2$, $\mu(L_m) = 0$.

Next, we study the edge-magic deficiency of ladder $L_m$ and chain graphs whose blocks are combination of $C_4$ and $L_m$ with some strings. In [6], Figueroa-Centeno et al. proved that the ladder $L_m$ is super edge-magic for any odd $m$ and suspected that $L_m$ is super edge-magic for any even $m > 2$. Here, we can prove that $L_m$ is edge-magic for any $m \geq 2$ by showing its edge-magic deficiency is zero. The result is presented in Theorem 6.

Theorem 7. Let $p$ and $q \geq 1$ be integers.

(a) If $m \geq 2$ is an even integer and $F_1 = C[C_4^{(p)} , L_m , \epsilon_4^{(q)}] \text{ with string } (2^{(p-1)}, m, 2^{(q-1)})$, then $\mu(F_1) = 0$.

(b) If $m \geq 3$ is an odd integer and $F_2 = C[C_4^{(p)} , L_m , \epsilon_4^{(q)}] \text{ with string } (2^{(p-1)}, m-1, 2^{(q-1)})$, then $\mu(F_2) = 0$.

Proof. (a) First, we introduce a constant $\lambda$ as follows: $\lambda = 1$, if $m$ is odd and $\lambda = 2$, if $m$ is even. Next, we define $F_1$ as a graph with $V(F_1) = \{a_i, b_i : 1 \leq i \leq p\} \cup \{c_i, d_i : 1 \leq i \leq p+1\} \cup \{u_i, v_i : 1 \leq j \leq m\} \cup \{x_j, y_j : 1 \leq j \leq q\} \cup \{z_i, t_i : 1 \leq i \leq q+1\}$, where $c_{p+1} = v_1$ and $u_m = z_1$, and $E(H) = \{c_1a_1, c_1b_1, a_1a_2, b_1b_2, a_1c_2, b_1d_2, a_2c_3, b_2d_3, \ldots, a_{p+1}c_p, b_{p+1}d_p\}$. The cut vertices of $F_1$ are $c_{i+1}, 2 \leq i \leq p$, and $z_{1}, 1 \leq i \leq q+1$.

Next, define a bijection $f_2 : E(F_1) \cup E(F_2) \to \{1, 2, 3, \ldots, 7(p+q) + 5m - 2\}$ as follows:

\[
f_1(x) = \begin{cases} 
4(p+q) + 3m + i - 1, & \text{if } x = a_i, 1 \leq i \leq p, \\
p + q + m + i, & \text{if } x = b_i, 1 \leq i \leq p, \\
i, & \text{if } x = c_i, 1 \leq i \leq p + 1, \\
5p + 4q + 3m + 1 \frac{1}{2}(j - 1), & \text{if } x = u_j, j \text{ is odd}, \\
p + j, & \text{if } x = u_j, j \text{ is even}, \\
p + j, & \text{if } x = v_j, j \text{ is odd}, \\
2p + q + m + j \frac{1}{2}, & \text{if } x = v_j, j \text{ is even}, \\
5p + 4q + y_1 + t, & \text{if } x = x_t, 1 \leq t \leq q, \\
2p + q + y_2 + t, & \text{if } x = y_t, 1 \leq t \leq q, \\
5p + m + t - 1, & \text{if } x = z_t, 1 \leq t \leq q + 1, \\
4(p+q) + 3m + 1 - 2i, & \text{if } x = c_ia_i, 1 \leq i \leq p, \\
7(p+q) + 5m - 2i, & \text{if } x = c_ia_i, 1 \leq i \leq p, \\
4(p+q) + 3m - 2i, & \text{if } x = c_ia_i, 1 \leq i \leq p, \\
7(p+q) + 5m - 1 - 2i, & \text{if } x = c_ia_i, 1 \leq i \leq p, \\
2p + 4q + 3m + 1 \frac{1}{2}(j + 1), & \text{if } x = u_{j+1}, j \text{ is odd}, \\
2p + 4q + 3m + 1 \frac{1}{2}(j), & \text{if } x = u_{j+1}, j \text{ is even}, \\
5p + 7q + 5m + 1 \frac{1}{2}(j + 1), & \text{if } x = v_{j+1}, j \text{ is odd}, \\
5p + 7q + 5m - 1 \frac{1}{2}(j + 1), & \text{if } x = v_{j+1}, j \text{ is even}, \\
5p + 7q + 5m - 3 \frac{1}{2}, & \text{if } x = u_{j+1}, j \text{ is odd}, \\
5p + 4q + y_2 - 2t, & \text{if } x = z_t x_t, 1 \leq t \leq q, \\
5p + 7q + y_2 - 2t, & \text{if } x = z_t y_t, 1 \leq t \leq q, \\
2p + 4q + y_2 - 2t, & \text{if } x = z_t x_t, 1 \leq t \leq q, \\
5p + 7q + y_2 - 2t, & \text{if } x = z_t y_t, 1 \leq t \leq q, \\
5p + 4q + y_2 - 2t, & \text{if } x = z_t x_t, 1 \leq t \leq q, \\
5p + 7q + y_2 - 2t, & \text{if } x = z_t y_t, 1 \leq t \leq q,
\end{cases}
\]

where $y_1 = (3/2)(\lambda - 1)(7m - 2) - (1/2)(\lambda - 2)(7m - 1), y_2 = (1/2)(\lambda - 1)(3m - 1), y_3 = (1/2)(\lambda - 1)(3m - 1), y_5 = (1/2)(\lambda - 1)(3m + 2) - (1/2)(\lambda - 2)(3m + 1), y_6 = (1/2)(\lambda - 1)(7m) - (1/2)(\lambda - 2)(7m + 1)$.

It can be checked that, for every edge $xy \in E(F_1)$, $f(x) + f(xy) + f(y) = 8(p+q) + 6m$.

(b) We define $F_2$ as graph with $V(F_2) = V(F_1)$, where $c_{p+1} = v_1$ and $v_m = z_1$, and $E(F_2) = E(F_1)$. Under this definition, the cut vertices of $F_2$ are $c_{i+1}, 2 \leq i \leq p + 1$, and $z_{1}, 1 \leq i \leq q+1$. Next, we define a bijection $f_2 : V(F_2) \cup E(F_2) \to \{1, 2, 3, \ldots, 7(p+q) + 5m - 2\}$, where $f_2(x) = f_1(x)$ for all $x \in V(F_2) \cup E(F_2)$. It can be checked that $f_2$ is an edge-magic labeling of $F_2$ with the magic constant $8(p+q) + 6m$.

Open Problem 3. Let $p$ and $q \geq 1$ be integers.

(a) If $m \geq 3$ is an odd integer, find the super edge-magic deficiency of $C[C_4^{(p)}, L_m, \epsilon_4^{(q)}]$ with string $(2^{(p-1)}, m, 2^{(q-1)})$.

(b) If $m \geq 2$ is an even integer, find the super edge-magic deficiency of $C[C_4^{(p)}, L_m, \epsilon_4^{(q)}]$ with string $(2^{(p-1)}, m - 1, 2^{(q-1)})$. 

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Theorem 8. Let \( p, q \geq 2 \) and \( r \geq 1 \) be integers.

(a) If \( m \geq 2 \) is an even integer and \( H_1 = C\{c^{(p,q)}_4\} \) with string \( (2^{(p-2)}, 1^{(2)}, 2^{(r-1)}), m, 2^{(r-1)}) \), then \( \mu(H_1) = 0 \).

(b) If \( m \geq 3 \) is an odd integer and \( H_2 = C\{c^{(p,q)}_r\} \) with string \( (2^{(p-2)}, 1^{(2)}, 2^{(r-1)}), m-1, 2^{(r-1)} \), then \( \mu(H_2) = 0 \).

Proof. (a) First, we define \( H_1 \) as a graph with \( V(H_1) = \{a_i; 1 \leq i \leq 2p\} \cup \{b_i; 1 \leq i \leq p + 1\} \cup \{u_i; 1 \leq j \leq 2q\} \cup \{v_i; 1 \leq j \leq q + 1\} \cup \{w_i; 1 \leq s \leq 2m\} \cup \{x_i; 1 \leq t \leq 2r\} \cup \{y_i; 1 \leq t \leq r + 1\} \), where \( a_{2p} = u_1, v_{q+1} = w_1 \), and \( w_{2m} = y_1 \), and \( E(H_1) = \{b_i a_{i+1}, b_i a_{i+1}, a_i b_{i+1}, a_i b_{i+1}, a_i b_{i+1}; 1 \leq i \leq p\} \cup \{v_j u_{i,j}, v_j u_{i,j}, u_j v_{j+1}, u_j v_{j+1}; 1 \leq j \leq q\} \cup \{w_i w_{i+1}, w_i w_{i+1}, v_{i+1} w_{i+2}; 1 \leq s \leq m-1\} \cup \{w_i w_{i+1}, v_{i+1} w_{i+2}; 1 \leq s \leq m\} \cup \{y_i x_r, x_r y_{r+1}, x_{r+1} y_{r+1}, x_{r+1} y_{r+1}; 1 \leq t \leq r\} \).

Next, define a bijection \( g_1 : V(H_1) \rightarrow \{1, 2, 3, \ldots, 7(p + q + r) + 5m - 2\} \) as follows:

\[
g_1(z) = \begin{cases} 
6p + 7(q + r) + 5m + i - 2, & \text{if } z = a_i, 1 \leq i \leq p, \\
3p + q + r + m + 1 + i, & \text{if } z = a_{p+i}, 1 \leq i \leq p, \\
i, & \text{if } z = b_i, 1 \leq i \leq p + 1, \\
4p + q + r + m + j, & \text{if } z = u_j, 1 \leq j \leq q, \\
\left(4(p + q + r) + 3m + 1 + j, \right) & \text{if } z = v_j, 1 \leq j \leq q + 1, \\
p + q + 1 + s, & \text{if } z = w_j, s \text{ is odd}, \\
4p + 2q + r + m + \frac{1}{2}s, & \text{if } z = w_j, s \text{ is even}, \\
4p + 5q + 4r + 3m + \frac{1}{2}(s - 1), & \text{if } z = w_{m+s}, s \text{ is odd}, \\
p + q + 1 + s, & \text{if } z = w_{m+s}, s \text{ is even}, \\
\left(4p + 2q + r + y_2 + t, \right) & \text{if } z = x_t, 1 \leq t \leq r, \\
4p + 5q + 4r + y_1 + t, & \text{if } z = x_{t+1}, 1 \leq t \leq r, \\
p + q + m + t, & \text{if } z = y_t, 1 \leq t \leq r + 1, \\
3p + q + r + n + 3 - 2t, & \text{if } z = b_t a_i, 1 \leq i \leq p, \\
6p + 7(q + r) + 5m - 2i, & \text{if } z = b_t a_{i+1}, 1 \leq i \leq p, \\
3p + q + r + m + 2 - 2i, & \text{if } z = a_t b_{i+1}, 1 \leq i \leq p, \\
6p + 7(q + r) + 5m - 2i, & \text{if } z = a_t b_{i+1}, 1 \leq i \leq p, \\
4p + 7(q + r) + 5m - 2j, & \text{if } z = v_j u_{i,j}, 1 \leq j \leq q, \\
4p + 7(q + r) + 5m - 2j, & \text{if } z = u_j v_{j+1}, 1 \leq j \leq q, \\
4p + 7(q + r) + 5m - 2j, & \text{if } z = u_j v_{j+1}, 1 \leq j \leq q, \\
4p + 5q + 7r + 5m - \frac{1}{2}(3s + 1), & \text{if } z = w_j w_{i+1}, s \text{ is odd}, \\
4p + 5q + 7r + 5m - \frac{1}{2}(3s + 2), & \text{if } z = w_j w_{i+1}, s \text{ is even}, \\
4p + 2q + 4r + 3m - \frac{1}{2}(3s + 1), & \text{if } z = w_j w_{i+1}, s \text{ is odd}, \\
4p + 2q + 4r + 3m - \frac{1}{2}(3s + 1), & \text{if } z = w_j w_{i+1}, s \text{ is even}, \\
4p + 2q + 4r + 3m - \frac{1}{2}(3s - 1), & \text{if } z = w_j w_{i+1}, s \text{ is odd}, \\
4p + 2q + 4r + 3m - \frac{1}{2}(3s - 1), & \text{if } z = w_j w_{i+1}, s \text{ is even}, \\
4p + 5q + 7r + 5m - \frac{1}{2}s, & \text{if } z = w_j w_{i+1}, s \text{ is even}, \\
4p + 5q + 7r + y_4 - 2t, & \text{if } z = y_t x_r, 1 \leq t \leq r, \\
4p + 2q + 4r + y_3 - 2t, & \text{if } z = y_t x_r, 1 \leq t \leq r, \\
4p + 5q + 7r + y_6 - 2t, & \text{if } z = x_t y_{r+1}, 1 \leq t \leq r, \\
4p + 2q + 4r + y_5 - 2t, & \text{if } z = x_t y_{r+1}, 1 \leq t \leq r,
\end{cases}
\]
where \( \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \), and \( \lambda \) are defined as in the proof of Theorem 7. It can be checked that, for every edge \( xy \in E(H_1) \), 
\[ g_1(x)+g_1(xy)+g_1(y) = 9p+8(q+r)+6m+1. \]
Hence \( \mu(H_1) = 0. \)

An illustration of the labeling defined in the proof of Theorem 8 is given in Figure 1(b).

(b) We define \( H_2 \) as graph with \( V(H_2) = V(H_1) \), where \( a_{2p} = u_1, v_{q+1} = w_1, \) and \( w_m = y_1 \), and \( E(H_2) = E(H_1) \). It can be checked that \( g_2 : V(H_2) \cup E(H_2) \rightarrow \{1, 2, 3, \ldots, 7(p+q+r)+5m-2\} \) defined by \( g_2(x) = g_1(x) \), for all \( x \in V(H_2) \cup E(H_2) \), is an edge-magic labeling of \( H_2 \) with the magic constant \( 9p+8(q+r)+6m+1. \)

\[ \square \]

**Open Problem 4.** Let \( p, q \geq 2 \) and \( r \geq 1 \) be integers.

(a) If \( m \geq 3 \) is an odd integer, find the edge-magic deficiency of \( C[4 \times 4, 4 \times 4, L_m, 4 \times 4] \) with string \((2^{(p-2)}, 1^{(2)}, 2^{(r-1)}, m, 2^{(r-1)})\).

(b) If \( m \geq 2 \) is an even integer, find the edge-magic deficiency of \( C[4 \times 4, 4 \times 4, L_m, 4 \times 4] \) with string \((2^{(p-2)}, 1^{(2)}, 2^{(r-1)}, m-1, 2^{(r-1)})\).

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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