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Research Article

On Killing Forms and Invariant Forms of Lie-Yamaguti Superalgebras

Patricia L. Zoungrana[1,2] and A. Nourou Issa[1,2]

[1] *Département de Mathématiques de la Décision, Université Ouaga 2, 12 BP 412, Ouagadougou 12, Burkina Faso*
[2] *Département de Mathématiques, Université d'Abomey-Calavi, 01 BP 4521, Cotonou, Benin*

Correspondence should be addressed to Patricia L. Zoungrana; patibffr@yahoo.fr

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The notions of the Killing form and invariant form in Lie algebras are extended to the ones in Lie-Yamaguti superalgebras and some of their properties are investigated. These notions are also $\mathbb{Z}_2$-graded generalizations of the ones in Lie-Yamaguti algebras.

1. Introduction

A *Lie-Yamaguti algebra* is a triple $(T, *, [\cdot,\cdot,\cdot])$ consisting of a vector space T, a bilinear map $* : T \times T \to T$, and a trilinear map $[\cdot,\cdot,\cdot] : T \times T \times T \to T$ such that

(LY1) $x * y = -y * x$,

(LY2) $[x, y, z] = -[y, x, z]$,

(LY3) $\circlearrowleft_{x,y,z}\{((x * y) * z) + [x, y, z]\} = 0$,

(LY4) $\circlearrowleft_{x,y,z}[x * y, z, u] = 0$,

(LY5) $[x, y, u * v] = [x, y, u] * v + u * [x, y, v]$,

(LY6) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$,

for all u, v, x, y, z, in T, where $\circlearrowleft_{x,y,z}$ denotes the sum over cyclic permutation of x, y, z. The bilinear map $*$ sometimes will be denoted by juxtaposition. If $x * y = 0$, $\forall x, y \in T$, one gets a *Lie triple system* $(T, [\cdot,\cdot,\cdot])$, while $[x, y, z] = 0$ in $(T, *, [\cdot,\cdot,\cdot])$ induces a Lie algebra $(T, *)$.

Lie-Yamaguti algebras were introduced by Yamaguti [1] (who formerly called them "generalized Lie triple systems") in an algebraic study of the characteristic properties of the torsion and curvature of a homogeneous space with canonical connection [2]. Later on, these algebraic objects were called "Lie triple algebras" [3] and the terminology of "Lie-Yamaguti algebras" is introduced in [4] for these algebras. For further development of the theory of Lie-Yamaguti algebras one may refer, for example, to [5–8]. From the standard enveloping Lie algebra of a given Lie-Yamaguti algebra, the notions of the Killing-Ricci form and the invariant form of a Lie-Yamaguti algebra are introduced and studied in [9]. Further properties of invariant forms of Lie-Yamaguti algebras were considered in [10].

Lie superalgebras as a $\mathbb{Z}_2$-graded generalization of Lie algebras are considered in [11, 12] while a $\mathbb{Z}_2$-graded generalization of Lie triple systems (called Lie supertriple systems) was first considered in [13]. For an application of Lie supertriple systems in physics, one may refer to [14]. Next, *Lie-Yamaguti superalgebras* as a $\mathbb{Z}_2$-graded generalization of Lie-Yamaguti algebras were first considered in [15].

Definition 1 (see [16]). A Lie-Yamaguti superalgebra is a $\mathbb{Z}_2$-graded vector space $T = T_0 \oplus T_1$ with a binary operation denoted by juxtaposition satisfying $T_iT_j \subseteq T_{i+j}$ and a ternary operation $[\cdot,\cdot,\cdot]$ satisfying $[T_i, T_j, T_k] \subseteq T_{i+j+k}$ $(i, j, k \in \mathbb{Z}_2)$ such that

(LYS1) $xy = -(-1)^{\overline{x}\,\overline{y}}yx$,

(LYS2) $[x, y, z] = -(-1)^{\overline{x}\,\overline{y}}[y, x, z]$,

(LYS3) $\circlearrowleft_{x,y,z}(-1)^{\overline{x}\,\overline{z}}\{((xy)z) + [x, y, z]\} = 0$,

(LYS4) $\circlearrowleft_{x,y,z}(-1)^{\overline{x}\,\overline{z}}[xy, z, u] = 0$,

(LYS5) $[x, y, uv] = [x, y, u]v + (-1)^{\overline{u}(\overline{x}+\overline{y})}u[x, y, v]$,

(LYS6) $[u,v,[x,y,z]] = [[u,v,x],y,z] + (-1)^{\overline{x}(\overline{u}+\overline{v})}[x,[u,v,y],z] + (-1)^{(\overline{x}+\overline{y})(\overline{u}+\overline{v})}[x,y,[u,v,z]]$,

for all u,v,x,y,z, in T.

Observe that T_0 is a Lie-Yamaguti algebra.

As a part of the general theory of superalgebras, the notion of the Killing form of Lie algebras is extended to the one of Lie triple systems (see [17] and references therein), Lie superalgebras [12], and next Lie supertriple systems [18] (see also [19]).

In this paper we define and study the Killing form and invariant form of Lie-Yamaguti superalgebras as a generalization of the ones of both of Lie-Yamaguti algebras [9] and Lie supertriple systems [18, 19] including Lie superalgebras [12].

The paper is organized as follows. In Section 2 we record some useful results on Lie-Yamaguti superalgebras (see [16]). In Section 3 the Killing form of a Lie-Yamaguti superalgebra is defined (see Theorem 10 and Definition 11) and some of its properties are investigated (Proposition 13, Theorem 14, and Corollary 15). In Section 4 the invariant form of a Lie-Yamaguti superalgebra is defined (Definition 16) and, under some conditions, it is shown (Theorem 21) that the Killing form of a Lie-Yamaguti superalgebra T is nondegenerate if and only if the standard enveloping Lie superalgebra of T is semisimple.

All vector spaces and algebras are finite-dimensional over a fixed ground field $\mathbb{K}$ of characteristic 0.

2. Some Basics on Lie-Yamaguti Superalgebras

We give here some definitions and results which can be found in [11, 12, 16].

A superalgebra over $\mathbb{K}$ is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$, where $A_iA_j \subseteq A_{i+j}$, $(i,j \in \mathbb{Z}_2)$. The subspaces A_0 and A_1 are called the even and the odd parts of the superalgebra and so are called the elements from A_0 and from A_1, respectively. Below, all the elements are assumed to be homogeneous, that is, either even or odd, and for a homogeneous element $x \in A_i$, $i = 0,1$, the notation $\overline{x} = i$ is used and means the parity of x.

Let G be the *Grassmann algebra* over $\mathbb{K}$ generated by the elements $1, e_1, \ldots, e_n$ such that $e_i^2 = 0$, $e_ie_j = -e_je_i$ for $i \neq j$. The elements $1, e_{i_1}e_{i_2}\cdots e_{i_r}$, $i_1 < i_2 < \cdots < i_r$ form a basis of G. Denote by G_0 (resp., G_1) the span of the products of even length (resp., odd length) in the generators. The product of zero e_i's is by convention equal to 1. Then $G = G_0 \oplus G_1$ is an associative and supercommutative superalgebra; that is, $g_1g_2 = (-1)^{\overline{g}_1\overline{g}_2}g_2g_1$, where $g_1, g_2 \in G_0 \cup G_1$. Let $A = A_0 \oplus A_1$ be a superalgebra. Consider the graded tensor product $G \otimes A$ which becomes a superalgebra with the product given by $(x \otimes g_1)(y \otimes g_2) = (-1)^{\overline{x}\,\overline{g}_2}xy \otimes g_1g_2$, for homogeneous elements $g_1, g_2 \in G$, $x, y \in A$ and grading given by $(G \otimes A)_0 = G_0 \otimes A_0 \oplus G_1 \otimes A_1$, $(G \otimes A)_1 = G_0 \otimes A_1 \oplus G_1 \otimes A_0$. The subalgebra $G(A) = (G \otimes A)_0 = G_0 \otimes A_0 \oplus G_1 \otimes A_1$ is called the *Grassmann envelope* of the superalgebra A.

Having in mind that if V is a homogeneous variety of algebras [20], a superalgebra $A = A_0 \oplus A_1$ is called a V-superalgebra, if its Grassmann envelope $G(A)$ belongs to V, we can state the following proposition.

Proposition 2. *A superalgebra $T = T_0 \oplus T_1$ equipped with bilinear and trilinear products verifying $T_iT_j \subseteq T_{i+j}$ and $[T_i, T_j, T_k] \subseteq T_{i+j+k}$ is a Lie-Yamaguti superalgebra if its Grassmann envelope $G(T) = G_0 \otimes T_0 \oplus G_1 \otimes T_1$ is a Lie-Yamaguti algebra under the following products:*

$$\begin{aligned}
&(x \otimes g_1)(y \otimes g_2) = (-1)^{\overline{x}\,\overline{y}}\, xy \otimes g_1g_2;\\
&[x \otimes g_1, y \otimes g_2, z \otimes g_3] \\
&\quad = (-1)^{\overline{x}\,\overline{y}+\overline{y}\,\overline{z}+\overline{x}\,\overline{z}}\,[x,y,z] \otimes g_1g_2g_3.
\end{aligned} \tag{1}$$

Proof. The proof is straightforward by using the fact that, for any element $x \otimes g$ in $G(T)$, we have $\overline{x} = \overline{g}$. □

Example 3. (1) Lie superalgebras are Lie-Yamaguti superalgebras with $[x,y,z] = 0$.

(2) If $xy = 0$ for any $x, y \in T_0 \cup T_1$ then (LYS2), (LYS3), and (LYS6) define a Lie supertriple system.

(3) Let $M = M_0 \oplus M_1$ be a Malcev superalgebra; that is, for any x, y, z, t in T,

$$\begin{aligned}
xy &= -(-1)^{\overline{x}\,\overline{y}}\, yx;\\
-(-1)^{\overline{y}\,\overline{z}}(xz)(yt) &= ((xy)z)t \\
&\quad + (-1)^{\overline{x}(\overline{y}+\overline{z}+\overline{t})}((yz)t)x \\
&\quad + (-1)^{(\overline{x}+\overline{y})(\overline{z}+\overline{t})}((zt)x)y \\
&\quad + (-1)^{\overline{t}(\overline{x}+\overline{y}+\overline{z})}((tx)y)z.
\end{aligned} \tag{2}$$

It is shown in [16] that M becomes a Lie-Yamaguti superalgebra if we set $[x,y,z] = x(yz) - (-1)^{\overline{x}\,\overline{y}}y(xz) + (xy)z$. Conversely, if on a Malcev superalgebra $(M,\cdot)$ we define a trilinear operation by $[x,y,z] = x \cdot yz - (-1)^{\overline{x}\,\overline{y}}y \cdot xz + xy \cdot z$ then $(M,\cdot,[\cdot,\cdot,\cdot])$ is a Lie-Yamaguti superalgebra.

Definition 4. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. A graded subspace $H = H_0 \oplus H_1$ of T is a graded Lie-Yamaguti subalgebra of T if $H_iH_j \subseteq H_{i+j}$ and $[H_i, H_j, H_k] \subseteq H_{i+j+k}$ for any $i, j, k \in \mathbb{Z}_2$.

Definition 5. A graded subalgebra of a Lie-Yamaguti superalgebra T is an invariant graded subalgebra (resp., an ideal) of T if $[T,T,H] \subseteq H$ (resp., $TH \subseteq H$ and $[T,H,T] \subseteq H$).

If H is an ideal of T, it is an invariant graded subalgebra of T. Obviously the center $Z(T)$ of a Lie-Yamaguti superalgebra T defined by $Z(T) = \{x \in T,\ xy = 0 \text{ and } [x,y,z] = 0,\ \forall y,z \in T\}$ is an ideal of T.

Definition 6. Let $T = T_0 \oplus T_1$ and $T' = T_0' \oplus T_1'$ be Lie-Yamaguti superalgebras. A linear map $f : T \to T'$ is said to be of degree r if $f(T_i) \subseteq T'_{r+i}$ for all $r, i \in \mathbb{Z}_2$.

Definition 7. Let $T = T_0 \oplus T_1$ and $T' = T_0' \oplus T_1'$ be Lie-Yamaguti superalgebras. A linear map $f : T \to T'$ is called a homomorphism of Lie-Yamaguti superalgebras if

(1) f preserves the grading, that is, $f(T_i) \subseteq T'_i$, $i \in \mathbb{Z}_2$;

(2) $f(xy) = f(x)f(y)$;

(3) $f([x, y, z]) = [f(x), f(y), f(z)]$ for any $x, y, z \in T_0 \cup T_1$.

Recall [11] that if $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space then, if we set $\mathrm{End}_r(V) = \{f \in \mathrm{End}(V)/f(V_i) \subseteq V_{r+i}\}$, we obtain an associative superalgebra $\mathrm{End}(V) = \mathrm{End}_0(V) \oplus \mathrm{End}_1(V)$; $\mathrm{End}_r(V)$ consists of the linear mappings of V into itself which are homogeneous of degree r. The bracket $[f, g] = fg - (-1)^{\overline{f}\,\overline{g}} gf$ makes $\mathrm{End}(V)$ into a Lie superalgebra which we denote by $l(V)$ or $l(m, n)$ where $m = \dim V_0$ and $n = \dim V_1$. Let $e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}$ be a basis of V. In this basis the matrix of $a \in l(m, n)$ is expressed as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, α being an $(m \times m)$-, δ an $(n \times n)$-, β an $(m \times n)$-, and γ an $(n \times m)$-matrix. The matrices of even elements have the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and those of odd ones $\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$. For $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, the supertrace of a is defined by $\mathrm{str}(a) = \mathrm{tr}\,\alpha - \mathrm{tr}\,\delta$ and does not depend on the choice of a homogeneous basis. We have $\mathrm{str}([a, b]) = 0$ that is $\mathrm{str}(ab) = (-1)^{\overline{a}\overline{b}}\mathrm{str}(ba)$ and $\mathrm{str}(aba^{-1}) = \mathrm{str}(b)$.

Definition 8. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra; $D \in \mathrm{End}_r(T)$ is a superderivation of T if, for any $x, y, z \in T_0 \cup T_1$,

$$\begin{aligned} D(xy) &= D(x)\,y + (-1)^{r\overline{x}}\,xD(y); \\ D([x, y, z]) &= [D(x), y, z] + (-1)^{r\overline{x}}\,[x, D(y), z] \\ &\quad + (-1)^{r(\overline{x}+\overline{y})}\,[x, y, D(z)]. \end{aligned} \tag{3}$$

Let $D_r(T)$ consist of all the superderivations of degree r and $D(T) = D_0(T) \oplus D_1(T)$. It is easy to check that $D(T)$ is a graded subalgebra of $\mathrm{End}(T)$ called the Lie superalgebra of superderivations of T.

Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. For any $x, y \in T_0 \cup T_1$, denote by $D_{x,y}$ the endomorphism of T defined by $D_{x,y}(z) = [x, y, z]$ for any $z \in T$. We have, for any $x, y \in T_0 \cup T_1$, $r \in \mathbb{Z}_2$, $D_{x,y}(T_r) \subseteq T_{r+\overline{x}+\overline{y}}$; that is, $D_{x,y}$ is a linear map of degree $\overline{x} + \overline{y}$. Moreover, it comes from (LYS5) and (LYS6) that

$$D_{x,y}(zw) = D_{x,y}(z)\,w + (-1)^{\overline{z}(\overline{x}+\overline{y})}\,zD_{x,y}(w); \tag{4}$$

$$\begin{aligned} D_{x,y}([z, v, w]) &= \left[D_{x,y}(z), v, w\right] \\ &\quad + (-1)^{\overline{z}(\overline{x}+\overline{y})}\left[z, D_{x,y}(v), w\right] \\ &\quad + (-1)^{(\overline{z}+\overline{v})(\overline{x}+\overline{y})}\left[z, v, D_{x,y}(w)\right] \end{aligned} \tag{5}$$

for any $x, y, z, v, w \in T_0 \cup T_1$. It follows that $D_{x,y}$ is a superderivation of T called an *inner superderivation* of T.

Let $D(T, T)$ be the vector space spanned by all $D_{x,y}$ $(x, y \in T)$.

We can define naturally a $\mathbb{Z}_2$-gradation by setting $D(T, T) = D_0(T, T) \oplus D_1(T, T)$, where $D_r(T, T)$ consists of the superderivation $D_{x,y}$ of degree r. From (5) we also have that, for any $x, y, z, v, w \in T_0 \cup T_1$,

$$\left[D_{x,y}, D_{z,v}\right] = D_{[x,y,z],v} + (-1)^{\overline{z}(\overline{x}+\overline{y})}\,D_{z,[x,y,v]}. \tag{6}$$

It is clear from (6) that $D(T, T)$ is a $\mathbb{Z}_2$-graded Lie subalgebra of $D(T)$ called the *Lie superalgebra of all inner superderivations of T*.

Now, let $(T, \cdot, [\cdot, \cdot, \cdot])$ be a Lie-Yamaguti superalgebra.

Set $L_i(T) = T_i \oplus D_i(T, T)$, $i = 0, 1$, and define a new bracket operation in $L(T) = L_0(T) \oplus L_1(T) = T \oplus D(T, T)$ as follows: for any $x, y \in T_0 \cup T_1$, $D_1, D_2 \in D_0(T, T) \cup D_1(T, T)$,

$$\begin{aligned} [x, y] &= xy + D_{x,y}; \\ [D, x] &= -(-1)^{\overline{x}\overline{D}}\,[x, D] = D(x); \\ [D_1, D_2] &= D_1D_2 - (-1)^{\overline{D}_1\overline{D}_2}\,D_2D_1. \end{aligned} \tag{7}$$

Theorem 9. *Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. Then*

(1) *$L(T)$ is a Lie superalgebra called the standard enveloping Lie superalgebra of T and $D(T, T)$ becomes a graded subalgebra of $L(T)$.*

(2) *If H is an ideal of T then $H \oplus D(T, H)$ is an ideal of $T \oplus D(T, T)$.*

Proof. The bracket $[\cdot, \cdot]$ is bilinear by definition and $XY = -(-1)^{\overline{X}\,\overline{Y}}YX$ for any $X, Y \in L(T)$ by (LYS1) and (LYS2). Jacobi's superidentity follows from (LTS3–6).

(2) is obvious. □

3. Killing Forms of Lie-Yamaguti Superalgebras

The definition of the Killing form given here for Lie-Yamaguti superalgebras stems from [9] in the case of Lie-Yamaguti algebras and extends the one given in [18] for Lie supertriple systems. Let $T = T_0 \oplus T_1$ be an n-dimensional Lie-Yamaguti superalgebra. Denote by α the Killing form of the standard enveloping Lie superalgebra $L(T) = (T_0 \oplus D_0(T, T)) \oplus (T_1 \oplus D_1(T, T))$. Consider the bilinear form β of T obtained by restricting α to $T \times T$. For any x, y, z in T, define the endomorphisms L_x and $R_{x,y}$ of the vector space T by $L_x(y) = xy$ and $R_{x,y}(z) = (-1)^{\overline{z}(\overline{x}+\overline{y})}[z, x, y] = (-1)^{\overline{z}(\overline{x}+\overline{y})}D_{z,x}(y)$. It is clear that $R_{x,y}$ is of degree $\overline{x} + \overline{y}$ and $[D_{z,t}, R_{x,y}] = R_{[z,t,x],y} + (-1)^{(\overline{z}+\overline{t})\overline{x}}R_{x,[z,t,y]}$.

Theorem 10. *For $x, y \in T$, we have*

$$\beta(x, y) = \mathrm{str}\left(L_xL_y\right) + \mathrm{str}\left(R_{x,y} + (-1)^{\overline{x}\,\overline{y}}\,R_{y,x}\right). \tag{8}$$

Proof. Let $\{a_i\}$, $\{b_i\}$, $\{u_i\}$, $\{v_i\}$ be bases of T_0, T_1, $D_0(T,T)$, $D_1(T,T)$, respectively. For these bases, we express the operations of T and $D(T,T)$ as follows:

$$\begin{aligned}
a_i a_j &= \sum_l S_{ij}^l a_l; \quad a_i a_j \in T_0,\\
a_i b_j &= \sum_l T_{ij}^l b_l; \quad a_i b_j \in T_1,\\
b_i b_j &= \sum_l R_{ij}^l a_l; \quad b_i b_j \in T_0,\\
D_{a_i,a_j} &= \sum_\alpha D_{ij}^\alpha u_\alpha; \quad D_{a_i,a_j} \in D_0(T,T),\\
D_{a_i,b_j} &= \sum_\alpha C_{ij}^\alpha v_\alpha; \quad D_{a_i,b_j} \in D_1(T,T),\\
D_{b_i,b_j} &= \sum_\alpha X_{ij}^\alpha u_\alpha; \quad D_{b_i,b_j} \in D_0(T,T),\\
[u_\alpha, a_i] &= u_\alpha(a_i) = \sum_j K_{\alpha i}^j a_j;\\
[v_\alpha, a_i] &= v_\alpha(a_i) = \sum_j L_{\alpha i}^j b_j;\\
[u_\alpha, b_i] &= u_\alpha(b_i) = \sum_j H_{\alpha i}^j b_j;\\
[v_\alpha, b_i] &= v_\alpha(b_i) = \sum_j Q_{\alpha i}^j a_j.
\end{aligned} \tag{9}$$

To prove the theorem, it suffices to show that $\beta(a_i,a_j) = \alpha(a_i,a_j)$, $\beta(a_i,b_j) = \alpha(a_i,b_j)$ and $\beta(b_i,b_j) = \alpha(b_i,b_j)$. Since $(L_{a_i}L_{b_j})(T_0) \subseteq T_1$ and $(L_{a_i}L_{b_j})(T_1) \subseteq T_0$, we have $\mathrm{str}(L_{a_i}L_{b_j}) = 0$. Also, $R_{a_i,b_j}(T_1) \subseteq T_0$ and $R_{a_i,b_j}(T_0) \subseteq T_1$ give $\mathrm{str}(R_{a_i,b_j} + R_{b_j,a_i}) = 0$ and then $\beta(a_i,b_j) = 0 = \alpha(a_i,b_j)$ because of the consistency property of α ($a_i \in T_0 \oplus D_0(T,T)$, $b_j \in T_1 \oplus D_1(T,T)$). Hence, it remains to show that $\beta(a_i,a_j) = \alpha(a_i,a_j)$ and $\beta(b_i,b_j) = \alpha(b_i,b_j)$. The operations in T and the identities (7) imply the following:

$$\begin{aligned}
[a_i,[a_j,a_k]] &= [a_i, a_j a_k + D_{a_j,a_k}]\\
&= \left[a_i, \sum_m S_{jk}^m a_m + \sum_\alpha D_{jk}^\alpha u_\alpha\right]\\
&= \sum_m S_{jk}^m \left(a_i a_m + D_{a_i,a_m}\right) - \sum_\alpha D_{jk}^\alpha \sum_l K_{\alpha i}^l a_l\\
&= \sum_m S_{jk}^m \left(\sum_l S_{im}^l a_l + \sum_\alpha D_{im}^\alpha u_\alpha\right)\\
&\quad - \sum_{\alpha,l} D_{jk}^\alpha K_{\alpha i}^l a_l\\
&= \sum_{m,l} S_{jk}^m S_{im}^l a_l + \sum_{m,\alpha} S_{jk}^m D_{im}^\alpha u_\alpha\\
&\quad - \sum_{\alpha,l} D_{jk}^\alpha K_{\alpha i}^l a_l.
\end{aligned} \tag{10}$$

In a similar way, we get

$$\begin{aligned}
[a_i,[a_j,b_k]] &= \sum_{m,l} T_{jk}^m T_{im}^l b_l + \sum_{m,\alpha} T_{jk}^m C_{im}^\alpha v_\alpha\\
&\quad - \sum_{\alpha,l} C_{jk}^\alpha L_{\alpha i}^l b_l,\\
[a_i,[a_j,u_\alpha]] &= -[a_i, u_\alpha(a_j)] = -\left[a_i, \sum_m K_{\alpha j}^m a_m\right]\\
&= -\sum_{m,l} K_{\alpha j}^m S_{im}^l a_l - \sum_{m,\beta} K_{\alpha j}^m D_{im}^\beta u_\beta,\\
[a_i,[a_j,v_\alpha]] &= -[a_i, v_\alpha(a_j)] = -\left[a_i, \sum_m L_{\alpha j}^m b_m\right]\\
&= -\sum_{m,l} L_{\alpha j}^m T_{im}^l b_l - \sum_{m,\beta} L_{\alpha j}^m C_{im}^\beta v_\beta.
\end{aligned} \tag{11}$$

Also,

$$\begin{aligned}
L_{a_i}L_{a_j}(a_k) &= a_i(a_j a_k) = a_i\left(\sum_p S_{jk}^p a_p\right)\\
&= \sum_p S_{jk}^p (a_i a_p) = \sum_{p,l} S_{jk}^p S_{ip}^l a_l,\\
L_{a_i}L_{a_j}(b_k) &= a_i(a_j b_k) = a_i\left(\sum_p T_{jk}^p b_p\right)\\
&= \sum_p T_{jk}^p (a_i b_p) = \sum_{p,l} T_{jk}^p T_{ip}^l b_l,\\
R_{a_i,a_j}(a_k) &= [a_k, a_i, a_j] = D_{a_k,a_i}(a_j) = \sum_\alpha D_{ki}^\alpha u_\alpha(a_j)\\
&= \sum_\alpha D_{ki}^\alpha \sum_m K_{\alpha j}^m a_m = \sum_{\alpha,m} D_{ki}^\alpha K_{\alpha j}^m a_m;\\
R_{a_i,a_j}(b_k) &= [b_k, a_i, a_j] = D_{b_k,a_i}(a_j)\\
&= -\sum_\alpha C_{ik}^\alpha v_\alpha(a_j) = -\sum_\alpha C_{ik}^\alpha \sum_m L_{\alpha j}^m b_m\\
&= -\sum_{\alpha,m} C_{ik}^\alpha L_{\alpha j}^m b_m.
\end{aligned} \tag{12}$$

By interchanging i and j, we have

$$\begin{aligned}
R_{a_j,a_i}(a_k) &= \sum_{\alpha,m} D_{kj}^\alpha K_{\alpha i}^m a_m;\\
R_{a_j,a_i}(b_k) &= -\sum_{\alpha,m} C_{jk}^\alpha L_{\alpha i}^m b_m.
\end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}\beta(a_i,a_j) &= \alpha(a_i,a_j)\\ &= \sum_{m,k} S_{jk}^m S_{im}^k - \sum_{\alpha,k} D_{jk}^\alpha K_{\alpha i}^k a_k - \sum_{m,k} T_{jk}^m T_{im}^k + \sum_{\alpha,k} C_{jk}^\alpha L_{\alpha i}^k\\ &\quad - \sum_{m,\alpha} K_{\alpha j}^m D_{im}^\alpha + \sum_{m,\alpha} L_{\alpha j}^m C_{im}^\alpha,\\ \operatorname{str}(L_{a_i}L_{a_j}) &+ \operatorname{str}(R_{a_j,a_i} + R_{a_j,a_i})\\ &= \sum_{p,k} S_{jk}^p S_{ip}^k - \sum_{p,k} T_{jk}^p T_{ip}^k + \sum_{\alpha,k} D_{ki}^\alpha K_{\alpha j}^k + \sum_{\alpha,k} D_{kj}^\alpha K_{\alpha i}^k\\ &\quad + \sum_{\alpha,k} C_{ik}^\alpha L_{\alpha j}^k + \sum_{\alpha,k} C_{jk}^\alpha L_{\alpha i}^k = \beta(a_i,a_j).\end{aligned} \tag{14}$$

It remains to show that $\beta(b_i,b_j) = \operatorname{str}(L_{b_i}L_{b_j}) + \operatorname{str}(R_{b_j,b_i} + R_{b_j,b_i})$,

$$\begin{aligned}[b_i,[b_j,b_k]] &= [b_i, b_jb_k + D_{b_j,b_k}]\\ &= \left[b_i, \sum_m R_{jk}^m a_m + \sum_\alpha X_{jk}^\alpha u_\alpha\right]\\ &= -\sum_{m,l} R_{jk}^m T_{mi}^l b_l - \sum_{m,\alpha} R_{jkmi}^{m\alpha} v_\alpha\\ &\quad - \sum_{\alpha,l} X_{jk}^\alpha H_{\alpha i}^l b_l.\end{aligned} \tag{15}$$

Likewise, we have

$$\begin{aligned}[b_i,[b_j,a_k]] &= -\sum_{m,l} T_{kj}^m R_{im}^l a_l - \sum_{m,\alpha} T_{kj}^m X_{im}^\alpha u_\alpha\\ &\quad - \sum_{\alpha,l} C_{kj}^\alpha Q_{\alpha i}^l a_l,\\ [b_i,[b_j,u_\alpha]] &= -[b_i, u_\alpha(b_j)] = -\left[b_i, \sum_l H_{\alpha j}^l b_l\right]\\ &= -\sum_l H_{\alpha j}^l R_{il}^m a_m - \sum_{l,\beta} H_{\alpha j}^l X_{il}^\beta u_\beta,\\ [b_i,[b_j,v_\alpha]] &= [b_i, v_\alpha(b_j)] = \left[b_i, \sum_l Q_{\alpha j}^l a_l\right]\\ &= -\sum_{l,m} Q_{\alpha j}^l T_{li}^m b_m - \sum_{l,\beta} Q_{\alpha j}^l C_{li}^\beta v_\beta.\end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned}\beta(a_i,a_j) &= \alpha(a_i,a_j)\\ &= -\sum_{m,k} T_{kj}^m R_{im}^k - \sum_{\alpha,k} C_{kj}^\alpha Q_{\alpha i}^k + \sum_{m,k} R_{jk}^m T_{mi}^k\\ &\quad + \sum_{\alpha,k} X_{jk}^\alpha H_{\alpha i}^k - \sum_{l,\alpha} H_{\alpha j}^l X_{il}^\alpha + \sum_{l,\alpha} Q_{\alpha j}^l C_{li}^\alpha.\end{aligned} \tag{17}$$

Now,

$$\begin{aligned}L_{b_i}L_{b_j}(a_k) &= b_i(b_ja_k) = -b_i\left(\sum_m T_{jk}^m b_m\right)\\ &= -\sum_m T_{jk}^m (b_ib_m) = -\sum_{m,l} T_{jk}^m R_{im}^l a_l;\\ L_{b_i}L_{b_j}(b_k) &= b_i(b_jb_k) = b_i\left(\sum_m R_{jk}^m a_m\right)\\ &= -\sum_m R_{jk}^m (a_mb_i) = -\sum_{m,l} R_{jk}^m T_{mi}^l b_l;\\ R_{b_i,b_j}(a_k) &= [a_k,b_i,b_j] = D_{a_k,b_i}(b_j) = \sum_\alpha C_{ki}^\alpha v_\alpha(b_j)\\ &= \sum_\alpha C_{ki}^\alpha \sum_l Q_{\alpha j}^l a_l = \sum_{\alpha,l} C_{ki}^\alpha Q_{\alpha j}^l a_l;\\ R_{b_i,b_j}(b_k) &= [b_k,b_i,b_j] = D_{b_k,b_i}(b_j) = \sum_\alpha X_{ki}^\alpha u_\alpha(b_j)\\ &= \sum_\alpha X_{ki}^\alpha \sum_l H_{\alpha j}^l b_l = \sum_{\alpha,l} X_{ki}^\alpha H_{\alpha j}^l b_l.\end{aligned} \tag{18}$$

By interchanging i and j, we have

$$\begin{aligned}R_{b_j,b_i}(a_k) &= \sum_{\alpha,l} C_{kj}^\alpha Q_{\alpha i}^l a_l;\\ R_{b_j,b_i}(b_k) &= \sum_{\alpha,l} X_{kj}^\alpha H_{\alpha i}^l b_l,\\ \operatorname{str}(L_{b_i}L_{b_j}) &+ \operatorname{str}(R_{b_j,b_i} + R_{b_j,b_i})\\ &= -\sum_{m,k} T_{jk}^m R_{im}^k + \sum_{m,k} R_{jk}^m T_{mi}^k + \sum_{\alpha,k} C_{ki}^\alpha Q_{\alpha j}^k - \sum_{\alpha,k} X_{ki}^\alpha H_{\alpha j}^k\\ &\quad - \sum_{\alpha,k} C_{kj}^\alpha Q_{\alpha i}^k + \sum_{\alpha,k} X_{kj}^\alpha H_{\alpha i}^k b_l = \beta(b_i,b_j).\end{aligned} \tag{19}$$

Hence the theorem is proved. □

Definition 11. The bilinear form β defined on the Lie-Yamaguti superalgebra $T = T_0 \oplus T_1$ by

$$\beta(x,y) = \operatorname{str}(L_xL_y) + \operatorname{str}(R_{x,y} + (-1)^{\bar{x}\bar{y}} R_{y,x}) \tag{20}$$

for $x, y \in T$ is called the Killing form of T.

Remark 12. Recall that if T is a Lie superalgebra, then the Killing form β on T is defined as $\beta(x,y) = \operatorname{str}(L_xL_y)$, $x, y \in T$. Likewise, if T is a Lie supertriple system (resp., a Lie-Yamaguti algebra), the Killing form on T is defined as $\beta(x,y) = \operatorname{str}(R_{x,y} + (-1)^{\bar{x}\bar{y}} R_{y,x})$ (resp., $\beta(x,y) = \operatorname{tr}(L_xL_y) + \operatorname{tr}(R_{x,y} + R_{y,x})$) with L_u and $R_{u,v}$ defined according to the considered structure on T. So if a Lie-Yamaguti superalgebra T is reduced to a Lie superalgebra (resp., a Lie supertriple system, a Lie-Yamaguti algebra), then β as defined in Definition 11 is the Killing form of the Lie superalgebra (resp., the Lie supertriple system, the Lie-Yamaguti algebra) T.

Proposition 13. *Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra with a Killing form denoted by β. Then,*

(1) $\beta(T_0, T_1) = 0$ *(consistence),*

(2) $\beta(x, y) = (-1)^{\overline{x}\,\overline{y}}\beta(y, x)$ *(supersymmetry),*

(3) $\beta(A(x), A(y)) = \beta(x, y)$, $A \in \mathrm{Aut}(T)$.

Proof. As $L_{T_0}L_{T_1}(T_1) \subseteq T_0$, $L_{T_0}L_{T_1}(T_0) \subseteq T_1$, $R_{T_0,T_1}(T_0) \subseteq T_1$ and $R_{T_0,T_1}(T_1) \subseteq T_0$ we can state that $\beta(T_0, T_1) = 0$.

(2) comes from the definition of β.

Now, for any A in $\mathrm{Aut}(T)$, x in T, $\overline{A(x)} = \overline{x}$, and

$$\begin{aligned}\beta(A(x), A(y)) &= \mathrm{str}\left(L_{A(x)}L_{A(y)}\right) \\ &\quad + \mathrm{str}\left(R_{A(x),A(y)} + (-1)^{\overline{x}\,\overline{y}} R_{A(y),A(x)}\right).\end{aligned} \tag{21}$$

As $A(xy) = A(x)A(y)$ then $AL_x(y) = L_{A(x)}A(y)$; that is, $AL_x = L_{A(x)}A$ and $AL_xA^{-1} = L_{A(x)}$.

Hence, $\mathrm{str}(L_{A(x)}L_{A(y)}) = \mathrm{str}(AL_xA^{-1}AL_yA^{-1}) = \mathrm{str}(AL_xL_yA^{-1}) = \mathrm{str}(L_xL_y)$. Also, $A[x, y, z] = [A(x), A(y), A(z)]$ gives $AR_{y,z}(x) = R_{A(y),A(z)}(A(x))$; that is, $AR_{y,z}A^{-1} = R_{A(y),A(z)}$. Then,

$$\begin{aligned}\beta(A(x), A(y)) &= \mathrm{str}\left(L_{A(x)}L_{A(y)}\right) \\ &\quad + \mathrm{str}\left(R_{A(x),A(y)} + (-1)^{\overline{x}\,\overline{y}} R_{A(y),A(x)}\right) \\ &= \mathrm{str}\left(L_xL_y\right) \\ &\quad + \mathrm{str}\left(AR_{x,y}A^{-1} + (-1)^{\overline{x}\,\overline{y}} AR_{y,x}A^{-1}\right) \\ &= \mathrm{str}\left(L_xL_y\right) + \mathrm{str}\left(A\left(R_{x,y} + (-1)^{\overline{x}\,\overline{y}} R_{y,x}\right)A^{-1}\right) \\ &= \mathrm{str}\left(L_xL_y\right) + \mathrm{str}\left(R_{x,y} + (-1)^{\overline{x}\,\overline{y}} R_{y,x}\right) \\ &= \beta(x, y).\end{aligned} \tag{22}$$

□

Now, let γ be a trilinear form in T given by $\gamma(x, y, z) = \mathrm{str}(D_{x,y}L_z)$ for any $x, y, z \in T$. We can easily see that, for any $x, y, z \in T$, $\gamma(x, y, z) = -(-1)^{\overline{x}\,\overline{y}}\gamma(y, x, z)$ and that γ vanishes identically if T is reduced to Lie superalgebra or Lie supertriple system.

Theorem 14. *Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra with a Killing form denoted by β. Then, β satisfies the identities*

$$\begin{aligned}&\beta(xy, z) + (-1)^{\overline{x}\,\overline{y}}\beta(y, xz) \\ &\quad = (-1)^{\overline{x}\,\overline{y}}\gamma(y, x, z) + (-1)^{(\overline{x}+\overline{y})\overline{z}}\gamma(z, x, y);\end{aligned} \tag{23}$$

$$\begin{aligned}&\beta(x, [y, z, w]) + (-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x, w, y], z) \\ &\quad = (-1)^{\overline{w}(\overline{y}+\overline{z})}\gamma(x, w, yz) - (-1)^{\overline{x}(\overline{y}+\overline{z})}\gamma(y, z, xw)\end{aligned} \tag{24}$$

for all $x, y, z \in T$.

Proof. The Killing form α of $L = (T_0 \oplus D_0(T, T)) \oplus (T_1 \oplus D_1(T, T))$ satisfies $\alpha(y, [x, z]) + (-1)^{\overline{x}\,\overline{y}}\alpha([x, y], z) = 0$; that is, $\alpha(y, [x, z]) = \alpha([y, x], z)$. But, using (7), we have

$$\begin{aligned}\alpha([x, y], z) &= \alpha\left(xy + D_{x,y}, z\right) \\ &= \alpha(xy, z) + \alpha\left(D_{x,y}, z\right) \\ &= \beta(xy, z) + \mathrm{str}\left(D_{x,y}L_z\right) \\ &= \beta(xy, z) + \gamma(x, y, z), \\ \alpha(y, [x, z]) &= \alpha(y, xz + D_{x,z}) \\ &= \alpha(y, xz) + \mathrm{str}\left(L_yD_{x,z}\right) \\ &= \beta(y, xz) + (-1)^{\overline{y}(\overline{x}+\overline{z})}\mathrm{str}\left(D_{x,z}L_y\right) \\ &= \beta(y, xz) + (-1)^{\overline{y}(\overline{x}+\overline{z})}\gamma(x, z, y).\end{aligned} \tag{25}$$

Then the identity $\alpha(y, [x, z]) + (-1)^{\overline{x}\,\overline{y}}\alpha([x, y], z) = 0$ gives $\beta(xy, z) + \gamma(x, y, z) + (-1)^{\overline{x}\,\overline{y}}\beta(y, xz) + (-1)^{\overline{y}\,\overline{z}}\gamma(x, z, y) = 0$ that is $\beta(xy, z) + (-1)^{\overline{x}\,\overline{y}}\beta(y, xz) = -\gamma(x, y, z) - (-1)^{\overline{y}\,\overline{z}}\gamma(x, z, y) = (-1)^{\overline{x}\,\overline{y}}\gamma(y, x, z) + (-1)^{(\overline{x}+\overline{y})\overline{z}}\gamma(z, x, y)$ and (23) is obtained.

From $\alpha([y, x], z) = \alpha(y, [x, z])$ we deduce $\alpha(x, [w, [y, z]]) = \alpha([x, w], [y, z]) = \alpha([[x, w], y], z)$ that is $-(-1)^{\overline{w}(\overline{y}+\overline{z})}\alpha(x, [[y, z], w]) = \alpha([[x, w], y], z)$ and

$$\alpha(x, [[y, z], w]) + (-1)^{\overline{w}(\overline{y}+\overline{z})}\alpha([[x, w], y], z) = 0. \tag{26}$$

Then, using (7) again and developing (26), we have $\alpha(x, [yz + D_{y,z}, w]) + (-1)^{\overline{w}(\overline{y}+\overline{z})}\alpha([xw + D_{x,w}, y], z) = 0$ and we get

$$\begin{aligned}&\alpha\left(x, (yz)w + D_{yz,w} + [y, z, w]\right) \\ &\quad + (-1)^{\overline{w}(\overline{y}+\overline{z})}\alpha\left((xw)y + D_{xw,y} + [x, w, y], z\right) \\ &= 0.\end{aligned} \tag{27}$$

This gives

$$\begin{aligned}&\beta(x, (yz)w) + \alpha\left(x, D_{yz,w}\right) + \beta(x, [y, z, w]) \\ &\quad + (-1)^{\overline{w}(\overline{y}+\overline{z})}\beta((xw)y, z) \\ &\quad + (-1)^{\overline{w}(\overline{y}+\overline{z})}\alpha\left(D_{xw,y}, z\right) \\ &\quad + (-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x, w, y], z) = 0, \\ &\beta(x, (yz)w) + \mathrm{str}\left(L_xL_{D_{yz,w}}\right) + \beta(x, [y, z, w]) \\ &\quad + (-1)^{\overline{w}(\overline{y}+\overline{z})}\beta((xw)y, z) \\ &\quad + (-1)^{\overline{w}(\overline{y}+\overline{z})}\mathrm{str}\left(L_{D_{xw,y}}L_z\right) \\ &\quad + (-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x, w, y], z) = 0.\end{aligned} \tag{28}$$

Thus,

$$\begin{aligned}&\beta(x,(yz)w)+(-1)^{\overline{x}(\overline{y}+\overline{z}+\overline{w})}\operatorname{str}\left(L_{D_{yz,w}}L_x\right)\\&\quad+\beta(x,[y,z,w])+(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta((xw)y,z)\\&\quad+(-1)^{\overline{w}(\overline{y}+\overline{z})}\operatorname{str}\left(L_{D_{xw,y}}L_z\right)\\&\quad+(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x,w,y],z)=0;\end{aligned}\tag{29}$$

that is,

$$\begin{aligned}&\beta(x,(yz)w)+(-1)^{\overline{x}(\overline{y}+\overline{z}+\overline{w})}\gamma(yz,w,x)\\&\quad+\beta(x,[y,z,w])+(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta((xw)y,z)\\&\quad+(-1)^{\overline{w}(\overline{y}+\overline{z})}\gamma(xw,y,z)\\&\quad+(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x,w,y],z)=0.\end{aligned}\tag{30}$$

This implies

$$\begin{aligned}&\beta(x,[y,z,w])+(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x,w,y],z)\\&\quad=-\beta(x,(yz)w)-(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta((xw)y,z)\\&\qquad-(-1)^{\overline{x}(\overline{y}+\overline{z}+\overline{w})}\gamma(yz,w,x)\\&\qquad-(-1)^{\overline{w}(\overline{y}+\overline{z})}\gamma(xw,y,z).\end{aligned}\tag{31}$$

But (23) gives

$$\begin{aligned}&(-1)^{\overline{w}\,\overline{x}}\beta(x,w(yz))+\beta(wx,yz)\\&\quad=(-1)^{\overline{w}\,\overline{x}}\gamma(x,w,yz)\\&\qquad+(-1)^{(\overline{w}+\overline{x})(\overline{y}+\overline{z})}\gamma(yz,w,x),\\&(-1)^{\overline{w}\,\overline{x}}\beta((xw)y,z)+\beta(wx,yz)\\&\quad=\gamma(wx,y,z)+(-1)^{(\overline{w}+\overline{x})(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}\gamma(z,y,wx).\end{aligned}\tag{32}$$

Then,

$$\begin{aligned}&-\beta(x,w(yz))-(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta((xw)y,z)\\&\quad=(-1)^{\overline{w}(\overline{y}+\overline{z})}\gamma(x,w,yz)\\&\qquad+(-1)^{\overline{x}(\overline{w}+\overline{y}+\overline{z})}\gamma(yz,w,x)\\&\qquad-(-1)^{\overline{w}(\overline{x}+\overline{y}+\overline{z})}\gamma(wx,y,z)\\&\qquad-(-1)^{\overline{x}(\overline{w}+\overline{y}+\overline{z})+\overline{y}\,\overline{z}}\gamma(z,y,wx).\end{aligned}\tag{33}$$

Hence, $\beta(x,[y,z,w])+(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x,w,y],z)=(-1)^{\overline{w}(\overline{y}+\overline{z})}\gamma(x,w,yz)-(-1)^{\overline{x}(\overline{y}+\overline{z})}\gamma(z,y,wx)$ and (24) is proved. □

Corollary 15. *Let $T=T_0\oplus T_1$ be a Lie-Yamaguti superalgebra with a Killing form denoted by β. Then, β satisfies the following for $x,y,z\in T$:*

$$([y,z,x],w)+(-1)^{\overline{x}(\overline{y}+\overline{z})}\beta(x,[y,z,w])=0;\tag{34}$$

$$\begin{aligned}&\beta\left(R_{w,y}(x),z\right)-(-1)^{(\overline{w}+\overline{y})\overline{x}+\overline{w}\,\overline{y}}\beta\left(x,R_{y,w}(z)\right)\\&\quad=(-1)^{\overline{x}(\overline{w}+\overline{y})}\gamma(x,w,yz)\\&\qquad+(-1)^{\overline{w}(\overline{y}+\overline{z})+\overline{x}\,\overline{z}}\gamma(y,z,xw).\end{aligned}\tag{35}$$

Proof. Using (24) we have

$$\begin{aligned}&\beta([y,z,x],w)\\&\quad=(-1)^{(\overline{w}+\overline{x})(\overline{y}+\overline{z})}\beta([x,w,y],z)\\&\qquad-(-1)^{(\overline{w}+\overline{x})(\overline{y}+\overline{z})}\gamma(x,w,yz)\\&\qquad-(-1)^{\overline{w}\,\overline{x}}\gamma(y,z,wx);\end{aligned}\tag{36}$$

$$\begin{aligned}&(-1)^{\overline{x}(\overline{y}+\overline{z})}\beta(x,[y,z,w])\\&\quad=-(-1)^{(\overline{w}+\overline{x})(\overline{y}+\overline{z})}\beta([x,w,y],z)\\&\qquad+(-1)^{(\overline{w}+\overline{x})(\overline{y}+\overline{z})}\gamma(x,w,yz)\\&\qquad+(-1)^{\overline{w}\,\overline{x}}\gamma(y,z,wx).\end{aligned}\tag{37}$$

By adding memberwise (36) and (37) we obtain the identity (34).

Also, the identity (37) is equivalent to $-(-1)^{\overline{y}\,\overline{z}}\beta(x,[z,y,w])+(-1)^{\overline{w}(\overline{y}+\overline{z})}\beta([x,w,y],z)=(-1)^{\overline{w}(\overline{y}+\overline{z})}\gamma(x,w,yz)-(-1)^{\overline{x}(\overline{y}+\overline{z})}\gamma(y,z,xw)$. Then, we obtain $(-1)^{\overline{w}\,\overline{y}+\overline{x}(\overline{w}+\overline{y})}\beta(R_{w,y}(x),z,z)-\beta(x,R_{y,w}(z))=(-1)^{\overline{w}\,\overline{y}}\gamma(w,x,yz)-(-1)^{\overline{x}(\overline{y}+\overline{z})+\overline{w}\,\overline{z}}\gamma(y,z,xw)$ that is $\beta(R_{w,y}(x),z)-(-1)^{(\overline{w}+\overline{y})\overline{x}+\overline{w}\,\overline{y}}\beta(x,R_{y,w}(z))=(-1)^{\overline{x}(\overline{w}+\overline{y})}\gamma(x,w,yz)+(-1)^{\overline{w}(\overline{y}+\overline{z})+\overline{x}\,\overline{z}}\gamma(y,z,xw)$ and the remaining assertion is proved. □

4. Invariant Forms of Lie-Yamaguti Superalgebras

In this section we introduced the concept of invariant forms of Lie-Yamaguti superalgebras as generalizations of those of Lie superalgebras and Lie supertriple systems.

Definition 16. An invariant form b of a Lie-Yamaguti superalgebra $T=T_0\oplus T_1$ is a supersymmetric bilinear form on T satisfying the identities

$$b(xy,z)+(-1)^{\overline{x}\,\overline{y}}b(y,xz)=0;\tag{38}$$

$$b([x,w,y],z)+(-1)^{\overline{w}(\overline{y}+\overline{z})}b(x,[y,z,w])=0\tag{39}$$

for all x,y,z,w in T.

Remark 17. (1) If $\gamma=0$, the Killing form of T is an invariant form of T.

(2) If T is reduced to a Lie supertriple system (resp., a Lie superalgebra, a Lie-Yamaguti algebra), then b is reduced to an invariant form of a Lie supertriple system [19] (resp., a Lie superalgebra [12], a Lie-Yamaguti algebra [10]).

Definition 18. Let b be an invariant form of a Lie-Yamaguti superalgebra T and S a subset of T. The orthogonal $S^{\perp}$ of S with respect to b is defined by $S^{\perp} = \{x \in T,\ b(x, y) = 0,\ \forall y \in S\}$. The invariant form b is nondegenerate if $T^{\perp} = \{0\}$.

Lemma 19. *Let b be an invariant form of a Lie-Yamaguti superalgebra T. Then, for any x, y, z, w in T, we have*

$$b([x, y, w], z) + (-1)^{(\overline{x}+\overline{y})\overline{z}} b(w, [x, y, z]) = 0. \tag{40}$$

Proof. By interchanging y and w in (39) we have

$$b([x, y, w], z) + (-1)^{\overline{y}(\overline{w}+\overline{z})} b(x, [w, z, y]) = 0 \tag{41}$$

that is $(-1)^{(\overline{x}+\overline{y})\overline{z}} b(z, [x, y, w]) + (-1)^{\overline{y}(\overline{w}+\overline{z})+\overline{w}\,\overline{z}} b(x, [w, z, y]) = 0$ by supersymmetry. Also by switching z and w in (41), we obtain $b([x, y, z], \widetilde{w}) + (-1)^{\overline{y}(\overline{w}+\overline{z})} b(x, [z, w, y]) = 0$ that is

$$b([x, y, z], w) - (-1)^{\overline{y}(\overline{w}+\overline{z})+\overline{w}\,\overline{z}} b(x, [w, z, y]) = 0. \tag{42}$$

Thus adding (41) and (42) we get (40) whence the lemma. □

Lemma 20. *Let b be an invariant form of a Lie-Yamaguti superalgebra T. Then,*

(1) $(T + [T, T, T])^{\perp} = Z(T)$ *if b is nondegenerate;*

(2) *if H is an ideal of T then $H^{\perp}$ is an ideal of T. In particular, $T^{\perp}$ is an ideal of T.*

Proof. Consider x in $(T + [T, T, T])^{\perp}$. Then, for any $u, v, w \in T$, we have $b(x, uv) = 0$ and $b(x, [u, v, w]) = 0$. This implies, by (38) and (39), that $(-1)^{\overline{u}\,\overline{x}} b(ux, v) = 0$ and $(-1)^{\overline{w}(\overline{u}+\overline{v})} b([x, w, u], v) = 0$ that is $b(ux, v) = 0$ and $b([x, w, u], v) = 0$. As b is nondegenerate, we get $ux = 0$ and $[x, w, u] = 0$ for any $u, v, w \in T$. This gives $x \in Z(T)$.

Conversely, if $x \in Z(T)$, we have, for any $u, v, u', v', w' \in T$, $b(x, uv + [u', v', w']) = b(x, uv) + b(x, [u', v', w']) = 0$ and $x \in (T + [T, T, T])^{\perp}$ whence $(T + [T, T, T])^{\perp} = Z(T)$.

Now, suppose that H is an ideal of T that is $TH \subseteq H$ and $[T, H, T] \subseteq H$; then for any $x, y \in T$, $u \in H^{\perp}$, and $h \in H$, we have $b(xu, h) = -(-1)^{\overline{x}\,\overline{u}} b(u, xh) = 0$ and $b([x, u, y], h) = -(-1)^{\overline{x}\,\overline{u}} b([u, x, y], h) = (-1)^{\overline{x}(\overline{y}+\overline{h})+\overline{x}\,\overline{u}} b(u, [y, h, x]) = 0$. Then $TH^{\perp} \subseteq H^{\perp}$ and $[T, H^{\perp}, T] \subseteq H^{\perp}$ which proves (2). □

We are now ready to prove the following theorem.

Theorem 21. *Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra with $\gamma = 0$. Then the Killing form β is nondegenerate if and only if the standard enveloping Lie superalgebra $L(T) = T \oplus D(T, T)$ is a semisimple Lie superalgebra.*

Proof. Let α be the Killing form of the Lie superalgebra $L(T)$. If $\gamma = 0$, we have, for any $x, y, z \in T$, $\gamma(x, y, z) = \mathrm{str}(D_{x,y} L_z) = 0$ and

$$\alpha(D_{x,y}, z) = 0. \tag{43}$$

Then, using the invariance of α and (43), we have, for any $x, y, z, w \in T$: $\alpha([x, y], D_{z,w}) = \alpha(x, [y, D_{z,w}])$; that is, by (7),

$\alpha(xy + D_{x,y}, D_{z,w}) = -(-1)^{\overline{y}(\overline{z}+\overline{w})} \alpha(x, [z, w, y])$ and $\alpha(D_{x,y}, D_{z,w}) = -(-1)^{\overline{y}(\overline{z}+\overline{w})} \alpha(x, [z, w, y])$. This gives

$$\alpha(D_{x,y}, D_{z,w}) = -(-1)^{\overline{y}(\overline{z}+\overline{w})} \beta(x, [z, w, y]). \tag{44}$$

Thus, if β is nondegenerate, the restriction of α on $D(T, T) \times D(T, T)$ is nondegenerate and α is nondegenerate.

Now, suppose that β is degenerate. Then by the lemma above, $T^{\perp}$ is an ideal of T so $T^{\perp} \oplus D(T, T^{\perp})$ is a nonzero ideal of T.

Using the identities (43) and (44) we get

$$\begin{aligned} &\alpha(T^{\perp} \oplus D(T, T^{\perp}), T \oplus D(T, T)) \\ &\quad = \alpha(T^{\perp}, T) + \alpha(T^{\perp}, D(T, T)) + \alpha(D(T, T^{\perp}), T) \\ &\qquad + \alpha(D(T, T^{\perp}), D(T, T)) \\ &\quad = \alpha(T^{\perp}, T) + \beta(T, [T, T^{\perp}, T]) = 0. \end{aligned} \tag{45}$$

It comes that α is degenerate and $T \oplus D(T, T)$ is not semisimple which proves the theorem. □

The results of this paper could be used for a study of the structure of a pair consisting of a semisimple Lie superalgebra and its semisimple graded subalgebra.

Competing Interests

The authors declare that they have no competing interests.

References

[1] K. Yamaguti, "On the Lie triple system and its generalization," *Journal of Science of the Hiroshima University, Series A*, vol. 21, pp. 107–113, 1957/1958.

[2] K. Nomizu, "Invariant affine connections on homogeneous spaces," *American Journal of Mathematics*, vol. 76, pp. 33–65, 1954.

[3] M. Kikkawa, "Geometry of homogeneous Lie loops," *Hiroshima Mathematical Journal*, vol. 5, no. 2, pp. 141–179, 1975.

[4] M. K. Kinyon and A. Weinstein, "Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces," *American Journal of Mathematics*, vol. 123, no. 3, pp. 525–550, 2001.

[5] P. Benito, A. Elduque, and F. Martín-Herce, "Irreducible Lie-Yamaguti algebras," *Journal of Pure and Applied Algebra*, vol. 213, no. 5, pp. 795–808, 2009.

[6] P. Benito, A. Elduque, and F. Martín-Herce, "Irreducible Lie-Yamaguti algebras of generic type," *Journal of Pure and Applied Algebra*, vol. 215, no. 2, pp. 108–130, 2011.

[7] J. Lin, L. Y. Chen, and Y. Ma, "On the deformation of Lie-Yamaguti algebras," *Acta Mathematica Sinica (English Series)*, vol. 31, no. 6, pp. 938–946, 2015.

[8] T. Zhang and J. Li, "Deformations and extensions of Lie-Yamaguti algebras," *Linear and Multilinear Algebra*, vol. 63, no. 11, pp. 2212–2231, 2015.

[9] M. Kikkawa, "On Killing-Ricci forms of Lie triple algebras," *Pacific Journal of Mathematics*, vol. 96, no. 1, pp. 153–161, 1981.

[10] M. Kikkawa, "Remarks on invariant forms of Lie triple algebras," *Memoirs of the Faculty of Science and Engineering, Shimane University*, vol. 16, pp. 23–27, 1982.

[11] V. G. Kac, "Lie superalgebras," *Advances in Mathematics*, vol. 26, no. 1, pp. 8–96, 1977.

[12] M. Scheunert, *The Theory of Lie Superalgebras*, Springer, Berlin, Germany, 1979.

[13] H. Tilgner, "A graded generalization of Lie triples," *Journal of Algebra*, vol. 47, no. 1, pp. 190–196, 1977.

[14] S. Okubo, "Parastatistics as Lie-supertriple systems," *Journal of Mathematical Physics*, vol. 35, no. 6, pp. 2785–2803, 1994.

[15] M. F. Ouédraogo, *Sur les Superalgèbres triples de Lie, Thèse de Doctorat 3è cycle Mathématiques [Ph.D. thesis]*, Université de Ouagadougou, Ouagadougou, Burkina Faso, 1999.

[16] P. L. Zoungrana, "A note on Lie-Yamaguti superalgebras," *Far East Journal of Mathematical Sciences*, vol. 100, no. 1, pp. 1–18, 2016.

[17] T. S. Ravisankar, "Some remarks on Lie triple systems," *Kumamoto Journal of Science (Mathematics)*, vol. 11, pp. 1–8, 1974.

[18] S. Okubo and N. Kamiya, "Quasi-classical Lie superalgebras and Lie supertriple systems," *Communications in Algebra*, vol. 30, no. 8, pp. 3825–3850, 2002.

[19] Z. Zhixue and J. Peipei, "The Killing forms and decomposition theorems of Lie supertriple systems," *Acta Mathematica Scientia. Series B*, vol. 29, no. 2, pp. 360–370, 2009.

[20] I. P. Shestakov, "Prime Mal'tsev superalgebras," *Matematicheskii Sbornik*, vol. 182, no. 9, pp. 1357–1366, 1991 (Russian).