Research Article

On Killing Forms and Invariant Forms of Lie-Yamaguti Superalgebras

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The notions of the Killing form and invariant form in Lie algebras are extended to the ones in Lie-Yamaguti superalgebras and some of their properties are investigated. These notions are also Z2-graded generalizations of the ones in Lie-Yamaguti algebras.

1. Introduction

A Lie-Yamaguti algebra is a triple \((T, *, [\cdot, \cdot, \cdot])\) consisting of a vector space \(T\), a bilinear map \(\ast : T \times T \to T\), and a trilinear map \([\cdot, \cdot, \cdot] : T \times T \times T \to T\) such that

\begin{align*}
\text{(LY1)} & \quad x \ast y = -y \ast x, \\
\text{(LY2)} & \quad [x, y, z] = -[y, x, z], \\
\text{(LY3)} & \quad \sigma_{x,y,z}(([x \ast y] \ast z) + [x, y, z]) = 0, \\
\text{(LY4)} & \quad \sigma_{x,y,z}[[x \ast y, z, u] = 0, \\
\text{(LY5)} & \quad [x, y, u \ast v] = [x, y, u] \ast v + u \ast [x, y, v], \\
\text{(LY6)} & \quad [u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]],
\end{align*}

for all \(u, v, x, y, z\) in \(T\), where \(\sigma_{x,y,z}\) denotes the sum over cyclic permutation of \(x, y, z\). The bilinear map \(\ast\) sometimes will be denoted by juxtaposition. If \(x \ast y = 0, \forall x, y \in T\), one gets a Lie triple system \((T, [\cdot, \cdot, \cdot])\), while \([x, y, z] = 0\) in \((T, \ast, [\cdot, \cdot, \cdot])\) induces a Lie algebra \((T, \ast)\).

Lie-Yamaguti algebras were introduced by Yamaguti [1] (who formerly called them “generalized Lie triple systems”) in an algebraic study of the characteristic properties of the torsion and curvature of a homogeneous space with canonical connection [2]. Later on, these algebraic objects were called “Lie triple algebras” [3] and the terminology of “Lie-Yamaguti algebras” is introduced in [4] for these algebras. For further development of the theory of Lie-Yamaguti algebras one may refer, for example, to [5–8]. From the standard enveloping Lie algebra of a given Lie-Yamaguti algebra, the notions of the Killing-Ricci form and the invariant form of a Lie-Yamaguti algebra are introduced and studied in [9]. Further properties of invariant forms of Lie-Yamaguti algebras were considered in [10].

Lie superalgebras as a \(\mathbb{Z}_2\)-graded generalization of Lie algebras are considered in [11, 12] while a \(\mathbb{Z}_2\)-graded generalization of Lie triple systems (called Lie supertriple systems) was first considered in [13]. For an application of Lie supertriple systems in physics, one may refer to [14]. Next, Lie-Yamaguti superalgebras as a \(\mathbb{Z}_2\)-graded generalization of Lie-Yamaguti algebras were first considered in [15].

Definition 1 (see [16]). A Lie-Yamaguti superalgebra is a \(\mathbb{Z}_2\)-graded vector space \(T = T_0 \oplus T_1\) with a binary operation denoted by juxtaposition satisfying \(T_IT_j \subseteq T_{i+j}\) and a ternary operation \([\cdot, \cdot, \cdot]\) satisfying \([T_i, T_j, T_k] \subseteq T_{i+j+k}\) \((i, j, k \in \mathbb{Z}_2)\) such that

\begin{align*}
\text{(LYS1)} & \quad xy = -(-1)^{xy}yx, \\
\text{(LYS2)} & \quad [x, y, z] = -(-1)^{xy}[y, x, z], \\
\text{(LYS3)} & \quad \sigma_{x,y,z}((-1)^{xy}(xy)z) + [x, y, z] = 0, \\
\text{(LYS4)} & \quad \sigma_{x,y,z}((-1)^{xy}xyu) = 0, \\
\text{(LYS5)} & \quad [x, y, uv] = [x, y, u]v + (-1)^{xy}u[x, y, v],
\end{align*}
Observing that \( T_0 \) is a Lie-Yamaguti algebra.

As a part of the general theory of superalgebras, the notion of the Killing form of Lie algebras is extended to the one of Lie triple systems (see [17] and references therein), Lie superalgebras [12], and next Lie supertriple systems [18] (see [19]).

In this paper we define and study the Killing form and invariant form of a Lie-Yamaguti superalgebra in some conditions, it is shown (Theorem 21) that the Killing form of a Lie-Yamaguti superalgebra is defined (see Theorem 10 and Definition 11) and some of its properties are investigated (Proposition 13, Theorem 14, and Corollary 15). In Section 4 the invariant form of a Lie-Yamaguti superalgebra is defined (Definition 16) and, under some conditions, it is shown (Theorem 21) that the Killing form of a Lie-Yamaguti superalgebra \( T \) is nondegenerate if and only if the standard enveloping Lie superalgebra of \( T \) is semisimple.

All vector spaces and algebras are finite-dimensional over a fixed ground field \( k \) of characteristic 0.

2. Some Basics on Lie-Yamaguti Superalgebras

We give here some definitions and results which can be found in [11, 12, 16].

A superalgebra over \( k \) is a \( Z_2 \)-graded algebra \( A = A_0 \oplus A_1 \), where \( A_iA_j \subseteq A_{i+j} \). The subspaces \( A_0 \) and \( A_1 \) are called the even and the odd parts of the superalgebra and so are called the elements from \( A_0 \) and from \( A_1 \), respectively.

Below, all elements assumed to be homogeneous, that is, either even or odd, and for a homogeneous element \( x \in A_i, i = 0, 1 \), the notation \( x = i \) is used and means the parity of \( x \).

Let \( G \) be the Grassmann algebra over \( k \) generated by the elements \( e_1, e_2, \ldots, e_n \) such that \( e_i^2 = 0, e_i e_j = -e_j e_i \) for \( i \neq j \). The elements \( 1, e_1, e_2, \ldots, e_n, i_1 < i_2 < \cdots < i_m \) form a basis of \( G \). Denote by \( G_0 \) (resp., \( G_1 \) ) the span of the products of even length (resp., odd length) in the generators. The product of zero \( e_i \)'s is by convention equal to 1. Then \( G = G_0 \oplus G_1 \) is an associative and supercommutative superalgebra; that is, \( g_1 g_2 = (-1)^{i_1i_2} g_2 g_1 \), where \( g_1, g_2 \in G_0 \cup G_1 \). Let \( A = A_0 \oplus A_1 \) be a superalgebra. Consider the graded tensor product \( G \otimes A \) which becomes a superalgebra with the product given by \( (x \otimes g_1)(y \otimes g_2) = (-1)^{x y} xy \otimes g_1 g_2 \) for homogeneous elements \( g_1, g_2 \in G, x, y \in A \) and grading given by \( (G \otimes A)_0 = G_0 \otimes A_0 \oplus G_1 \otimes A_1 \) and \( (G \otimes A)_1 = G_0 \otimes A_1 \oplus G_1 \otimes A_0 \). The subalgebra \( G(A) = (G \otimes A)_0 = G_0 \otimes A_0 \oplus G_1 \otimes A_1 \) is called the Grassmann envelope of the superalgebra \( A \).

Having in mind that if \( V \) is a homogeneous variety of algebras [20], a superalgebra \( A = A_0 \oplus A_1 \) is called a \( V \)-superalgebra, if its Grassmann envelope \( G(A) \) belongs to \( V \), we can state the following proposition.

**Proposition 2.** A superalgebra \( T = T_0 \oplus T_1 \) equipped with bilinear and trilinear products verifying \( T_1 T_1 \subseteq T_{1+2} \) and \( [T_1, T_1, T_1] \subseteq T_{1+j+k} \) is a Lie-Yamaguti superalgebra if its Grassmann envelope \( G(T) = G_0 \oplus G_1 \otimes G_1 \) is a Lie-Yamaguti superalgebra under the following products:

\[
\begin{align*}
(x \otimes g_1)(y \otimes g_2) &= (-1)^{x y} xy \otimes g_1 g_2; \\
(x \otimes g_1, y \otimes g_2, z \otimes g_3) &= (-1)^{x y z} x y z \otimes g_1 g_2 g_3.
\end{align*}
\]

**Proof.** The proof is straightforward by using the fact that, for any element \( x \otimes g \) in \( G(T) \), we have \( x = \overline{x} \).

**Example 3.** (1) Lie superalgebras are Lie-Yamaguti superalgebras with \( [x, y, z] = 0 \).

(2) If \( xy = 0 \) for any \( x, y \in T_0 \cup T_1 \), then (LYS2), (LYS3), and (LYS6) define a Lie supertriple system.

(3) Let \( M = M_0 \oplus M_1 \) be a Malcev superalgebra; that is, for any \( x, y, z, t \) in \( T \),

\[
\begin{align*}
xy &= -(-1)^{x y} yx; \\
-(-1)^{x y z} (x y z) t &= ((x y) z) t \\
&+ (-1)^{x y z t} (y z t) x \\
&+ (-1)^{x y z t} ((z t) x) y \\
&+ (-1)^{x y z t} ((t x) y) z.
\end{align*}
\]

It is shown in [16] that \( M \) becomes a Lie-Yamaguti superalgebra if we set \( [x, y, z] = x(y z) - (-1)^{x y} y(x z) + (x y) z \).

Conversely, if on a Malcev superalgebra \( (M, \cdot) \) we define a trilinear operation by \( [x, y, z] = x \cdot y \cdot z - (-1)^{x y} y \cdot x z + x y \cdot z \) then \( (M, \cdot, [\cdot, \cdot, \cdot]) \) is a Lie-Yamaguti superalgebra.

**Definition 4.** Let \( T = T_0 \oplus T_1 \) be a Lie-Yamaguti superalgebra. A graded subspace \( H = H_0 \oplus H_1 \) of \( T \) is a graded Lie-Yamaguti subalgebra of \( T \) if \( H_i H_j \subseteq H_{i+j} \) and \( [H_i, H_j, H_k] \subseteq H_{i+j+k} \) for any \( i, j, k \in Z_2 \).

**Definition 5.** A graded subalgebra of a Lie-Yamaguti superalgebra \( T \) is an invariant graded subalgebra (resp., an ideal) of \( T \) if \( [T, T, H] \subseteq H \) (resp., \( TH \subseteq H \) and \( [T, H, T] \subseteq H \)).

If \( H \) is an ideal of \( T \), it is an invariant graded subalgebra of \( T \). Obviously the center \( Z(T) \) of a Lie-Yamaguti superalgebra \( T \) defined by \( Z(T) = \{ x \in T, xy = 0 \} \) and \( [x, y, z] = 0, \forall y, z \in T \) is an ideal of \( T \).

**Definition 6.** Let \( T = T_0 \oplus T_1 \) and \( T' = T'_0 \oplus T'_1 \) be Lie-Yamaguti superalgebras. A linear map \( f : T \rightarrow T' \) is said to be of degree \( r \) if \( f(T_i) \subseteq T'_r \) for all \( r, i \in Z_2 \).

**Definition 7.** Let \( T = T_0 \oplus T_1 \) and \( T' = T'_0 \oplus T'_1 \) be Lie-Yamaguti superalgebras. A linear map \( f : T \rightarrow T' \) is called a homomorphism of Lie-Yamaguti superalgebras if
(1) $f$ preserves the grading, that is, $f(T_i) \subseteq T'_i$, $i \in \mathbb{Z}_2$;
(2) $f(xy) = f(x)f(y)$;
(3) $f([x, y, z]) = [f(x), f(y), f(z)]$ for any $x, y, z \in T_0 \cup T_1$.

Recall [11] that if $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space then, if we set $\text{End}_n(V) = \{f \in \text{End}(V) | f(V) \subseteq V_i\}$, we obtain an associative superalgebra $\text{End}_n(V) = \text{End}_0(V) \oplus \text{End}_1(V)$; $\text{End}_n(V)$ consists of the linear mappings of $V$ into itself which are homogeneous of degree $n$. The bracket $[f, g] = fg - (-1)^{|f||g|}gf$ makes $\text{End}(V)$ into a Lie superalgebra which we denote by $\mathbb{L}(V)$ or $l(m, n)$ where $m = \dim V_0$ and $n = \dim V_1$. Let $e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}$ be a basis of $V$. In this basis the matrix of $a \in l(m, n)$ is expressed as $( \begin{pmatrix} a_{ij} \end{pmatrix} )$, $a$ being an $(m \times n)$- matrix of even elements and those of odd ones $( \begin{pmatrix} 0 & \gamma \beta \\ \delta & 0 \end{pmatrix} )$. For any $a = ( \begin{pmatrix} a_{ij} \end{pmatrix} )$, the supertrace of $a$ is defined by $\text{str}(a) = \text{tr}a - \text{tr}a^\dagger$ and does not depend on the choice of a homogeneous basis. We have $\text{str}(ab) = -(-1)^{|a||b|} \text{str}(ba)$ and $\text{str}(aba^{-1}) = \text{str}(b)$.

**Definition 8.** Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra; $D \in \text{End}_n(T)$ be a superderivation of degree $n$, for any $x, y, z \in T_0 \cup T_1$,

$$D(xy) = D(x) y + (-1)^{|x||y|} x D(y);$$

$$D([x, y, z]) = [D(x), y, z] + (-1)^{|x||y|}[x, D(y), z] + (-1)^{|x||y||z|}[x, y, D(z)].$$

Let $D_n(T)$ consist of all the superderivations of degree $n$ and $D(T) = D(T) \oplus D_n(T)$. It is easy to check that $D(T)$ is a graded subalgebra of $\text{End}(T)$ called the Lie superalgebra of superderivations of $T$.

Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. For any $x, y, z \in T_0 \cup T_1$, denote by $D_{x,y}$ the endomorphism of $T$ defined by $D_{x,y}(z) = [x, y, z]$ for any $z \in T$. We have, for any $x, y, z \in T_0 \cup T_1$, $r \in \mathbb{Z}_2$, $D_{x,y}(T_r) \subseteq T_{r+\eta}$, where $D_{x,y}$ is a linear map of degree $\bar{x} + \bar{y}$. Moreover, it comes from (L5S5) and (L5S6) that

$$D_{x,y}(zw) = D_{x,y}(z) w + (-1)^{|z||y|} z D_{x,y}(w);$$

$$D_{x,y}([z, w, v]) = [D_{x,y}(z), v, w] + (-1)^{|z||v||w|}[z, D_{x,y}(w), v] + (-1)^{|z||y||w|}[z, v, D_{x,y}(w)].$$

for any $x, y, z, v, w \in T_0 \cup T_1$. It follows that $D_{x,y}$ is a superderivation of $T$ called an inner superderivation of $T$.

Let $D(T, T)$ be the vector space spanned by all $D_{x,y}$, $x, y \in T$.

We can define naturally a $\mathbb{Z}_2$-gradation by setting $D(T, T) = D(T) \oplus D(T, T)$, where $D(T, T)$ consists of the superderivation $D_{x,y}$ of degree $r$. From (5) we also have that, for any $x, y, z, v, w \in T_0 \cup T_1$,

$$[D_{x,y}, D_{z,v}] = D_{[x,y,z],v} + (-1)^{|x||y||z|} D_{z,[x,y,v]}. \tag{6}$$

It is clear from (6) that $D(T, T)$ is a $\mathbb{Z}_2$-graded Lie superalgebra of $D(T)$ called the Lie superalgebra of all inner superderivations of $T$.

Now, let $(T, \cdot, \cdot, \cdot)$ be a Lie-Yamaguti superalgebra.
Set $L_1(T) = T \oplus D(T, T), i = 0, 1$, and define a new bracket operation in $L(T) = L_0(T) \oplus L_1(T) = T \oplus D(T, T)$ as follows: for any $x, y \in T_0 \cup T_1$, $D_1, D_2 \in D_0(T, T) \cup D_1(T, T)$,

$$[x, y] = xy + D_{x,y};$$

$$[D_1, x] = -(-1)^{|x||y|}[x, D] = D(x); \tag{7}$$

$$[D_1, D_2] = D_1 D_2 - (-1)^{|y||z|} D_2 D_1.$$ 

**Theorem 9.** Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. Then

(1) $L(T)$ is a Lie superalgebra called the standard enveloping Lie superalgebra of $T$ and $D(T, T)$ becomes a graded subalgebra of $L(T)$.

(2) If $H$ is an ideal of $T$ then $H \oplus D(T, H)$ is an ideal of $\text{End}(T)$.

**Proof.** The bracket $[\cdot, \cdot]$ is bilinear by definition and $XY = (-1)^{|X||Y|} YX$ for any $X, Y \in L(T)$ by (LYS1) and (LYS2). Jacobi’s superidentity follows from (LTS3–6). (2) is obvious. \(\square\)

### 3. Killing Forms of Lie-Yamaguti Superalgebras

The definition of the Killing form given here for Lie-Yamaguti superalgebras stems from [9] in the case of Lie-Yamaguti algebras and extends the one given in [18] for Lie supertriple systems. Let $T = T_0 \oplus T_1$ be an $n$-dimensional Lie-Yamaguti superalgebra. Denote by $\beta$ the Killing form of the standard enveloping Lie superalgebra $L(T) = (T_0 \oplus D(T, T)) \oplus (T_1 \oplus D(T, T))$. Consider the bilinear form $\beta$ of $T$ obtained by restricting $\alpha$ to $T \times T$. For any $x, y, z \in T$, define the endomorphisms $L_x$ and $R_{x,y}$ of the vector space $T$ by $L_x(y) = xy$ and $R_{x,y}(z) = (-1)^{|x||y|}[z, x, y] = (-1)^{|x||y||z|} D_{x,y}(y)$. It is clear that $R_{x,y}$ is of degree $\bar{x} + \bar{y}$ and $[D_{x,y}, R_{x,y}] = R_{y,x} R_{x,y} + (-1)^{|x||y|} R_{x,[x,y]}$.

**Theorem 10.** For $x, y \in T$, we have

$$\beta(x, y) = \text{str}(L_x L_y) + \text{str}(R_{x,y} + (-1)^{|x||y|} R_{y,x}). \tag{8}$$
Proof. Let \( \{a_i\}, \{b_i\}, \{u_i\}, \{v_i\} \) be bases of \( T_0, T_1, D_0(T, T), D_1(T, T) \), respectively. For these bases, we express the operations of \( T \) and \( D(T, T) \) as follows:

\[
a_i a_j = \sum_i s_{ij}^a a_i; \quad a_i a_j \in T_0,
\]

\[
a_i b_j = \sum_i t_{ij}^a b_i; \quad a_i b_j \in T_1,
\]

\[
b_i b_j = \sum_i k_{ij}^b a_i; \quad b_i b_j \in T_0,
\]

\[
D_{a,aj} = \sum_{\alpha} D_{ij}^\alpha u_{\alpha}; \quad D_{a,aj} \in D_0(T, T),
\]

\[
D_{a,bj} = \sum_{\alpha} D_{ij}^\alpha v_{\alpha}; \quad D_{a,bj} \in D_1(T, T),
\]

\[
D_{b,aj} = \sum_{\alpha} D_{ij}^\alpha w_{\alpha}; \quad D_{b,aj} \in D_0(T, T),
\]

\[
[u_{\alpha}, a_i] = u_{\alpha}(a_i) = \sum_j K_{ij}^\alpha a_j;
\]

\[
[v_{\alpha}, a_i] = v_{\alpha}(a_i) = \sum_j L_{ij}^\alpha b_j;
\]

\[
[u_{\alpha}, b_i] = u_{\alpha}(b_i) = \sum_j H_{ij}^\alpha b_j;
\]

\[
[v_{\alpha}, b_i] = v_{\alpha}(b_i) = \sum_j Q_{ij}^\alpha a_j.
\]

To prove the theorem, it suffices to show that \( \beta(a_i, a_j) = \alpha(a_i, a_j) \), \( \beta(a_i, b_j) = \alpha(a_i, b_j) \), and \( \beta(b_i, b_j) = \alpha(b_i, b_j) \). Since \( (L_a L_b)(T_0) \subseteq T_1 \) and \( (L_a L_b)(T_1) \subseteq T_0 \), we have \( \text{str}(L_a L_b) = 0 \). Also, \( R_{a,bj}(T_1) \subseteq T_0 \) and \( R_{a,bj}(T_0) \subseteq T_1 \) give \( \text{str}(R_{a,bj} + R_{b,aj}) = 0 \) and then \( \beta(a_i, b_j) = 0 = \alpha(a_i, b_j) \) because of the consistency property of \( \alpha(a_i \in T_0 \oplus D_0(T, T), b_j \in T_1 \oplus D_1(T, T)) \). Hence, it remains to show that \( \beta(a_i, a_j) = \alpha(a_i, a_j) \) and \( \beta(b_i, b_j) = \alpha(b_i, b_j) \). The operations in \( T \) and the identities (7) imply the following:

\[
[a_i, [a_j, a_k]] = [a_i, a_j a_k + D_{a,aj}]
\]

\[
= [a_i, \sum_{\alpha} c_{\alpha} a_m + \sum_{\alpha} D_{\alpha}^a u_{\alpha}]
\]

\[
= \sum_{\alpha} S_{\alpha}^m (a_m a_i + D_{a,am}) - \sum_{\alpha} D_{\alpha}^a \sum_{\alpha} K_{ij}^\alpha a_i
\]

\[
= \sum_{\alpha} S_{\alpha}^m \left( \sum_{\alpha} S_{ij}^\alpha a_i + \sum_{\alpha} D_{\alpha m}^a u_{\alpha} \right)
\]

\[
- \sum_{\alpha} D_{\alpha j}^a \sum_{\alpha} K_{ij}^\alpha a_i
\]

\[
= \sum_{\alpha} S_{\alpha}^m \sum_{\alpha} S_{ij}^\alpha a_i + \sum_{\alpha} D_{\alpha m}^a u_{\alpha}
\]

\[
- \sum_{\alpha} D_{\alpha j}^a \sum_{\alpha} K_{ij}^\alpha a_i.
\]

In a similar way, we get

\[
\sum_{\alpha} D_{\alpha j}^a \sum_{\alpha} K_{ij}^\alpha a_i.
\]

By interchanging \( i \) and \( j \), we have

\[
R_{\alpha,aj} (a_k) = \sum_{\alpha} D_{\alpha j}^a K_{aj}^m a_m.
\]

\[
R_{\alpha,aj} (b_k) = \sum_{\alpha} \sum_{\alpha} C_{\alpha j}^a \sum_{\alpha} D_{\alpha j}^a b_{m}.
\]
Therefore,

\[ \beta(a_i, a_j) = \alpha(a_i, a_j) \]

\[ = \sum_{m,k} s_{jk}^m a_k + \sum_{m,k} D_{jk}^a a_{a_k} - \sum_{m,k} T_{jk}^{m} a_k + \sum_{a,k} C_{jk}^a a_i \]

\[
\text{str } (L_a L_b) + \text{tr } (R_{a,b} + R_{b,a}) \\
= \sum_{p,k} C_{p}^{a_{k}} a_k + \sum_{p,k} D_{p,k}^a a_k + \sum_{a,k} D_{jk}^a a_k + \sum_{a,k} C_{jk}^a a_k \]

\[ \beta(\beta(x, y)) = \beta(x, y) \]

It remains to show that \( \beta(b_i, b_j) = \text{str}(L_b L_{b_j}) + \text{str}(R_{b,j} + R_{b,j}) \),

\[ [b_i, [b_j, a_k]] = [b_i, [b_j, a_k]] + D_{b_i b_j} \]

\[ = [b_i, \sum_{m,k} R_{jk}^m a_k + \sum_{a,k} C_{jk}^a a_k] \]

\[ = -\sum_{m,k} R_{jk}^m a_k + \sum_{a,k} C_{jk}^a a_k \]

\[ = -\sum_{a,k} C_{jk}^a a_k \]

Likewise, we have

\[ [b_i, [b_j, a_k]] = -[b_i, [b_j, a_k]] \]

\[ = -[b_i, [b_j, a_k]] \]

\[ = -\sum_{a,k} C_{jk}^a a_k \]

Therefore,

\[ \beta(a_i, a_j) = \alpha(a_i, a_j) \]

\[ = -\sum_{m,k} s_{jk}^m a_k - \sum_{a,k} C_{jk}^a a_k + \sum_{m,k} R_{jk}^m a_k \]

\[ + \sum_{a,k} C_{jk}^a a_k \]

Now,

\[ L_b L_{b_j}(a_k) = b_j(b_k a_k) = -b_j \left( \sum_{m} T_{jk}^m b_m \right) \]

\[ = -\sum_{m} T_{jk}^m (b_j b_m) = -\sum_{m} T_{jk}^m b_m a_k; \]

\[ L_b L_{b_j}(b_k) = b_j(b_k b_j) = \left( \sum_{m} R_{jk}^m b_m \right) \]

\[ = -\sum_{m} R_{jk}^m (a_k b_m) = -\sum_{m} R_{jk}^m a_k b_m; \]

By interchanging \( i \) and \( j \), we have

\[ \text{R}_{b,b_j}(a_k) = \sum_{a,\alpha} C_{a}^{\alpha} Q^\alpha a_k; \]

\[ \text{R}_{b,b_j}(b_k) = \sum_{x,\alpha} X_{x}^{\alpha} H_{x}^b a_k; \]

\[ \text{str} (L_b L_{b_j}) + \text{tr } (R_{b,j} + R_{b,j}) \]

\[ = -\sum_{m} R_{jk}^m a_k + \sum_{m} C_{jk}^a a_k - \sum_{a} \sum_{a} C_{jk}^a T_{jk}^m b_m \]

\[ - \sum_{a} C_{jk}^a Q^\alpha a_k + \sum_{a} X_{x}^{\alpha} H_{x}^b b_k \]

Hence the theorem is proved.

**Definition II.** The bilinear form \( \beta \) defined on the Lie-Yamaguti superalgebra \( T = T_0 \oplus T_1 \) by

\[ \beta(x, y) = \text{str} (L_x L_y) + \text{tr } (R_{x,y} + (-1)^{x} R_{y,x}) \]

for \( x, y \in T \) is called the Killing form of \( T \).

**Remark 12.** Recall that if \( T \) is a Lie superalgebra, then the Killing form \( \beta \) on \( T \) is defined as \( \beta(x, y) = \text{str}(L_x L_y) \), \( x, y \in T \). Likewise, if \( T \) is a Lie supertriple system (resp., a Lie-Yamaguti algebra), the Killing form on \( T \) is defined as \( \beta(x, y) = \text{str}(R_{x,y} + (-1)^{x} R_{y,x}) \) (resp., \( \beta(x, y) = \text{tr}(L_x L_y) + \text{tr}(R_{x,y} + R_{y,x}) \)) with \( L_u \) and \( R_{a,v} \) defined according to the considered structure on \( T \). So if a Lie-Yamaguti superalgebra \( T \) is reduced to a Lie superalgebra (resp., a Lie supertriple system, the Lie-Yamaguti algebra) then \( \beta \) as defined in Definition II is the Killing form of the Lie superalgebra (resp., the Lie supertriple system, the Lie-Yamaguti algebra) \( T \).
Proposition 13. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra with a Killing form denoted by $\beta$. Then,

(1) $\beta(T_0, T_1) = 0$ (consistence),
(2) $\beta(x, y) = (−1)^{\overline{x}\overline{y}} \beta(y, x)$ (supersymmetry),
(3) $\beta(A(x), A(y)) = \beta(x, y)$, $A \in \text{Aut}(T)$.

Proof. As $L_T, L_T, (T_0) \subseteq T_0$, $L_T, L_T, (T_0) \subseteq T_1$ and $R_T, R_T, (T_1) \subseteq T_0$ we can see that $\beta(T_0, T_1) = 0$.

(2) comes from the definition of $\beta$.

Now, for any $A \in \text{Aut}(T)$, $x$ in $T$, $A(x) = \bar{x}$, and

$$\beta(A(x), A(y)) = \text{str} \left( L_{A(x)} L_{A(y)} \right) + \text{str} \left( R_{A(x), A(y)} + (-1)^{\overline{x}\overline{y}} R_{A(y), A(x)} \right).$$

As $A(xy) = A(x)A(y)$ then $AL_x(x) = L_{A(x)} A(y)$; that is, $AL_x = L_{A(x)} A$ and $AL_x A^{-1} = L_{A(x)}$.

Hence, $\text{str}(L_{A(x)} L_{A(y)}) = \text{str}(AL_x A^{-1} AL_y A^{-1}) = \text{str}(AR_{x,y} A^{-1} + (-1)^{\overline{x}\overline{y}} R_{x,y} A^{-1})$.

(21)

From $\text{str} \left( L_{A(x)} L_{A(y)} \right) = \text{str} \left( R_{A(x), A(y)} + (-1)^{\overline{x}\overline{y}} R_{A(y), A(x)} \right)$, that is, $AR_{x,y} A^{-1} = R_{A(x), A(y)}$, Then,

$$\beta(A(x), A(y)) = \text{str} \left( L_{A(x)} L_{A(y)} \right) + \text{str} \left( R_{A(x), A(y)} + (-1)^{\overline{x}\overline{y}} R_{A(y), A(x)} \right)$$

$$= \text{str} \left( L_{A(x)} L_{A(y)} \right) + \text{str} \left( R_{A(x), A(y)} + (-1)^{\overline{x}\overline{y}} R_{A(y), A(x)} \right)$$

$$= \beta(x, y).$$

(22)

Now, let $y$ be a trilinear form in $T$ given by $\gamma(x, y, z) = \text{str}(D_{x,y} L_z)$ for any $x, y, z \in T$. We can easily see that, for any $x, y, z \in T$, $\gamma(x, y, z) = (−1)^{\overline{y}\overline{z}} \gamma(x, y, z)$ and that $\gamma$ vanishes identically if $T$ is reduced to Lie superalgebra or Lie superalgebra system.

Theorem 14. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra with a Killing form denoted by $\beta$. Then, $\beta$ satisfies the identities

$$\beta(xy, z) + (−1)^{\overline{x}\overline{y}} \beta(y, xz)$$

$$= (−1)^{\overline{y}} \gamma(x, y, z) + (−1)^{\overline{y}\overline{z}} \gamma(z, x, y);$$

(23)

$$\beta([x, y, w]) + (−1)^{\overline{x}\overline{y}} \beta([x, w, y]) + (−1)^{\overline{x}\overline{y}} \beta([x, w, y]) + (−1)^{\overline{x}\overline{y}} \beta([x, w, y]) + (−1)^{\overline{x}\overline{y}} \beta([x, w, y]) + (−1)^{\overline{x}\overline{y}} \beta([x, w, y])$$

(24)

for all $x, y, z \in T$.

Proof. The Killing form $\alpha$ of $L = (T_0 \oplus D_0(T, T)) \oplus (T_1 \oplus D_1(T, T))$ satisfies $\alpha(y, [x, z]) = (−1)^{\overline{y}} \alpha([x, y], z) = 0$; that is, $\alpha(y, [x, z]) = \alpha([x, y], z)$. But, using (7), we have

$$\alpha([x, y], z) = \alpha(xy + D_{x,y} z)$$

$$= \alpha(xy, z) + \alpha(D_{x,y} z)$$

$$= \beta(xy, z) + \beta(D_{x,y} L_z)$$

$$= \beta(xy, z) + \gamma(x, y, z),$$

$$\alpha(y, [x, z]) = \alpha(y, xz + D_{x,z})$$

$$= \alpha(y, xz) + \beta(L_y D_{x,z})$$

$$= \beta(y, xz) + (−1)^{\overline{y}} \alpha \beta(D_{x,z} L_y)$$

$$= \beta(y, xz) + (−1)^{\overline{y}} \alpha \beta(D_{x,z} L_y)$$

Then the identity $\alpha(y, [x, z]) = (−1)^{\overline{y}} \alpha([x, y], z) = 0$ gives $\beta(xy, z) + \gamma(x, y, z) + (−1)^{\overline{y}} \beta(y, xz) + (−1)^{\overline{y}} \gamma(x, y, z) = 0$ that is $\beta(xy, z) + (−1)^{\overline{y}} \beta(y, xz) = −\gamma(x, y, z) − (−1)^{\overline{y}} \gamma(x, y, z) = (−1)^{\overline{y}} \gamma(x, y, z)$ and (23) is obtained.

From $\alpha([x, y], z) = \alpha([x, y], z)$ we deduce $\alpha([x, w], [y, z]) = \alpha([x, w], [y, z])$ that is $−(−1)^{\overline{y}\overline{z}} \alpha\beta([x, w], [y, z]) = 0$ and (26) is obtained.

Then, using (7) again and developing (26), we have $\alpha\beta([x, [y, z], w]) + (−1)^{\overline{y}\overline{z}} \alpha\beta([xw + D_{xw}, y], z) = 0$ and we get

$$\alpha\beta([x, [y, z], w]) + (−1)^{\overline{y}\overline{z}} \alpha\beta([xw + D_{xw}, y], z) = 0$$

$$\alpha\beta([x, (yz) w] + D_{y, w} + [y, z, w])$$

$$+ (−1)^{\overline{y}\overline{z}} \alpha\beta([xw + D_{xw}, y], z)$$

$$= 0.$$
Thus,
\[
\begin{align*}
\beta(x, (yz)w) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \text{str}(L_{D_{xyw}} L_x) \\
&+ \beta(x, [y, z, w]) + (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \text{str}(L_{D_{xyw}} L_z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0;
\end{align*}
\]
that is,
\[
\begin{align*}
\beta(x, (yz)w) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(yz, w, x) \\
&+ \beta(x, [y, z, w]) + (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(xy, y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

Corollary 15. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra with a Killing form denoted by $\beta$. Then, $\beta$ satisfies the following for $x, y, z \in T$:
\[
\begin{align*}
\beta(xy, [y, z, w]) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0;
\end{align*}
\]
where
\[
\begin{align*}
\beta([y, z, x], w) &+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((yw) x, z) \\
&+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta([y, x, w], z) = 0.
\end{align*}
\]

This implies
\[
\begin{align*}
\beta(x, [y, z, w]) &+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(yz, w, x) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(xy, y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

Proof. Using (24) we have
\[
\begin{align*}
\beta([y, z, x], w) &+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0,
\end{align*}
\]
where
\[
\begin{align*}
\beta((xw) y, z) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(yz, w, x) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(xy, y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

By adding memberwise (36) and (37) we obtain the identity (34).

Also, the identity (37) is equivalent to
\[
\begin{align*}
\beta([y, z, x], w) &+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

Then, we obtain
\[
\begin{align*}
\beta((xw) y, z) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(yz, w, x) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(xy, y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

Hence, $\beta(x, [y, z, w]) + (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) = (-1)^{\frac{|y|+|z|+|w|}{2}} \gamma(yz, w, x)$ and (24) is proved.

4. Invariant Forms of Lie-Yamaguti Superalgebras

In this section we introduced the concept of invariant forms of Lie-Yamaguti superalgebras as generalizations of those of Lie superalgebras and Lie supertriple systems.

Definition 16. An invariant form $b$ of a Lie-Yamaguti superalgebra $T = T_0 \oplus T_1$ is a supersymmetric bilinear form on $T$ satisfying the identities
\[
\begin{align*}
\beta((xw) y, z) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

Remark 17. (1) If $\gamma = 0$, the Killing form of $T$ is an invariant form of $T$. 

By adding memberwise (36) and (37) we obtain the identity (34).

Also, the identity (37) is equivalent to
\[
\begin{align*}
\beta((xw) y, z) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(yz, w, x) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(xy, y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

By adding memberwise (36) and (37) we obtain the identity (34).

Also, the identity (37) is equivalent to
\[
\begin{align*}
\beta((xw) y, z) &+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(yz, w, x) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \gamma(xy, y, z) \\
&+ (-1)^{\frac{|x|+|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]

Then, we obtain
\[
\begin{align*}
\beta(x, [y, z, w]) &+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta((xw) y, z) \\
&+ (-1)^{\frac{|y|+|z|+|w|}{2}} \beta([x, w, y], z) = 0.
\end{align*}
\]
(2) If $T$ is reduced to a Lie supertriple system (resp., a Lie superalgebra, a Lie-Yamaguti algebra), then $b$ is reduced to an invariant form of a Lie supertriple system [19] (resp., a Lie superalgebra [12], a Lie-Yamaguti algebra [10]).

Definition 18. Let $b$ be an invariant form of a Lie-Yamaguti superalgebra $S$ and $T$ a subset of $T$. The orthogonal $S^\perp$ of $S$ with respect to $b$ is defined by $S^\perp = \{x \in T, \ b(x, y) = 0, \forall y \in S\}$. The invariant form $b$ is nondegenerate if $T^\perp = \{0\}$.

Lemma 19. Let $b$ be an invariant form of a Lie-Yamaguti superalgebra $T$. Then, for any $x, y, z, \in T$, we have
\[ b\left([x, y, w], z\right) + (-1)^{(x+y)z} b\left(w, [x, y, z]\right) = 0. \] (40)

Proof. By interchanging $y$ and $w$ in (39) we have
\[ b\left([x, y, w], z\right) + (-1)^{(x+y)z} b\left(x, [y, w, z]\right) = 0 \] (41)
that is $(-1)^{(x+y)z} b\left(z, [x, y, w]\right) = 0$ by supersymmetry. Also by switching $z$ and $w$ in (41), we obtain
\[ b\left([x, y, z], w\right) + (-1)^{(x+y)z} b\left(y, [z, w, x]\right) = 0 \] (42)
Thus adding (41) and (42) we get (40) whence the lemma. □

Lemma 20. Let $b$ be an invariant form of a Lie-Yamaguti superalgebra $T$. Then,
1. $(T + [T, T, T])^\perp = Z(T)$ if $b$ is nondegenerate; 
2. If $H$ is an ideal of $T$ then $H^\perp$ is an ideal of $H$. In particular, $T^\perp$ is an ideal of $T$.

Proof. Consider $x$ in $(T + [T, T, T])^\perp$. Then, for any $u, v, w \in T$, we have $b(x, uv) = 0$ and $b(x, [u, v, w]) = 0$. This implies, by (38) and (39), that $(-1)^{xy} b(xu, v) = 0$ and $(-1)^{xy} b(x, [u, v, w]) = 0$.

Conversely, if $x \in Z(T)$, we have, for any $u, v, u', v', w' \in T$, $b(x, uv + [u, v, w']) = b(x, uv) + b(x, [u, v, w']) = 0$ and $b(x, [u, v, w]) = 0$. Thus $x \in Z(T)$.

Now, suppose that $H$ is an ideal of $T$ that is $TH \subseteq H$ and $[T, H, T] \subseteq H$; then for any $y, x, u, \in T, u \in H^\perp$, and $h \in H$, we have $b(xu, h) = (-1)^{xy} b(x, uh) = 0$ and $b(x, y, h) = (-1)^{xy} b(x, y, h) = (-1)^{xy} b(x, y, h) = 0$. Then $TH^\perp \subseteq H^\perp$ and $[[T, H, T], T] \subseteq H^\perp$ which proves (2). □

We are now ready to prove the following theorem.

Theorem 21. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra with $y = 0$. Then the Killing form $\alpha$ is nondegenerate if and only if the standard enveloping Lie superalgebra $L(T) = T \oplus D(T, T)$ is a semisimple Lie superalgebra.

Proof. Let $\alpha$ be the Killing form of the Lie superalgebra $L(T)$. If $y = 0$, we have, for any $x, y, z \in T, y(x, y, z) = \text{str}(D_{x,y}L_z) = 0$ and
\[ \alpha\left(D_{x,y}, z\right) = 0. \] (43)

Then, using the invariance of $\alpha$ and (43), we have, for any $x, y, z, w \in T: \alpha([x, y], D_{z,w}) = \alpha(x, [y, D_{z,w}])$; that is, by (7),
\[ \alpha(xy + D_{x,y}, D_{z,w}) = (-1)^{(x+y)z} \alpha(x, [z, w, y]) \] and
\[ \alpha(D_{x,y}, D_{z,w}) = (-1)^{(x+y)z} \alpha(x, [z, w, y]). \] This gives
\[ \alpha\left(D_{x,y}, D_{z,w}\right) = (-1)^{(x+y)z} \beta(x, [z, w, y]). \] (44)
Thus, if $\beta$ is nondegenerate, the restriction of $\alpha$ on $D(T, T) \times D(T, T)$ is nondegenerate and $\alpha$ is nondegenerate. Now, suppose that $\beta$ is degenerate. Then by the lemma above, $\perp^\perp$ is an ideal of $T$ so $T^\perp \oplus D(T, T^\perp)$ is a nonzero ideal of $T$.

Using the identities (43) and (44) we get
\[ \alpha\left(T^\perp \oplus D(T, T^\perp), T \oplus D(T, T)\right) = \alpha\left(T^\perp, T\right) + \alpha\left(T^\perp, D(T, T)\right) + \alpha\left(D(T, T^\perp), T\right) + \alpha\left(D(T, T^\perp), D(T, T)\right) \] (45)
\[ \alpha\left(T^\perp, T\right) + \beta\left(T, [T, T^\perp, T]\right) = 0. \]

It comes that $\alpha$ is degenerate and $T^\perp \oplus D(T, T)$ is not semisimple which proves the theorem. □

The results of this paper could be used for a study of the structure of a pair consisting of a semisimple Lie superalgebra and its semisimple graded subalgebra.

Competing Interests

The authors declare that they have no competing interests.

References


