Research Article

Compact Operators on the Bergman Spaces with Variable Exponents on the Unit Disc of $\mathbb{C}$

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Received 20 October 2017; Accepted 6 December 2017; Published 30 January 2018

Academic Editor: Nageswari Shanmugalingam

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We study the compactness of some classes of bounded operators on the Bergman space with variable exponent. We show that via extrapolation, some results on boundedness of the Toeplitz operators with general L^1 symbols and compactness of bounded operators on the Bergman spaces with constant exponents can readily be extended to the variable exponent setting. In particular, if S is a finite sum of finite products of Toeplitz operators with symbols from class BT , then S is compact if and only if the Berezin transform of S vanishes on the boundary of the unit disc.

1. Introduction and Statement of Results

Variable Lebesgue spaces are a generalization of the Lebesgue spaces that allow the exponents to be a measurable function and thus the exponent may vary. These spaces have many properties similar to the normal Lebesgue spaces, but they also differ in surprising and subtle ways. For this reason, the variable Lebesgue spaces have an intrinsic interest, but they are also very important in applications to partial differential equations and variational integrals with nonstandard growth conditions. See [1] for more details on the variable Lebesgue spaces.

Let Δ denote the unit disc in $\mathbb{C}$ and dA the normalized Lebesgue measure on Δ . For $1 \leq p < \infty$, the Bergman space $A^p = A^p(\Delta, dA)$ is the space of all analytic functions, f , on Δ such that

$$\|f\|_p^p = \int_{\Delta} |f(z)|^p dA(z) < \infty. \quad (1)$$

Let P be the Bergman projection from L^2 onto A^2 . Then P is an integral operator given by

$$P(f)(z) = \int_{\Delta} \frac{f(w)}{(1 - z\bar{w})^2} dA(w), \quad (2)$$

for each $z \in \Delta$ and $f \in L^2$. Here, the function $K(z, w) = 1/(1 - z\bar{w})^2$ is the reproducing kernel for A^2 . For

$f \in L^\infty$, the Toeplitz operator with symbol f is defined on A^p by

$$T_f g = P(fg), \quad g \in A^p. \quad (3)$$

Toeplitz operators are amongst the most widely studied classes of concrete operators and have attracted a lot of interest in recent years. The behaviour of these operators on the Hardy spaces, Bergman spaces, and Fock spaces has been studied widely and a lot of results are available in the literature. The characterization of compactness has been studied in [2–8] just to cite a few.

Given $\Omega \subset \mathbb{R}^n$, a measurable function $p : \Omega \rightarrow [1, \infty)$ will be called a variable exponent. If p is a variable exponent then we denote

$$\begin{aligned} p_+ &= p_\Omega^+ := \operatorname{ess\,sup}_{x \in \Omega} p(x), \\ p_- &= p_\Omega^- := \operatorname{ess\,inf}_{x \in \Omega} p(x). \end{aligned} \quad (4)$$

Let $\mathcal{P}(\Omega)$ denote the set of all variable exponents for which $p_+ < \infty$.

For a complex-valued measurable function $\phi : \Omega \rightarrow \mathbb{C}$, we define the modular $\rho_{p(\cdot)}$ by

$$\rho_{p(\cdot)}(\phi) := \int_{\Omega} |\phi(x)|^{p(x)} dx \quad (5)$$

and the norm

$$\|\phi\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left(\frac{\phi}{\lambda} \right) \leq 1 \right\}. \quad (6)$$

Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then the Lebesgue variable exponent space $L^{p(\cdot)}$ is the set of all complex-valued measurable functions $\phi : \Omega \rightarrow \mathbb{C}$ for which $\rho_{p(\cdot)}(\phi) < \infty$. If we equip $L^{p(\cdot)}$ with the norm given in (6), then $L^{p(\cdot)}$ becomes a Banach space. We note here that the condition $\rho_{p(\cdot)}(\phi) < \infty$ is not enough in general to define the variable exponent Lebesgue space (e.g., see chapter 2 of [1]).

It is known (e.g., see chapter 2 of [1]) that the dual of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$, where $1/p(\cdot) + 1/p'(\cdot) = 1$. A straightforward computation shows that

$$\begin{aligned} (p'(\cdot))_+ &= (p_-)' , \\ (p'(\cdot))_- &= (p_+)' . \end{aligned} \quad (7)$$

For simplicity, we will omit one set of parenthesis and write the left-hand side of each equality as $p'(\cdot)_+$ and $p'(\cdot)_-$. Throughout this work, we shall use $p'(\cdot)$ as the conjugate exponent of $p(\cdot)$ and if p is a constant in $(1, \infty)$ we shall use p' as the conjugate exponent of p . In other words, to study these spaces, some regularity conditions are imposed on the exponents. A function $p : \Omega \rightarrow \mathbb{C}$ is said to be log-Hölder continuous on Ω if there exists a positive constant $C_{\log}$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}}{\log(1/|x - y|)}, \quad (8)$$

for all $x, y \in \Omega$ with $|x - y| < 1/2$. It follows that

$$|p(x) - p(y)| \leq \frac{2lC_{\log}}{\log(2l/|x - y|)}, \quad (9)$$

for all $x, y \in \Omega$ with $|x - y| < l$. We denote by $\mathcal{P}^{\log}(\Omega)$ the exponents in $\mathcal{P}(\Omega)$ that are log-Hölder continuous on Ω . For $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$ and a given measurable function, f , define

$$\|f\|_{p(\cdot)}^* = \sup \left\{ \left| \int_{\Omega} f(x) g(x) dx \right| : \|g\|_{p(\cdot)} \leq 1 \right\}. \quad (10)$$

Theorem 2.34 of [1] shows that there exist constants C_1 and C_2 , depending on $p(\cdot)$, such that

$$C_1 \|f\|_{p(\cdot)} \leq \|f\|_{p(\cdot)}^* \leq C_2 \|f\|_{p(\cdot)}. \quad (11)$$

The next result which establishes a relationship between the Lebesgue spaces with exponents p_- , p_+ , and $p(\cdot)$ will be very useful in the rest of the work. It is Corollary 2.50 of [1].

Lemma 1. Suppose $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$ and $|\Omega| < \infty$. Then there exist constants c_1 and c_2 such that

$$c_1 \|f\|_{p_-} \leq \|f\|_{p(\cdot)} \leq c_2 \|f\|_{p_+}. \quad (12)$$

The study of variable exponent Bergman space, $A^{p(\cdot)}$, which is the space of analytic functions in $L^{p(\cdot)}$, has been introduced in [9]. There it was shown, amongst other things, that the Bergman projector P is bounded from $L^{p(\cdot)}$ onto $A^{p(\cdot)}$. Also in [10], the authors studied Carleson measures in such spaces.

In this paper, we will extend the results in [3, 7] on boundedness and compactness of operators for the Bergman spaces with constant exponents to the Bergman spaces with variable exponents.

For $z \in \Delta$, let φ_z be the analytic map of Δ onto Δ given by $\varphi_z(w) = (z - w)/(1 - \bar{z}w)$. We define the operator U_z on A^2 by

$$U_z f = (f \circ \varphi_z) \varphi_z', \quad z \in \Delta. \quad (13)$$

Then U_z is a unitary operator on A^2 . We shall show later that U_z is bounded on $A^{p(\cdot)}$. For S , a bounded operator on $A^{p(\cdot)}$, we define S_z by $S_z = U_z S U_z$.

If S is a bounded operator on $A^{p(\cdot)}$, then the Berezin transform of S is the function $\tilde{S}$ on Δ defined by

$$\tilde{S}(z) = \langle S k_z, k_z \rangle, \quad (14)$$

where $k_z(w) = (1 - |z|^2)K_z$ is the normalized Bergman kernel which also belongs to $A^{p(\cdot)}$ and $\langle \cdot, \cdot \rangle$ is the inner product of A^2 . We let

$$\begin{aligned} \tilde{f}(z) &:= \widetilde{T_f}(z) = \langle f k_z, k_z \rangle \\ &= \int_{\Delta} f(w) |k_z(w)|^2 dA(w) \end{aligned} \quad (15)$$

and set

$$BT := \left\{ f \in L^1 : \|f\|_{BT} = \sup_{z \in \Delta} |\tilde{f}(z)| < \infty \right\}. \quad (16)$$

Our first result gives some conditions for the boundedness of Toeplitz operators with L^1 symbols on the variable Bergman spaces.

Theorem 2. Suppose $p(\cdot) \in \mathcal{P}^{\log}(\Delta)$, $1 < p_0 \leq p_- \leq p_+ < \infty$, and $p_1 = \min(p_0, p'_0)$. Suppose $f \in L^1$ is such that

$$\begin{aligned} C_1 &= \sup_{z \in \Delta} \|P(f \circ \varphi_z)\|_q < \infty, \\ C_2 &= \sup_{z \in \Delta} \|P(\bar{f} \circ \varphi_z)\|_q < \infty \end{aligned} \quad (17)$$

for all $q > (p_1 + 1)/(p_1 - 1)$. Then T_f is bounded on $A^{p(\cdot)}$.

We note here that this result was proved in [3] in the Bergman spaces A^p , where p is a constant. We also have the following result on compactness.

Theorem 3. Suppose $p(\cdot) \in \mathcal{P}^{log}(\Delta)$, $1 < p_0 \leq p_- \leq p_+ < \infty$, and $p_1 = \min(p_0, p'_0)$. If S is a bounded operator on $A^{p(\cdot)}$ such that

$$\begin{aligned} C_1 &= \sup_{z \in \Delta} \|S_z 1\|_q < \infty, \\ C_2 &= \sup_{z \in \Delta} \|S_z^* 1\|_q < \infty \end{aligned} \quad (18)$$

for some $q > (p_1 + 1)/(p_1 - 1)$, then the following are equivalent:

- (1) S is compact on $A^{p(\cdot)}$,
- (2) $\tilde{S}(z) \rightarrow 0$ as $z \rightarrow \partial\Delta$,
- (3) for every $s \in [1, q]$, $\|S_z 1\|_s \rightarrow 0$ as $z \rightarrow \partial\Delta$,
- (4) $\|S_z 1\|_1 \rightarrow 0$ as $z \rightarrow \partial\Delta$.

This theorem is well known in the Bergman spaces with constant exponents; for example, see [3, 7]. However, the techniques here are different from those used in either of the papers for both the proof of boundedness and compactness. This is because their proofs depend on the use of Schur's test which does not hold in the variable Lebesgue space. However, using the Muckenhoupt weights we were able to develop some Schur-like tests from where we obtain the theory that builds upon the Rubio de Francia theory of extrapolation from the theory of weighted norm inequalities. The advantage of this approach is that it quickly yields to sufficient conditions for these operators to be bounded on variable Lebesgue spaces. Through such techniques, we are also able to obtain some norm estimates for bounded operators on the space $A^{p(\cdot)}$.

Similar to the work of Miao and Zheng [7], we consider the case of the algebra of Toeplitz operators generated by symbols in the class BT . To be precise, we have the following.

Theorem 4. Suppose $p(\cdot) \in \mathcal{P}^{log}(\Delta)$ and S is a finite sum of finite products of Toeplitz operators with symbols in the class BT . Then S is compact on $A^{p(\cdot)}$ if and only if $\tilde{S}(z) \rightarrow 0$ as $z \rightarrow \partial\Delta$.

This paper is organized as follows: in Section 2, we will study some basic concepts on the Muckenhoupt weights. Section 3 deals with the variable Bergman spaces and the proof of Theorem 2. In Section 4, we study some norm estimates on these spaces and in Section 5 we give the proof of the compactness results.

2. Muckenhoupt Weights A_1

Definition 5. Let Ω be a set. Then the function $d : \Omega \times \Omega \rightarrow \mathbb{R}^+$ is said to be a pseudodistance on Ω if it satisfies the following:

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$;
- (3) there exists a positive constant $K \geq 1$, such that, for all $x, y, z \in \Omega$,

$$d(x, y) \leq K(d(x, z) + d(z, y)). \quad (19)$$

For $x \in \Omega$ and $r > 0$, the set $B(x, r) = \{y \in \Omega : d(x, y) < r\}$ is called a pseudoball with centre x and radius r . If μ is a measure on Ω , then the triple (Ω, d, μ) is called a homogeneous space if Ω is endowed with the topology generated by the collection $\{B(x, r) : x \in \Omega, r > 0\}$ (that is, the topology generated by the pseudoballs) and μ satisfies the doubling property; there exists a constant δ such that, for all $x \in \Omega$ and $r > 0$, we have

$$0 < \mu(B(x, 2r)) \leq \delta \mu(B(x, r)) < \infty. \quad (20)$$

We now turn our attention to the case when $\Omega = \Delta$. By lemma 2.2 of [11], it is shown that the distance function d given on Δ by

$$d(z, w) = \begin{cases} ||z| - |w|| + \left|1 - \frac{z\bar{w}}{|z||w|}\right| & \text{if } z, w \in \Delta^*, \\ |z| + |w| & \text{otherwise} \end{cases} \quad (21)$$

is a pseudodistance on Δ , where $\Delta^* = \Delta \setminus \{0\}$. It is known (see [12]) that, at the boundary of Δ , d becomes the Koranyi distance. Also by Lemma 2 of [12], we have that for any pseudoball $B(w, r)$, $w \neq 0$, and $r \in (0, 2)$ we have that

$$|B(w, r)| \approx r^2, \quad (22)$$

where $|B|$ denotes the Lebesgue area measure of set B . Also observe that the pseudoball $B(0, 1) = \Delta$. It is known that (see [12]) (Δ, d, dA) is a homogeneous space.

Let f be a locally integrable function in Δ . Then the Hardy-Littlewood maximal function relative to the pseudodistance d is given by

$$Mf(z) = \sup_B \frac{1}{|B|} \int_B |f(w)| dA(w), \quad (23)$$

where the supremum is taken over all pseudoballs containing z .

Suppose $0 < \omega(z) < \infty$ almost everywhere on Δ . Then we say that ω is in the Muckenhoupt weight A_1 if

$$[\omega]_{A_1} = \text{ess sup}_{z \in \Delta} \frac{M\omega(z)}{\omega(z)} < \infty. \quad (24)$$

There are two equivalent definitions which are useful in practice. First, $\omega \in A_1$, if for almost every $z \in \Delta$,

$$M\omega(z) \leq [\omega]_{A_1} \omega(z). \quad (25)$$

It follows that if $\omega \in A_1$ then

$$[\omega]_{A_1} \omega(z) \geq M\omega(z) \geq \omega(\Delta), \quad (26)$$

and thus

$$1 \leq \frac{[\omega]_{A_1} \omega(z)}{\omega(\Delta)}. \quad (27)$$

Alternatively, $\omega \in A_1$ if for every pseudoball B we have that

$$\frac{1}{|B|} \int_B \omega(z) dA(z) \leq [\omega]_{A_1} \text{ess inf}_{u \in B} \omega(u). \quad (28)$$

For more details on the Muckenhoupt weights, see chapter 9 of [13] or chapter 4 of [1].

We will need some results on extrapolation. The following proposition is Theorem 5.24 of [1].

Proposition 6. Let $\Omega \in \mathbb{R}^n$ and suppose there is some $p_0 \geq 1$ and the family $\mathcal{F}$ such that for all $\omega \in A_1$,

$$\int_{\Omega} F(x)^{p_0} \omega(x) dx \leq C_0 \int_{\Omega} G(x)^{p_0} \omega(x) dx, \quad (29)$$

$$(F, G) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}^{log}(\Omega)$, if $1 \leq p_0 \leq p_- \leq p_+ < \infty$ and the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\Omega)$, then

$$\|F\|_{p(\cdot)} \leq C_{p(\cdot)} \|G\|_{p(\cdot)}, \quad (F, G) \in \mathcal{F}, \quad (30)$$

where $C_{p(\cdot)} = CC_0$ and C is some positive constant depending on the dimension of Ω .

The following is Theorem 3.16 of [1].

Proposition 7. Let $p \in \mathcal{P}^{log}(\Omega)$. Then the Hardy-Littlewood maximal operator function is bounded in $L^{p(\cdot)}(\Omega)$ and we have

$$\|Mf\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}. \quad (31)$$

3. Variable Exponent Bergman Spaces

Given $p(\cdot) \in \mathcal{P}^{log}(\Delta)$, we define the variable exponent Bergman space $A^{p(\cdot)}$ as the space of all analytic functions on Δ that belong to the variable exponent Lebesgue space $L^{p(\cdot)}$ with respect to the area measure dA on the unit disc. With this definition $A^{p(\cdot)}$ is a closed subspace of $L^{p(\cdot)}$. By Theorem 4.4 of [9], the Bergman projection, P , given by (2) is bounded from $L^{p(\cdot)}$ onto $A^{p(\cdot)}$ for any $p(\cdot) \in \mathcal{P}^{log}(\Delta)$. It is, thus, necessary to study the behaviour of Toeplitz operators on such spaces. Similar to the definition of Toeplitz operators on the Bergman spaces with constant exponent, we define the Toeplitz operator with symbol $f \in L^\infty$ on $A^{p(\cdot)}$ by

$$T_f g = P(fg), \quad g \in A^{p(\cdot)}. \quad (32)$$

Lemma 8. The operator U_z is bounded on $A^{p(\cdot)}$ for $p(\cdot) > 1$.

Proof. Let $1 < p_0 \leq p_- \leq p_+ < \infty$ and $\omega \in A_1$. Then

$$\begin{aligned} & \int_{\Delta} |U_z f(\xi)|^{p_0} \omega(\xi) dA(\xi) \\ & \leq \int_{\Delta} |U_z f(\xi)|^{p_0} M\omega(\xi) dA(\xi) \end{aligned} \quad (33)$$

since $\omega(\xi) \leq M\omega(\xi)$ for almost every $\xi \in \Delta$. Now, for $0 < \epsilon < \omega(\Delta)$ there is a pseudoball B containing ξ such that

$$M\omega(\xi) \leq \frac{\omega(B)}{|B|} + \epsilon \leq \frac{\omega(\Delta)}{|B|} + \omega(\Delta). \quad (34)$$

It follows that

$$\begin{aligned} & \int_{\Delta} |U_z f(\xi)|^{p_0} M\omega(\xi) dA(\xi) \\ & \leq \left(\frac{\omega(\Delta)}{|B|} + \omega(\Delta) \right) \int_{\Delta} |U_z f(\xi)|^{p_0} dA(\xi). \end{aligned} \quad (35)$$

Also

$$\begin{aligned} & \int_{\Delta} |U_z f(\xi)|^{p_0} dA(\xi) = \int_{\Delta} |f \circ \varphi_z(\xi) \varphi'_z(\xi)|^{p_0} dA(\xi) \\ & = \int_{\Delta} |f(\xi) \varphi'_z(\varphi_z(\xi))|^{p_0} |\varphi'_z(\xi)|^2 dA(\xi) \\ & = \int_{\Delta} |f(\xi)|^{p_0} |\varphi'_z(\varphi_z(\xi))|^{p_0-2} dA(\xi) \\ & \leq \frac{2^{p_0}}{(1-|z|^2)^{p_0-2}} \int_{\Delta} |f(\xi)|^{p_0} dA(\xi) \\ & \leq \frac{[\omega]_{A_1} 2^{p_0}}{\omega(\Delta) (1-|z|^2)^{p_0-2}} \int_{\Delta} |f(\xi)|^{p_0} \omega(\xi) dA(\xi), \end{aligned} \quad (36)$$

where the last inequality comes from (27). This shows that

$$\begin{aligned} & \int_{\Delta} |U_z f(\xi)|^{p_0} \omega(\xi) dA(\xi) \\ & \leq \frac{(|B|+1) [\omega]_{A_1} 2^{p_0}}{|B| (1-|z|^2)^{p_0-2}} \int_{\Delta} |f(\xi)|^{p_0} \omega(\xi) dA(\xi). \end{aligned} \quad (37)$$

It follows that the family $\{|U_z f|, |f|\} : f \in L^{p_0}\}$ satisfies inequality (29). Also, by Proposition 7 the maximal function Mf belongs to $L^{(p(\cdot)/p_0)'}$. Thus by Proposition 6 U_z is bounded on $A^{p(\cdot)}$. $\square$

Remark 9. We just want to give an alternative argument to obtain the estimate (35), and this argument has different effects and may be useful in applications.

We recall that if ω is locally integrable in Δ , then

$$\lim_{r \rightarrow 0} \frac{1}{|B(z, r)|} \int_{B(z, r)} |\omega(z) - \omega(\xi)| dA(\xi) = 0. \quad (38)$$

The proof of this statement can easily be adapted from that of Theorem 1.3 of [14]. We use this statement as follows:

Let $\omega \in A_1$ and $p_0 > 1$. Then for any $\epsilon > 0$, we can find $R > 0$ such that

$$\frac{1}{|B(z, r)|} \int_{B(z, r)} |\omega(z) - \omega(\xi)| dA(\xi) < \epsilon \omega(\Delta), \quad (39)$$

for all $r \in (0, R)$. Now, if we fix such R then for $0 < r < R$ we have

$$\begin{aligned} & \int_{\Delta} |U_z f(\xi)|^{p_0} \omega(\xi) dA(\xi) = \int_{\Delta} |U_z f(\xi)|^{p_0} \left(\omega(\xi) \right. \\ & \quad \left. - \frac{1}{|B(\xi, r)|} \int_{B(\xi, r)} \omega(\zeta) dA(\zeta) \right. \\ & \quad \left. + \frac{1}{|B(\xi, r)|} \int_{B(\xi, r)} \omega(\zeta) dA(\zeta) \right) dA(\xi) \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\Delta} |U_z f(\xi)|^{p_0} \frac{1}{|B(\xi, r)|} \\
&\cdot \int_{B(\xi, r)} |\omega(\xi) - \omega(\zeta)| dA(\zeta) dA(\xi) \\
&+ \int_{\Delta} |U_z f(\xi)|^{p_0} \frac{1}{|B(\xi, r)|} \int_{B(\xi, r)} \omega(\zeta) dA(\zeta) dA(\xi) \\
&\leq \omega(\Delta) (\epsilon + C_r) \int_{\Delta} |U_z f(\xi)|^{p_0} dA(\xi),
\end{aligned} \quad (40)$$

where the constant C_r does not depend on $\omega \in A_1$, and thus (35) holds. We may also use this same argument in some parts of the proofs of Theorem 2 and Proposition 17 by replacing similar statements that give rise to the estimates (34) in the proof of Lemma 8.

Let

$$Tf(z) = \int_{\Delta} f(w) K(z, w) dA(w), \quad z \in \Delta, \quad (41)$$

where $K : \Delta \times \Delta \rightarrow \mathbb{C}$ is a kernel function. We give a Schur-type lemma that will be useful in our work.

Lemma 10. *Let $\omega \in A_1$, $p > 1$, and $1/p + 1/p' = 1$. If there exist positive constants C_1 and C_2 that depends on $[\omega]_{A_1}$ and not ω , and a nonnegative measurable function, h , such that*

$$\int_{\Delta} |K(\xi, z)| h(z)^{p'} dA(z) \leq C_1 h(\xi)^{p'} \quad (42)$$

for almost every $\xi \in \Delta$ and

$$\omega(\Delta)^{-1} \int_{\Delta} |K(\xi, z)| h(\xi)^p \omega(\xi) dA(z) \leq C_2 h(z)^p \quad (43)$$

for almost every $z \in \Delta$, then

$$\begin{aligned}
&\int_{\Delta} |Tf(z)|^p \omega(z) dA(z) \\
&\leq C_1 C_2 \int_{\Delta} |f(z)|^p \omega(z) dA(z).
\end{aligned} \quad (44)$$

Proof. Using Hölder's inequality, we have

$$\begin{aligned}
|Tf(z)| &\leq \int_{\Delta} |K(\xi, z)| h(\xi) h(\xi)^{-1} |f(\xi)| dA(\xi) \\
&\leq C'_1 h(z) \left\{ \int_{\Delta} |K(\xi, z)| h(\xi)^{-p} |f(\xi)|^p dA(\xi) \right\}^{1/p} \\
&\leq C'_1 h(z) \left\{ \int_{\Delta} |K(\xi, z)| h(\xi)^{-p} |f(\xi)|^p \right. \\
&\quad \cdot \frac{[\omega]_{A_1} \omega(\xi)}{\omega(\Delta)} dA(\xi) \left. \right\}^{1/p},
\end{aligned} \quad (45)$$

where the second inequality comes from (42) and the third inequality is from (27). Also, Fubini's theorem gives

$$\begin{aligned}
&\int_{\Delta} |Tf(z)|^p \omega(z) dA(z) \\
&\leq C_1 \int_{\Delta} \frac{[\omega]_{A_1} h(z)^p}{\omega(\Delta)} \left\{ \int_{\Delta} |K(\xi, z)| h(\xi)^{-p} |f(\xi)|^p \right. \\
&\quad \cdot \omega(\xi) dA(\xi) \left. \right\} \omega(z) dA(z) \\
&= C_1 [\omega]_{A_1} \int_{\Delta} |f(\xi)|^p h(\xi)^{-p} \int_{\Delta} |K(\xi, z)| h(z)^p \\
&\quad \cdot \frac{\omega(z)}{\omega(\Delta)} dA(z) \omega(\xi) dA(\xi) \leq C_1 C_2 [\omega]_{A_1} \\
&\quad \cdot \int_{\Delta} |f(z)|^p \omega(z) dA(z),
\end{aligned} \quad (46)$$

where we have used (43) to get the last inequality. $\square$

Proposition 11. *Let $p(\cdot) \in \mathcal{P}^{log}(\Delta)$, $p \in (1, \infty)$ be such that $1 < p \leq p_- \leq p_+ < \infty$. Suppose the function $K : \Delta \times \Delta \rightarrow \mathbb{C}$ satisfies the hypothesis of Lemma 10. Then there is a constant C such that*

$$\|Tf\|_{p(\cdot)} \leq CC_1 C_2 \|f\|_{p(\cdot)}, \quad (47)$$

for all $f \in L^{p(\cdot)}$.

Proof. Let $\omega \in A_1$. Then by Lemma 10, we obtain (29) for the family $\{|Tf|, |f|\} : f \in L^{p_0}(\Delta)$, where $1 < p_0 \leq p_- \leq p_+ < \infty$. Also by Proposition 7, the maximal function Mf belongs to $L^{(p(\cdot)/p_0)'}(\Delta)$ and thus by Proposition 6 we get the required estimates. $\square$

In the application of Lemma 10, we may assume that $\text{ess inf}_{z \in \Delta} \omega(z) \geq 1$ for $\omega \in A_1$, as the following lemma shows.

Lemma 12. *Let $\omega \in A_1$ be such that $\text{ess inf}_{z \in \Delta} \omega(z) < 1$ and $\mu = \delta \omega$ for any $\delta > 1$ such that $\text{ess inf}_{z \in \Delta} \mu(z) \geq 1$. If the hypothesis of Lemma 10 holds for the weight μ , then the conclusion of Lemma 10 holds for the weight ω .*

Proof. By (27), we have that $1 \leq ([\mu]_{A_1}/\mu(\Delta))\mu(z)$ for almost every $z \in \Delta$. Now by Proposition 9.1.5 of [13], we have that $[\mu]_{A_1} = [\delta \omega]_{A_1} = [\omega]_{A_1}$. It follows that $1 \leq ([\omega]_{A_1}/\mu(\Delta))\mu(z)$ for almost every $z \in \Delta$. Thus we have that the constants $C_1 = [\omega]_{A_1}$ and C_2 are independent of μ and, hence, independent of ω . Now since the hypothesis of Lemma 10 holds for the weight μ , we have that

$$\int_{\Delta} |Tf(z)|^p \mu(z) dA(z) \leq C_1 C_2 \|f\mu^{1/p}\|_p^p. \quad (48)$$

Thus,

$$\delta \int_{\Delta} |Tf(z)|^p \omega(z) dA(z) \leq \delta C_1 C_2 \|f\omega^{1/p}\|_p^p. \quad (49)$$

It follows that

$$\int_{\Delta} |Tf(z)|^p \omega(z) dA(z) \leq C_1 C_2 \|f\omega^{1/p}\|_p^p, \quad (50)$$

which gives the result. $\square$

The next lemma will be used frequently and is well known; see, for example, Lemma 3.10 of [15] for the proof.

Lemma 13. Suppose $a < 1$ and $a + b < 2$. Then

$$\sup_{z \in \Delta} \int_{\Delta} \frac{dA(w)}{(1 - |w|^2)^a |1 - \bar{z}w|^b} < \infty. \quad (51)$$

Proof of Theorem 2. Let $g \in A^{p(\cdot)}$. Then

$$\begin{aligned} T_f g(z) &= \langle T_f g, K_z \rangle = \langle g, T_{\bar{f}} K_z \rangle \\ &= \int_{\Delta} g(\xi) \overline{T_{\bar{f}} K_z(\xi)} dA(\xi). \end{aligned} \quad (52)$$

Now, let $h(z) = (1 - |z|^2)^{-\epsilon}$ for

$$\max\left(\frac{2}{qp_0}, \frac{2}{qp'_0}\right) < \epsilon < \min\left(\frac{1}{q'p'_0}, \frac{1}{q'p_0}\right). \quad (53)$$

Then using the identity

$$T_{\bar{f}} K_z(\xi) = P(\bar{f} \circ \varphi_z)(\varphi_z(\xi)) K_z(w), \quad z, \xi \in \Delta \quad (54)$$

we have

$$\begin{aligned} &\int_{\Delta} |T_{\bar{f}} K_z(\xi)| h(\xi)^{p'_0} dA(\xi) \\ &= \int_{\Delta} |P(\bar{f} \circ \varphi_z)(\varphi_z(\xi)) K_z(\xi)| (1 - |z|^2)^{-\epsilon p'_0} dA(\xi) \\ &= \frac{1}{(1 - |z|^2)^{\epsilon p'_0}} \int_{\Delta} \frac{|P(\bar{f} \circ \varphi_z)(\xi)| dA(\xi)}{(1 - |\xi|^2)^{\epsilon p'_0} |1 - \bar{z}\xi|^{(2-2\epsilon p'_0)q'}} \\ &\leq \frac{\|P(\bar{f} \circ \varphi_z)\|_q}{(1 - |z|^2)^{\epsilon p'_0}} \left(\int_{\Delta} \frac{dA(\xi)}{(1 - |\xi|^2)^{\epsilon p'_0 q'} |1 - \bar{z}\xi|^{(2-2\epsilon p'_0)q'}} \right)^{1/q'}. \end{aligned} \quad (55)$$

Also,

$$\sup_{z \in \Delta} \int_{\Delta} \frac{dA(\xi)}{(1 - |\xi|^2)^{\epsilon p'_0 q'} |1 - \bar{z}\xi|^{(2-2\epsilon) p'_0 q'}} < \infty \quad (56)$$

provided that $\epsilon < 1/p'_0 q'$ and $\epsilon p'_0 q' + 2q' - 2\epsilon p'_0 q' < 2$. That is, $2/p'_0 q' < \epsilon < 1/p'_0 q'$ which holds from the choice of ϵ .

Now, observe that

$$T_{\bar{f}} K_z(\xi) = \langle T_{\bar{f}} K_z, K_{\xi} \rangle = \langle K_z, T_f K_{\xi} \rangle = \overline{T_f K_{\xi}(z)}. \quad (57)$$

Thus, for each $\omega \in A_1$, we have

$$\begin{aligned} &\int_{\Delta} |T_{\bar{f}} K_z(\xi)| h(z)^{p_0} \frac{\omega(z)}{\omega(\Delta)} dA(z) \\ &= \int_{\Delta} |T_f K_{\xi}(z)| h(z)^{p_0} \frac{\omega(z)}{\omega(\Delta)} dA(z) \\ &\leq \frac{1}{\omega(\Delta)} \int_{\Delta} |T_f K_{\xi}(z)| h(z)^{p_0} M\omega(z) dA(z) \end{aligned} \quad (58)$$

since $\omega(z) \leq M\omega(z)$ for almost every $z \in \Delta$. For any $0 < \epsilon_1 < \omega(\Delta)$, there is a pseudoball B containing z such that

$$M\omega(z) \leq \frac{\omega(B)}{|B|} + \epsilon_1 \leq \frac{\omega(\Delta)}{|B|} + \epsilon_1 \leq \frac{\omega(\Delta)}{|B|} + \omega(\Delta). \quad (59)$$

Substitute this in (58) to obtain

$$\begin{aligned} &\int_{\Delta} |T_f K_{\xi}(z)| h(z)^{p_0} \frac{\omega(z)}{\omega(\Delta)} dA(z) \\ &\leq (|B|^{-1} + 1) \int_{\Delta} |T_f K_{\xi}(z)| h(z)^{p_0} dA(z). \end{aligned} \quad (60)$$

Using the identity (54), we have

$$\begin{aligned} &\int_{\Delta} |T_{\bar{f}} K_z(\xi)| h(\xi)^{p_0} \frac{\omega(\xi)}{\omega(\Delta)} dA(\xi) \leq C' \int_{\Delta} |T_f K_{\xi}(z)| \\ &\cdot \frac{1}{(1 - |z|^2)^{\epsilon p_0}} dA(z) \\ &= C' \int_{\Delta} |P(\bar{f} \circ \varphi_{\xi})(\varphi_{\xi}(z)) K_{\xi}(z)| \\ &\cdot (1 - |z|^2)^{-\epsilon p_0} dA(\xi) = \frac{C'}{(1 - |\xi|^2)^{-\epsilon p_0}} \\ &\cdot \int_{\Delta} |P(\bar{f} \circ \varphi_{\xi})(z) K_{\xi}(z)| (1 - |z|^2)^{-\epsilon p_0} dA(\xi) \\ &= \frac{C'}{(1 - |\xi|^2)^{-\epsilon p_0}} \int_{\Delta} \frac{|P(\bar{f} \circ \varphi_{\xi})(z)| dA(z)}{(1 - |z|^2)^{\epsilon p_0} |1 - \bar{z}\xi|^{(2-2\epsilon p_0)q'}}, \end{aligned} \quad (61)$$

where the last equality is from the change of variable $z = \varphi_{\xi}$. By Hölder's inequality, we have that

$$\begin{aligned} &\int_{\Delta} \frac{|P(\bar{f} \circ \varphi_{\xi})(z)| dA(z)}{(1 - |z|^2)^{\epsilon p_0} |1 - \bar{z}\xi|^{(2-2\epsilon p_0)q'}} \leq \|P(\bar{f} \circ \varphi_z)\|_q \\ &\cdot \left(\int_{\Delta} \frac{dA(\xi)}{(1 - |w|^2)^{\epsilon p_0 q'} |1 - \bar{z}\xi|^{(2-2\epsilon p_0)q'}} \right)^{1/q'}, \\ &\sup_{\xi \in \Delta} \int_{\Delta} \frac{dA(\xi)}{(1 - |\xi|^2)^{\epsilon p_0 q'} |1 - \bar{z}\xi|^{(2-2\epsilon p_0)q'}} < \infty \end{aligned} \quad (62)$$

if $\epsilon < 1/p_0 q'$ and $\epsilon p_0 q' + 2q' - 2\epsilon p_0 q' < 2$; that is, $2/qp_0 < \epsilon < 1/p_0 q'$. Now if $\epsilon > 0$ is chosen to satisfy (53) we see that the hypothesis of Lemma 10 is satisfied and thus for every $\omega \in A_1$ we have

$$\begin{aligned} &\int_{\Delta} |T_f g(z)|^p \omega(z) dA(z) \leq C \sup_{z \in \Delta} \|P(f \circ \varphi_z)\|_q \\ &\cdot \sup_{z \in \Delta} \|P(\bar{f} \circ \varphi_z)\|_q \int_{\Delta} |g(z)|^p \omega(z) dA(z) \\ &= CC_1 C_2 \int_{\Delta} |g(z)|^p \omega(z) dA(z), \end{aligned} \quad (63)$$

where the constant C does not depend on ω but on $[\omega]_{A_1}$. Finally we apply Proposition 11 to $\{(|T_f g|, |g|) : g \in L^{p_0}\}$ to obtain the result. $\square$

For $f \in BT$ it is easy to show that

$$\begin{aligned} \sup_{z \in \Delta} \|P(f \circ \varphi_z)\|_q &< \infty, \\ \sup_{z \in \Delta} \|P(\bar{f} \circ \varphi_z)\|_q &< \infty \end{aligned} \quad (64)$$

for any $q > 0$. Thus, from Theorem 2, we immediately have the following.

Corollary 14. Suppose $f \in BT$. Then

- (1) T_f is bounded on $A^{p(\cdot)}$,
- (2) $T_{f \circ \varphi_z}$ is bounded on $A^{p(\cdot)}$, and

$$\|T_{f \circ \varphi_z}\|_{p(\cdot)} \leq C \|f\|_{BT}. \quad (65)$$

Proof. (1) is an immediate consequence of Theorem 2.

(2) follows from the fact that $\|f \circ \varphi_z\|_{BT} = \|f\|_{BT}$ and $\|T_{f \circ \varphi_z}\|_{p(\cdot)} \leq C \|f \circ \varphi_z\|_{BT}$, which is given by assertion (1). $\square$

We also have the following estimate for operators in the Toeplitz algebra. To be precise, we have the following.

Lemma 15. Let $S = T_{f_1} \cdots T_{f_n}$, $f_i \in BT$, $i = 1, \dots, n$. Then

$$\begin{aligned} \sup_{z \in \Delta} \|S_z 1\|_{p(\cdot)} &< \infty, \\ \sup_{z \in \Delta} \|S_z^* 1\|_{p(\cdot)} &< \infty \end{aligned} \quad (66)$$

for any $P \in \mathcal{P}^{log}(\Delta)$.

Proof. By assertion (2) of Corollary 14, we have that

$$\begin{aligned} \|S_z 1\|_{p(\cdot)} &= \|T_{f_1 \circ \varphi_z} \cdots T_{f_n \circ \varphi_z}\|_{p(\cdot)} \\ &\leq C \|f_1\|_{BT} \cdots \|f_n\|_{BT}. \end{aligned} \quad (67)$$

Also, since each $\bar{f}_j \in BT$ and $\|\bar{f}_j\|_{BT} = \|f_j\|_{BT}$, we have that

$$\begin{aligned} \|S_z 1\|_{p(\cdot)} &= \|T_{\bar{f}_1 \circ \varphi_z} \cdots T_{\bar{f}_n \circ \varphi_z}\|_{p(\cdot)} \\ &\leq C \|f_1\|_{BT} \cdots \|f_n\|_{BT}. \end{aligned} \quad (68)$$

This completes the proof of the lemma. $\square$

4. Norm Estimates

Lemma 16. Let $1 < p_0 < \infty$ and $p_1 = \min(p_0, p'_0)$ and suppose that S is a bounded operator on $A^{p(\cdot)}$ and $q > (p_1 + 1)/(p_1 - 1)$. If $\epsilon > 0$ satisfies (53), then for all $\omega \in A_1$ we have the following:

$$\int_{\Delta} \frac{|(SK_z)(\xi)|}{(1 - |\xi|^2)^{\epsilon p'_0}} dA(\xi) \leq \frac{C_1 \|S_z 1\|_q}{(1 - |z|^2)^{\epsilon p'_0}} \quad (69)$$

for all $z \in \Delta$ and

$$\omega(\Delta)^{-1} \int_{\Delta} \frac{|(SK_z)(\xi)|}{(1 - |z|^2)^{\epsilon p'_0}} \omega(z) dA(z) \leq \frac{C_2 \|S_z^* 1\|_q}{(1 - |z|^2)^{\epsilon p'_0}} \quad (70)$$

for all $\xi \in \Delta$, where the constant C_2 does not depend on ω .

Proof. Fix $z \in \Delta$. Then

$$SK_z = \frac{SU_z 1}{|z|^2 - 1} = \frac{U_z S_z 1}{|z|^2 - 1} = \frac{((S_z 1) \circ \varphi_z) \varphi'_z}{|z|^2 - 1}, \quad (71)$$

where the second equality comes from the definition of S_z and the third equality from the definition of U_z . Thus,

$$\begin{aligned} &\int_{\Delta} \frac{|(SK_z)(\xi)|}{(1 - |\xi|^2)^{\epsilon p'_0}} dA(\xi) \\ &= \frac{1}{1 - |z|^2} \int_{\Delta} \frac{|(S_z 1) \circ \varphi_z(\xi)| |\varphi'_z(\xi)|}{(1 - |\xi|^2)^{\epsilon p'_0}} dA(\xi) \\ &= \frac{1}{(1 - |z|^2)^{\epsilon p'_0}} \int_{\Delta} \frac{|(S_z 1)(\xi)|}{(1 - |\xi|^2)^{\epsilon p'_0} |1 - \bar{z}\xi|^{2-2\epsilon p'_0}} dA(\xi) \\ &\leq C \frac{\|S_z 1\|_q \|f_z\|_{q'}}{(1 - |z|^2)^{\epsilon p'_0}}, \end{aligned} \quad (72)$$

where $f_z(\xi) = 1/(1 - |\xi|^2)^{\epsilon p'_0} |1 - \bar{z}\xi|^{2-2\epsilon p'_0}$. By the choice of ϵ , we have that $\sup_{z \in \Delta} \|f_z\|_{q'} < \infty$ and (69) holds.

To prove (70), replace S by S^* in (69), interchange ξ and $\bar{\xi}$ in (69), and then use the equation

$$S^* K_{\bar{\xi}}(z) = \langle S^* K_{\bar{\xi}}, K_z \rangle = \langle K_{\bar{\xi}}, SK_z \rangle = \overline{SK_z(\xi)}. \quad (73)$$

Finally, we use the same argument as in the proof of Theorem 2 to obtain that there is a pseudoball B containing z such that $\omega(z) \leq M\omega(B) \leq \omega(\Delta)(|B|^{-1} + 1)$ and thus

$$\begin{aligned} &\omega(\Delta)^{-1} \int_{\Delta} \frac{|(SK_z)(\xi)|}{(1 - |z|^2)^{\epsilon p'_0}} \omega(z) dA(z) \\ &\leq (|B|^{-1} + 1) \int_{\Delta} \frac{|(S^* K_{\bar{\xi}})(z)|}{(1 - |z|^2)^{\epsilon p'_0}} dA(z). \end{aligned} \quad (74)$$

A similar argument as the one used to obtain the estimate (69) will give us (70). $\square$

Proposition 17. Let $p \in \mathcal{P}^{log}(\Delta)$, $1 < p_0 \leq p_- \leq p_+ < \infty$, and $p_1 = \min(p_0, p'_0)$ and suppose that S is a bounded operator on $A^{p(\cdot)}$. If

$$\begin{aligned} C_1 &= \sup_{z \in \Delta} \|S_z 1\|_q < \infty, \\ C_2 &= \sup_{z \in \Delta} \|S_z^* 1\|_q < \infty \end{aligned} \quad (75)$$

for some $q > (p_1 + 1)/(p_1 - 1)$, then there is a constant C such that

$$\|S\|_{p(\cdot)} \leq CC_1C_2. \quad (76)$$

Proof. For $f \in A^{p(\cdot)}$ and $w \in \Delta$, we have

$$\begin{aligned} (Sf)(\xi) &= \langle Sf, K_\xi \rangle = \langle f, S^* K_\xi \rangle \\ &= \int_{\Delta} f(z) \overline{(S^* K_\xi)(z)} dA(z) \\ &= \int_{\Delta} f(z) (SK_z)(\xi) dA(z), \end{aligned} \quad (77)$$

where the last equation follows from (69). Given that $\epsilon > 0$ that satisfies (53), we have by (69) that

$$\int_{\Delta} \frac{|(SK_z)(\xi)|}{(1 - |\xi|^2)^{\epsilon p_0}} dA(w) \leq \frac{C \|S_z 1\|_q}{(1 - |z|^2)^{\epsilon p_0}}. \quad (78)$$

In a similar manner, we use (73) and (70) to get that

$$\begin{aligned} \omega(\Delta)^{-1} \int_{\Delta} \frac{|(SK_z)(\xi)|}{(1 - |\xi|^2)^{\epsilon p_0}} \omega(z) dA(z) \\ = \omega(\Delta)^{-1} \int_{\Delta} \frac{|(S^* K_\xi)(z)|}{(1 - |z|^2)^{\epsilon p_0}} \omega(z) dA(z) \\ \leq \frac{C \|S_\xi^* 1\|_q}{(1 - |\xi|^2)^{\epsilon p_0}}. \end{aligned} \quad (79)$$

$$\int_{\Delta} |Sf(z)|^p \omega(z) dA(z) \leq C \int_{\Delta} |f(z)|^p \omega(z) dA(z)$$

for all $\omega \in A_1$ where the constant C depends on $[\omega]_{A_1}$ and not on ω . We now apply Proposition 11 to get the required result. $\square$

Lemma 18. (a) $\|K_z\|_{p(\cdot)}$ is equivalent to $(1 - |z|^2)^{-2+2/p(z)}$ for all $z \in \Delta$.

(b) $K_z/\|K_z\|_{p(\cdot)} \rightarrow 0$ weakly in $A^{p(\cdot)}$ as $z \rightarrow \partial\Delta$.

Proof. The assertion (a) is just Theorem 3.5 of [10].

(b) If $f \in A^{p(\cdot)}$, then assertion (a) implies

$$\left\langle f, \frac{K_z}{\|K_z\|_{p(\cdot)}} \right\rangle \cong f(z) (1 - |z|^2)^{-2+2/p(z)}. \quad (80)$$

Thus if f is a bounded function in $A^{p(\cdot)}$, then $\langle f, K_z/\|K_z\|_{p(\cdot)} \rangle \rightarrow 0$ as $z \rightarrow \partial\Delta$. The assertion follows from the fact that polynomials are dense in $A^{p(\cdot)}$. $\square$

5. Compact Operators on $A^{p(\cdot)}$

Theorem 19. Let $p \in \mathcal{P}^{log}(\Delta)$ be such that $1 < p_- \leq p_+ < \infty$ and suppose that

$$\int_{\Delta} \|K(\cdot, z)\|_{p(\cdot)}^{p'(\cdot)+} dA(z) < \infty. \quad (81)$$

Then the operator T given by (41) is compact on $L^{p(\cdot)}(\Delta)$.

Proof. Firstly we observe that if (81) holds and $g \in L^{p'(\cdot)}$, then the function $w \mapsto F(w)$, where

$$F(w) = \int_{\Delta} K(z, w) g(z) dA(z) \quad (82)$$

belongs to $L^{p'(\cdot)}(\Delta)$. Indeed,

$$\begin{aligned} \int_{\Delta} |F(w)|^{p'(\cdot)+} dA(w) \\ = \int_{\Delta} \left| \int_{\Delta} g(z) K(z, w) dA(z) \right|^{p'(\cdot)+} dA(w) \\ \leq C_{p(\cdot)} \int_{\Delta} \|g\|_{p'(\cdot)}^{p'(\cdot)+} \|K(\cdot, w)\|_{p(\cdot)}^{p'(\cdot)+} dA(w) < \infty. \end{aligned} \quad (83)$$

It follows from Lemma 1 that $\|F\|_{p'(\cdot)} \leq \|F\|_{p'(\cdot)+} < \infty$. Thus for $g \in L^{p'(\cdot)}$, we see that $F \in L^{p'(\cdot)}$. Now, suppose (φ_n) is a bounded sequence in $L^{p(\cdot)}$ such that $\varphi_n \rightarrow \varphi$ weakly in $L^{p(\cdot)}$. For $\epsilon > 0$ and any $g \in L^{p'(\cdot)}$, we can find that $N \in \mathbb{N}$ such that for $n \geq N$ we have

$$\left| \int_{\Delta} (\varphi_n(w) - \varphi(w)) \overline{g}(w) dA(w) \right| < \epsilon. \quad (84)$$

We will show that $\|T\varphi_n - T\varphi\|_{p(\cdot)} \rightarrow 0$ as $n \rightarrow \infty$. Now, given ϵ , we fix N such that (84) holds. It follows for any $n \geq N$ and (11) that

$$\begin{aligned} \|T\varphi_n - T\varphi\|_{p(\cdot)} &\leq C_{p(\cdot)} \|T\varphi_n - T\varphi\|_{p(\cdot)}^* \\ &= \sup \left\{ \left| \int_{\Delta} (T\varphi_n - T\varphi)(w) \overline{g}(w) dA(w) \right| : \|g\|_{p'(\cdot)} \leq 1 \right\} \\ &= \sup \left\{ \left| \int_{\Delta} \int_{\Delta} (\varphi_n(w) - \varphi(w)) K(w, z) dA(w) \overline{g}(z) dA(z) \right| : \right. \\ &\quad \left. \|g\|_{p'(\cdot)} \leq 1 \right\} \\ &= \sup \left\{ \left| \int_{\Delta} (\varphi_n(w) - \varphi(w)) \overline{\int_{\Delta} K(z, w) g(z) dA(z)} dA(w) \right| : \right. \\ &\quad \left. \|g\|_{p'(\cdot)} \leq 1 \right\} = \sup \left\{ \left| \int_{\Delta} (\varphi_n(w) - \varphi(w)) \overline{F(w)} dA(w) \right| : \right. \\ &\quad \left. \|g\|_{p'(\cdot)} \leq 1 \right\} < \epsilon. \end{aligned} \quad (85)$$

Thus $\|T\varphi_n - T\varphi\|_{p(\cdot)} \rightarrow 0$ as $n \rightarrow \infty$. Finally, it is shown in Corollary 2.81 of [1] that the variable Lebesgue space $L^{p(\cdot)}$ is reflexive if and only if $1 < p_- \leq p_+ < \infty$. We thus conclude that T is compact, since $L^{p(\cdot)}$ is reflexive. $\square$

We will need the power series formula for the Berezin transform of the bounded operator S on A^2 . From the definition of the reproducing kernel, we get that

$$k_z(w) = (1 - |z|^2) \sum_{n=0}^{\infty} (n+1) w^n \overline{z}^n \quad (86)$$

for $z, w \in \Delta$. To compute $\tilde{S}(z) = \langle Sk_z, k_z \rangle$, we first compute Sk_z by applying S to both sides of (86) and then take the inner product with k_z , again using (86), to obtain

$$\begin{aligned} \tilde{S}(z) &= (1 - |z|^2)^2 \sum_{m,j=0}^{\infty} (j+1)(m+1) \langle Sw^j, w^m \rangle \bar{z}^j z^m. \end{aligned} \quad (87)$$

Lemma 20. Suppose S is a bounded operator on $A^{p(\cdot)}$ such that

$$\sup_{z \in \Delta} \|S_z 1\|_q < \infty \quad (88)$$

for some $q > 1$. Then $\tilde{S}(z) \rightarrow 0$ as $z \rightarrow \partial\Delta$ if and only if for every $t \in [1, q]$, $\|S_z 1\|_t \rightarrow 0$ as $z \in \partial\Delta$.

Proof. Suppose for every $t \in [1, q]$, $\|S_z 1\|_t \rightarrow 0$ as $z \rightarrow \partial\Delta$. In particular, $\|S_z 1\|_1 \rightarrow 0$ as $z \in \partial\Delta$. Thus

$$|\tilde{S}(z)| = |\langle S_z 1, 1 \rangle| \leq \|S_z 1\|_1 \rightarrow 0 \quad (89)$$

as $z \in \partial\Delta$.

Suppose $\tilde{S}(z) \rightarrow 0$ as $z \rightarrow \partial\Delta$. Fix $t \in [1, q]$. We will show that $\|S_z 1\|_t \rightarrow 0$ as $z \in \partial\Delta$.

For $z \in \Delta$, $j, m = 0, 1, \dots$, an easy computation shows that

$$\int_{\Delta} (S_z w^j) \overline{w^m} dA(w) = \int_{\Delta} (S U_z w^j) \overline{U_z w^m} dA(w). \quad (90)$$

Since

$$\begin{aligned} U_z w^j &= (\varphi_z(w))^j \varphi'_z(w) \\ &= -(\varphi_z(w))^j (1 - |z|^2) K_z(w), \end{aligned} \quad (91)$$

we have that

$$\begin{aligned} \int_{\Delta} (S U_z w^j) \overline{U_z w^m} dA(w) &= (1 - |z|^2)^2 \\ &\cdot \int_{\Delta} S[(\varphi_z)^j K_z](w) \overline{(\varphi_z(w))^m K_z(w)} dA(w). \end{aligned} \quad (92)$$

It follows from (90) and (92) and Hölder's inequality that

$$\begin{aligned} |\langle S_z w^j, w^m \rangle| &= \left| \int_{\Delta} (S_z w^j) \overline{w^m} dA(w) \right| = (1 - |z|^2)^2 \\ &\cdot \left| \int_{\Delta} S[(\varphi_z)^j K_z](w) \overline{(\varphi_z(w))^m K_z(w)} dA(w) \right| \\ &\leq C_{p(\cdot)} (1 - |z|^2)^2 \|S[\varphi_z^j K_z]\|_{p(\cdot)} \|\varphi_z^m K_z\|_{p'(\cdot)} \\ &\leq C_{p(\cdot)} (1 - |z|^2)^2 \|S\|_{p(\cdot)} \|\varphi_z^j K_z\|_{p(\cdot)} \|\varphi_z^m K_z\|_{p'(\cdot)}. \end{aligned} \quad (93)$$

Now, let

$$\begin{aligned} S_1 &= \left\{ \lambda > 0 : \int_{\Delta} \left(\frac{|K_z(w)|}{\lambda} \right)^{p(w)} dA(w) \leq 1 \right\}, \\ S_2 &= \left\{ \lambda > 0 : \int_{\Delta} \left(\frac{|\varphi_z(w) K_z(w)|}{\lambda} \right)^{p(w)} dA(w) \leq 1 \right\}. \end{aligned} \quad (94)$$

Then $S_1 \subset S_2$ and thus $\inf S_1 \geq \inf S_2$. This shows that

$$\|\varphi_z^j K_z\|_{p(\cdot)} \leq \|K_z\|_{p(\cdot)} \quad (95)$$

for any $p(\cdot) \in P^{\log}(\Delta)$. This and Lemma 18(a) show that

$$\begin{aligned} |\langle S_z w^j, w^m \rangle| &\leq C_{p(\cdot)} (1 - |z|^2)^2 \|S\|_{p(\cdot)} \|K_z\|_{p(\cdot)} \|K_z\|_{p'(\cdot)} \\ &\leq C \|S\|_{p(\cdot)}; \end{aligned} \quad (96)$$

that is, $\langle S_z w^j, w^m \rangle$ is uniformly bounded in $z \in \Delta$ and $j, m = 0, 1, \dots$.

Now, we will show that for every nonnegative integer n

$$\langle S_z 1, w^n \rangle \rightarrow 0 \quad \text{as } z \rightarrow \partial\Delta. \quad (97)$$

If this is not true, then there is a sequence $z_k \in \Delta$ such that

$$\langle S_{z_k} 1, w^n \rangle \rightarrow a_{0n} \quad \text{as } z_k \rightarrow \partial\Delta \quad (98)$$

for some nonzero constant a_{0n} and $n \geq 1$. Since $\langle S_z w^j, w^m \rangle$ is uniformly bounded, we may assume without loss of generality that for each j and m

$$\langle S_z w^j, w^m \rangle \rightarrow a_{jm} \quad \text{as } z \rightarrow \partial\Delta, \quad (99)$$

for some constant a_{jm} . For $z, \xi \in \Delta$, we have

$$\begin{aligned} \tilde{S}(\varphi_z(\xi)) &= \tilde{S}_z(\xi) \\ &= (1 - |\xi|^2)^2 \sum_{j,m=0}^{\infty} (j+1)(m+1) \langle S_z w^j, w^m \rangle \bar{\xi}^j \xi^m, \end{aligned} \quad (100)$$

where the second equality comes from (87). Also, note that the power series in (100) converges uniformly for each $\xi \in \Delta$.

For each $\xi \in \Delta$, we know that $\varphi_{z_k}(\xi) \rightarrow \partial\Delta$ as $z_k \rightarrow \partial\Delta$. Thus $\tilde{S}(\varphi_{z_k}(\xi)) \rightarrow 0$ as $z_k \rightarrow \partial\Delta$ for each $\xi \in \Delta$. Replacing z by z_k in (100) and taking the limit as $z_k \rightarrow \partial\Delta$, we get

$$(1 - |\xi|^2)^2 \sum_{j,m=0}^{\infty} (j+1)(m+1) a_{jm} \bar{\xi}^j \xi^m = 0 \quad (101)$$

for each $\xi \in \Delta$. If

$$f(\xi) = \sum_{j,m=0}^{\infty} (j+1)(m+1) a_{jm} \bar{\xi}^j \xi^m, \quad (102)$$

then $f(\xi) = 0$ for all $\xi \in \Delta$. This gives

$$\left[\frac{\partial^m}{\partial \xi^m} \frac{\partial^j}{\partial \bar{\xi}^j} f \right] (0) = 0 \quad (103)$$

for each j and m . On the other hand, we have

$$\left[\frac{\partial^m}{\partial \xi^m} \frac{\partial^j}{\partial \bar{\xi}^j} f \right] (0) = ((j+1)!(m+1)!) a_{jm} \quad (104)$$

for each j and m . In particular, $a_{0n} = 0$ which is a contradiction. Hence, we obtain

$$\lim_{z \rightarrow \partial\Delta} \langle S_z 1, w^n \rangle = 0. \quad (105)$$

For $\xi \in \Delta$, we have

$$(S_z 1)(\xi) = \sum_{n=0}^{\infty} (n+1) \langle S_z 1, w^n \rangle \xi^n. \quad (106)$$

It is clear that for each fixed $\xi \in \Delta$, the power series above converges uniformly for $z \in \Delta$. This gives

$$\lim_{z \rightarrow \partial\Delta} (S_z 1)(\xi) = 0 \quad (107)$$

for each $\xi \in \Delta$. It follows that

$$\lim_{z \rightarrow \partial\Delta} |(S_z 1)(\xi)|^t = 0 \quad (108)$$

for each $\xi \in \Delta$. If $s = q/t$ then $s > 1$ and

$$\int_{\Delta} [|(S_z 1)(\xi)|^t]^s dA(\xi) = \|S_z 1\|_q^q \leq \sup_{z \in \Delta} \|S_z 1\|_q^q < \infty. \quad (109)$$

This shows that $\{|(S_z 1)|^t\}_{z \in \Delta}$ is uniformly integrable. By Vitali's Theorem or Exercise 11 on pages 133-134 of [16], we have that

$$\lim_{z \rightarrow \partial\Delta} \|S_z 1\|_{t(\cdot)} = 0. \quad (110)$$

This completes the proof of the lemma. $\square$

Proof of Theorem 3. If S is compact on $A^{p(\cdot)}$, then by Lemma 18(b),

$$\left\langle \frac{SK_z}{\|K_z\|_{p(\cdot)}}, \frac{K_z}{\|K_z\|_{p'(\cdot)}} \right\rangle \rightarrow 0 \quad (111)$$

as $z \rightarrow \partial\Delta$. Now by Lemma 18(a), we see that $\tilde{S}(z)$ is equivalent to $\langle SK_z / \|K_z\|_{p(\cdot)}, K_z / \|K_z\|_{p'(\cdot)} \rangle$ for $z \in \Delta$. Thus $\tilde{S}(z) \rightarrow 0$ as $z \rightarrow \partial\Delta$.

Suppose $\tilde{S}(z) \rightarrow 0$ as $z \rightarrow \partial\Delta$. By Lemma 20, we have that $\|S_z 1\|_t \rightarrow 0$ as $z \rightarrow \partial\Delta$ for every $t \in [1, q]$. We will show that S is compact on $A^{p(\cdot)}$. Fix $t \in (1, q)$ in the rest of the proof.

For $f \in A^{p(\cdot)}$, we have that

$$(Sf)(w) = \int_{\Delta} f(z) (SK_z)(w) dA(z). \quad (112)$$

For $0 < r < 1$, we define an operator $S_{[r]}$ on $A^{p(\cdot)}$ by

$$(S_{[r]}f)(w) = \int_{r\Delta} f(z) (SK_z)(w) dA(z). \quad (113)$$

Then $S_{[r]}$ is an integral operator with kernel $(SK_z)(w)\chi_{r\Delta}(z)$. We will first show that the operator $S_{[r]}$ is compact on $A^{p(\cdot)}$. By Theorem 19, we only need to show that

$$\begin{aligned} & \int_{\Delta} \|(SK_z)\chi_{r\Delta}(z)\|_{p(\cdot)}^{p'(\cdot)+} dA(z) \\ &= \int_{r\Delta} \|(SK_z)\|_{p(\cdot)}^{p'(\cdot)+} dA(z) < \infty. \end{aligned} \quad (114)$$

But

$$\begin{aligned} & \int_{r\Delta} \|(SK_z)\|_{p(\cdot)}^{p'(\cdot)+} dA(z) \\ & \leq \int_{r\Delta} \|S\|_{p(\cdot)}^{p'(\cdot)+} \|K_z\|_{p(\cdot)}^{p'(\cdot)+} dA(z) \\ & \leq C \|S\|_{p(\cdot)}^{p'(\cdot)+} \int_{r\Delta} \frac{dA(z)}{(1-|z|^2)^{(2-2/p(z))p'(\cdot)+}} \\ & \leq \frac{Cr^2 \|S\|_{p(\cdot)}^{p'(\cdot)+}}{(1-|r|^2)^{2p'(\cdot)+}}. \end{aligned} \quad (115)$$

This shows that

$$\int_{r\Delta} \|(SK_z)\|_{p(\cdot)}^{p'(\cdot)+} dA(z) < \infty, \quad (116)$$

and thus $S_{[r]}$ is compact on $A^{p(\cdot)}$. Hence, to prove that S is compact, we only need to show that $\|S - S_{[r]}\|_{p(\cdot)} \rightarrow 0$ as $r \rightarrow 1^-$.

If $r \in (0, 1)$ then $S - S_{[r]}$ is the integral operator with kernel

$$(SK_z)(w) \chi_{\Delta \setminus r\Delta}(z) \quad (117)$$

as can be seen from (77) and (113). The proof of Proposition 17 indicates that

$$\|S - S_{[r]}\|_{p(\cdot)} \leq CC_1 C_2, \quad (118)$$

where

$$\begin{aligned} C_1 &= \sup \{\|S_z 1\|_t : r \leq |z| < 1\}, \\ C_2 &= \sup \{\|S_z^* 1\|_t : z \in \Delta\}. \end{aligned} \quad (119)$$

We have shown that $C_1 \rightarrow 0$ as $r \rightarrow 1^-$ and the hypothesis of the theorem shows that $C_2 < \infty$. Thus, $\|S - S_{[r]}\|_{p(\cdot)} \rightarrow 0$ as $r \rightarrow 1^-$, which completes the proof. $\square$

Proof of Theorem 4. Suppose S is a finite sum of operators of the form $T_{f_1} \cdots T_{f_n}$, where each $f_j \in BT$. By Corollary 14 and Lemma 15, we have that S is bounded on $A^{p(\cdot)}$ and

$$\begin{aligned} & \sup_{z \in \Delta} \|S_z 1\|_q < \infty, \\ & \sup_{z \in \Delta} \|S_z^* 1\|_q < \infty \end{aligned} \quad (120)$$

for all $q > 0$. The conclusion then follows from Theorem 3. $\square$

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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