Research Article

The Regular Part of a Semigroup of Full Transformations with Restricted Range: Maximal Inverse Subsemigroups and Maximal Regular Subsemigroups of Its Ideals

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Let $T(X)$ be the full transformation semigroup on a set X . For a fixed nonempty subset Y of a set X , let $T(X, Y)$ be the semigroup consisting of all full transformations from X into Y . In a paper published in 2008, Sanwong and Sommanee proved that the set $F(X, Y) = \{\alpha \in T(X, Y) : X\alpha = Y\alpha\}$ is the largest regular subsemigroup of $T(X, Y)$. In this paper, we describe the maximal inverse subsemigroups of $F(X, Y)$ and completely determine all the maximal regular subsemigroups of its ideals.

1. Introduction

Let $T(X)$ be the set of all full transformations from a nonempty set X into itself. It is well-known that $T(X)$ is a regular semigroup under composition of functions; see [1, p. 63]. Moreover, every semigroup can be embedded in $T(X)$ for some nonempty set X ; see [1, p. 7]. In [2, 3], Schein posed the problem to determine all maximal inverse subsemigroups of $T(X)$; this problem is still unsolved. In 1976, Nichols [4] characterized one class of maximal inverse subsemigroup of $T(X)$. Later in 1978, Reilly [5] has generalized Nichols construction and obtained a much wider class of maximal inverse subsemigroups of $T(X)$. In 1999, Yang [6] described all of the maximal inverse subsemigroups of the finite symmetric inverse semigroup. Later in 2001, Yang [7] obtained the maximal subsemigroups of the finite singular transformation semigroups. In 2002, You [8] determined all the maximal regular subsemigroups of all ideals of the finite full transformation semigroup. In 2004, H. B. Yang and X. L. Yang [9] completely described the maximal subsemigroups of ideals of the finite full transformation semigroup. In 2014, Zhao et al. [10] showed that any maximal regular subsemigroup of ideals of the finite full transformation semigroup is idempotent generated. After that in 2015, East et al. [11] classified the maximal subsemigroups of $T(X)$ when X is an

infinite set containing certain subgroups of the symmetric group on X .

Let Y be a fixed nonempty subset of X and let $T(X, Y)$ be the subsemigroup of $T(X)$ of all elements with ranges contained in Y . In 1975, Symons [12] introduced and studied the semigroup $T(X, Y)$. He described all the automorphisms of $T(X, Y)$ and also determined the isomorphism theorem for two semigroups of type $T(X, Y)$. In 2005, Nenthein et al. [13] characterized the regular elements of $T(X, Y)$. In general, $T(X, Y)$ is not a regular semigroup. In 2008, Sanwong and Sommanee [14] defined

$$F(X, Y) = \{\alpha \in T(X, Y) : X\alpha = Y\alpha\} \quad (1)$$

and showed that $F(X, Y)$ is the largest regular subsemigroup of $T(X, Y)$. Obviously, $F(X, Y) = T(X)$ when $X = Y$. Hence, we may regard $F(X, Y)$ as a generalization of $T(X)$. The authors also characterized Green's relations on $T(X, Y)$ and gave one class of maximal inverse subsemigroups of $T(X, Y)$. Later in 2009, Sanwong et al. [15] described all maximal and minimal congruences on $T(X, Y)$. In 2011, Mendes-Gonçalves and Sullivan [16] obtained all the ideals of $T(X, Y)$. In the same year, Sanwong [17] described Green's relations and ideals and all maximal regular subsemigroups of $F(X, Y)$. The author also proved that every regular semigroup S can

be embedded in $F(S^1, S)$. After that, in 2013, Sommanee and Sanwong [18] computed the rank of $F(X, Y)$ when X is a finite set. They also obtained the rank and the idempotent rank of its ideals when X is finite. In 2014, Fernandes and Sanwong [19] calculated the rank of $T(X, Y)$ when X is finite. In 2016, L. Sun and J. Sun [20] characterised the natural partial order on $T(X, Y)$. In the same year, Tinpun and Koppitz [21] determined the relative rank of $T(X, Y)$ modulo the semigroup $S(X, Y)$ of all extensions of the bijections on Y when X is finite.

Let X be a chain and $\text{OT}(X)$ the full order-preserving transformation semigroup on X . In 2011, Dimitrova and Koppitz [22] characterized the maximal regular subsemigroups of ideals of $\text{OT}(X)$ when X is a finite chain. In 2015, Sommanee and Sanwong [23] investigated the regularity and Green's relations of the order-preserving transformation semigroup

$$\begin{aligned} OF(X, Y) &= F(X, Y) \cap \text{OT}(X) \\ &= \{\alpha \in F(X, Y) : \alpha \text{ is order-preserving}\}. \end{aligned} \quad (2)$$

They also proved that $OF(X, Y)$ is idempotent generated when Y is finite.

Let V be a vector space and let $T(V)$ denote the linear transformation semigroup from V into V . For a fixed subspace W of V , let $T(V, W)$ be the semigroup consisting of all linear transformations from V into W . Recently in 2017, Sommanee and Sangkhanan [24] determined all the maximal regular subsemigroups of the semigroup

$$Q = \{\alpha \in T(V, W) : V\alpha = W\alpha\}, \quad (3)$$

when W is a finite dimensional subspace of V over a finite field.

In this paper, we describe the maximal inverse subsemigroups of $F(X, Y)$ and completely determine all the maximal regular subsemigroups of its ideals.

2. Preliminaries and Notations

For all undefined notions, the reader is referred to [1].

For any set A , $|A|$ means the cardinality of the set A . If A is a subset of a semigroup S , then $\langle A \rangle$ denotes the subsemigroup of S generated by A . An element e of a semigroup S is called *idempotent* if $e^2 = e$. If U is a subset of a semigroup S , then $E(U)$ denotes the set of all idempotents in the set U . An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $a = axa$. The semigroup S is called *regular* if all its elements are regular. If a is an element of a semigroup S , we say that a' is an *inverse* of a if $a = aa'a$ and $a' = a'aa'$. We denote the set of inverses of an element a by $V(a)$. A semigroup S is called an *inverse semigroup* if every element a in S has a unique inverse a^{-1} in S ; it is equivalent to S being regular and idempotent elements commute. A proper (regular, inverse) subsemigroup M of a semigroup S is a *maximal (regular, inverse) subsemigroup* of S if whenever $M \subseteq T \subseteq S$ for some $(\text{regular, inverse})$ subsemigroup T of S , then $M = T$ or $T = S$.

The Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ on a semigroup S are defined as follows. For $a, b \in S$,

- (1) $a\mathcal{L}b$ if and only if $S^1a = S^1b$;
- (2) $a\mathcal{R}b$ if and only if $aS^1 = bS^1$;
- (3) $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$;
- (4) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

For each $a \in S$, we denote $\mathcal{L}$ -class, $\mathcal{R}$ -class, $\mathcal{H}$ -class, $\mathcal{D}$ -class, and $\mathcal{J}$ -class containing a by L_a , R_a , H_a , D_a , and J_a , respectively.

Lemma 1 (see [5, Lemma 1]). *Let M be a maximal inverse subsemigroup of a semigroup S . If M contains a minimum idempotent e , then $H_e \subseteq M$.*

Lemma 2. *Let S be any inverse semigroup. If e is a minimum idempotent in S , then $ea = ae$ for all $a \in S$.*

Proof. Assume that e is a minimum idempotent in S . Then $e \leq f$ for all $f \in E(S)$; that is, $e = ef = fe$. Let $a \in S$. Then $aea^{-1}, a^{-1}ea \in E(S)$ and $ea = e(aea^{-1})a = (aea^{-1})ea = e(ae^{-1}ea) = ae$. $\square$

Throughout the paper we assume that Y is a nonempty subset of a set X . Let $T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$, where $X\alpha$ denotes the image of α . Define

$$F(X, Y) = \{\alpha \in T(X, Y) : X\alpha = Y\alpha\}. \quad (4)$$

It is known that $F(X, Y)$ is the largest regular subsemigroup of $T(X, Y)$.

If $\alpha \in T(X)$ and $x \in X$, then the image of x under α is written as $x\alpha$. The set of all inverse images of x under α is denoted by $x\alpha^{-1}$.

Lemma 3 (see [17, Lemma 1]). *Let $\alpha \in T(X, Y)$. Then $\alpha \in F(X, Y)$ if and only if $\alpha\alpha^{-1} \cap Y \neq \emptyset$ for all $a \in X\alpha$.*

In [17], the author gave a complete description of Green's relations on $F(X, Y)$ as the following lemma.

Lemma 4. *Let $\alpha, \beta \in F(X, Y)$. Then*

- (1) $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$;
- (2) $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$, where $\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$;
- (3) $\alpha\mathcal{D}\beta$ if and only if $|X\alpha| = |X\beta|$;
- (4) $\mathcal{D} = \mathcal{J}$.

In addition, $\alpha\mathcal{H}\beta$ if and only if $X\alpha = X\beta$ and $\pi_\alpha = \pi_\beta$.

Lemma 5. *$F(X, Y)$ is an inverse semigroup if and only if $|Y| = 1$.*

Proof. If $|Y| = 1$, then $|F(X, Y)| = 1$ and so it is inverse. Conversely, assume that $F(X, Y)$ is an inverse semigroup. Let a and b be elements in Y . Then there exist constant maps X_a and X_b with ranges $\{a\}$ and $\{b\}$, respectively. Thus they are idempotents in $F(X, Y)$ such that $X_a = X_bX_a = X_aX_b = X_b$; it follows that $a = b$. Hence, $|Y| = 1$. $\square$

Proposition 6. M is a maximal inverse subsemigroup of $T(X, Y)$ if and only if M is a maximal inverse subsemigroup of $F(X, Y)$.

Proof. One direction is clear. Indeed, each inverse subsemigroup of $T(X, Y)$ is contained in $F(X, Y)$, the largest regular subsemigroup of $T(X, Y)$. Conversely, assume that M is a maximal inverse subsemigroup of $F(X, Y)$. Then M is an inverse subsemigroup of $T(X, Y)$. Let V be an inverse subsemigroup of $T(X, Y)$ such that $M \subseteq V \subsetneq T(X, Y)$. Since $F(X, Y)$ is the largest regular subsemigroup of $T(X, Y)$, it follows that $M \subseteq V \subseteq F(X, Y)$. We get that $M = V$ or $V = F(X, Y)$ by the maximality of M in $F(X, Y)$. If $F(X, Y) = V$, then $F(X, Y)$ is an inverse semigroup and so $|Y| = 1$ by Lemma 5; this implies that $T(X, Y) = F(X, Y)$. Therefore, M is a maximal inverse subsemigroup of $T(X, Y)$. $\square$

For each $a \in Y$, let

$$F_a = \{\alpha \in F(X, Y) : \alpha\alpha = a, \alpha \text{ is injective on } X \setminus a\alpha^{-1}\}. \quad (5)$$

The authors [14] showed that F_a is a maximal inverse subsemigroup of $T(X, Y)$. Thus by Proposition 6, we have that F_a is also a maximal inverse subsemigroup of $F(X, Y)$.

3. Maximal Inverse Subsemigroups

In this section, we write $G(A)$ for the permutation group on a set A .

Let e be any idempotent in $T(X)$. In 1978, Reilly defined I_e to be the set of those elements $\alpha \in T(X)$ which satisfy the following three conditions:

- (1) The restriction of α to Xe is a permutation of Xe ; that is, $\alpha|_{Xe} \in G(Xe)$.
- (2) For $x \in X$, if $x\alpha \notin Xe$, then $|(x\alpha)\alpha^{-1}| = 1$.
- (3) $\alpha e = e\alpha$.

Lemma 7 (see [5, Theorem 2]). *Let e be an idempotent in $T(X)$ and let I_e be defined as above. Then I_e is a maximal inverse subsemigroup of $T(X)$ with minimum idempotent e .*

Notice that if e is a constant map with range $\{a\} \subseteq X$, then $I_e = I_a$, where

$$I_a = \{\alpha \in T(X) : \alpha\alpha = a, \alpha \text{ is injective on } X \setminus a\alpha^{-1}\}; \quad (6)$$

see [4, 5] for more details.

Now, let e be any idempotent in $F(X, Y)$ and define

$$F_e = I_e \cap F(X, Y). \quad (7)$$

Since $e \in I_e \cap F(X, Y) = F_e$, we obtain $F_e \neq \emptyset$ and so F_e is a subsemigroup of $F(X, Y)$. We note that if $X = Y$, then $F(X, Y) = T(X)$ and $F_e = I_e \cap F(X, Y) = I_e \cap T(X) = I_e$.

Here, we aim to prove that F_e is the maximal inverse subsemigroups of $F(X, Y)$ with minimum idempotent e .

To prove our main result we need the following five lemmas.

Lemma 8. *Let e be any idempotent in $F(X, Y)$ and $\alpha \in F(X, Y)$. Then $\alpha \in F_e$ if and only if*

- (1) $\alpha|_{Xe} \in G(Xe)$,
- (2) for $y \in Y$, if $y\alpha \notin Xe$, then $|(y\alpha)\alpha^{-1}| = 1$,
- (3) $\alpha e = e\alpha$.

Proof. One direction is clear. Conversely, assume that conditions (1), (2), and (3) hold. We prove that $\alpha \in I_e$. Suppose that $x \in X$ and $x\alpha \notin Xe$. Since $\alpha \in F(X, Y)$, it follows that $x\alpha = y\alpha$ for some $y \in Y$. By condition (2), we get that $|(y\alpha)\alpha^{-1}| = 1$. Thus $|(x\alpha)\alpha^{-1}| = 1$ and so $\alpha \in I_e$. $\square$

Lemma 9. *Let e be any idempotent in $F(X, Y)$ and $\alpha \in F(X, Y)$. Then condition (2) in Lemma 8 is equivalent to the statement*

$$(X \setminus Y)\alpha \subseteq Xe, \quad \alpha \text{ is injective on } Y \setminus (Xe)\alpha^{-1}. \quad (8)$$

Proof. Assume that, for $y \in Y$, if $y\alpha \notin Xe$, then $|(y\alpha)\alpha^{-1}| = 1$. We prove $(X \setminus Y)\alpha \subseteq Xe$. Let $x\alpha \in (X \setminus Y)\alpha$ for some $x \in X \setminus Y$. Since $(X \setminus Y)\alpha \subseteq X\alpha = Y\alpha$, there exists $y \in Y$ such that $x\alpha = y\alpha$. Thus $x, y \in (y\alpha)\alpha^{-1}$. If $y\alpha \notin Xe$, then $|(y\alpha)\alpha^{-1}| = 1$ by the assumption. Whence $x = y$, this is a contradiction. Hence $y\alpha \in Xe$; that is, $x\alpha \in Xe$. Next, we prove α is injective on $Y \setminus (Xe)\alpha^{-1}$. Let $w, z \in Y \setminus (Xe)\alpha^{-1}$ be such that $w\alpha = z\alpha$. Then $w \in Y$ and $w\alpha \notin Xe$. Since $w, z \in (w\alpha)\alpha^{-1}$, we obtain $w = z$ by our assumption.

Conversely, suppose that $(X \setminus Y)\alpha \subseteq Xe$ and α is injective on $Y \setminus (Xe)\alpha^{-1}$. Let $y \in Y$ and $y\alpha \notin Xe$. To show that $|(y\alpha)\alpha^{-1}| = 1$, we let $u, v \in (y\alpha)\alpha^{-1}$. Then $u\alpha = v\alpha = y\alpha \notin Xe$; this implies that $u, v \notin (Xe)\alpha^{-1}$. Since $(X \setminus Y)\alpha \subseteq Xe$, we obtain $u, v \in Y$. Thus $u, v \in Y \setminus (Xe)\alpha^{-1}$ and so $u = v$ since α is injective on $Y \setminus (Xe)\alpha^{-1}$. $\square$

Lemma 10. *Let α be any element in $T(X)$ and let e be an idempotent in $T(X)$. If there is $\alpha' \in V(\alpha)$ such that $e = e\alpha'\alpha = \alpha\alpha'e$, $e\alpha = \alpha e$ and $e\alpha' = \alpha'e$, then $\alpha|_{Xe} \in G(Xe)$.*

Proof. Assume that the conditions hold. Since

$$\begin{aligned} Xe &= X(e\alpha'\alpha) = X(e\alpha')\alpha = X(\alpha'e)\alpha \subseteq (Xe)\alpha \\ &= X(e\alpha) = X(\alpha e) \subseteq Xe, \end{aligned} \quad (9)$$

it follows that $(Xe)\alpha = Xe$; that is, α maps Xe onto Xe . We prove that $\alpha|_{Xe}$ is injective. Let $x, y \in Xe$ be such that $x\alpha = y\alpha$. Then

$$\begin{aligned} x &= xe = x(\alpha\alpha'e) = (x\alpha)\alpha'e = (y\alpha)\alpha'e = y(\alpha\alpha'e) \\ &= ye = y. \end{aligned} \quad (10)$$

Hence $\alpha|_{Xe} : Xe \rightarrow Xe$ is bijective. $\square$

Lemma 11. *Let e be an idempotent in $F(X, Y)$. Then F_e is an inverse subsemigroup of $F(X, Y)$ with minimum idempotent e .*

Proof. Let α be any element in F_e . Then $\alpha \in I_e$. Since I_e is an inverse semigroup, there exists $\beta \in I_e$ such that $\alpha = \alpha\beta\alpha$

and $\beta = \beta\alpha\beta$. We prove $\beta \in F(X, Y)$. Since $\beta = \beta\alpha\beta$, we obtain $X\beta = X\beta\alpha\beta \subseteq X\alpha\beta \subseteq Y\beta \subseteq X\beta$; that is, $X\beta = Y\beta$. To show $X\beta \subseteq Y$, we let $x\beta \in X\beta$ for some $x \in X$. Since $\alpha \in F_e$, $(X \setminus Y)\alpha \subseteq Xe$ by Lemma 9. If $x\beta \in X \setminus Y$, then $(x\beta)\alpha \in (X \setminus Y)\alpha \subseteq Xe$. It follows that $x\beta = (x\beta\alpha)\beta \in (Xe)\beta = Xe \subseteq Y$; this is a contradiction. Hence, $x\beta \in Y$ and so $X\beta \subseteq Y$. Thus, $\beta \in I_e \cap F(X, Y) = F_e$. Whence F_e is a regular subsemigroup of I_e , it follows from Lemma 7 that F_e is an inverse subsemigroup of $F(X, Y)$ and e is the minimum idempotent in F_e . $\square$

Lemma 12. *Let e be an idempotent in $F(X, Y)$. If M is any inverse subsemigroup of $F(X, Y)$ such that $F_e \subseteq M$, then $E(M) \subseteq F_e$.*

Proof. Assume that M is an inverse subsemigroup of $F(X, Y)$ containing F_e . Let f be any idempotent in M . Since e, f are idempotents in M , we get $ef = fe$. We prove that $Xe \subseteq Xf$. Suppose that $Xe \not\subseteq Xf$. Then $Xef \subseteq Xe \cap Xf \subsetneq Xe$, so there exists $z \in Xe \setminus Xef$. We see that $zf \in Xef$ and $zf \neq z$. Now, we have $z, zef \in Xe$ such that $z \neq zef$. Let $\sigma \in G(Xe)$ be such that $z\sigma = zef$ and $(zef)\sigma = z$. We define $\alpha \in T(X)$ by

$$x\alpha = (xe)\sigma \quad \forall x \in X. \quad (11)$$

We see that $X\alpha = (Xe)\sigma = (Ye)\sigma = Y\alpha$ and $X\alpha = (Xe)\sigma = Xe \subseteq Y$; we obtain $\alpha \in F(X, Y)$. We prove $\alpha|_{Xe} \in G(Xe)$. Let $x, y \in X$ be such that $(xe)\alpha = (ye)\alpha$. Then $((xe)e)\sigma = ((ye)e)\sigma$; that is, $(xe)\sigma = (ye)\sigma$. Thus $xe = ye$ since σ is injective on Xe . Since $(Xe)\alpha = ((Xe)e)\sigma = (Xe)\sigma = Xe$, α is surjective on Xe . Hence, $\alpha|_{Xe} \in G(Xe)$. Since $y\alpha = (ye)\sigma \in Xe$ for all $y \in Y$, whence condition (2) in Lemma 8 holds. To prove $\alpha e = e\alpha$. Let $x \in X$. Since $(xe)\sigma \in Xe$ and e is an idempotent, we get that $x\alpha e = ((xe)\sigma)e = (xe)\sigma = ((xe)e)\sigma = x\alpha$. So, $\alpha \in F_e$. By Lemma 11, we have F_e as an inverse subsemigroup of $F(X, Y)$. Then there exists $\alpha^{-1} \in V(\alpha) \cap F_e$ and $\sigma^{-1} = \alpha^{-1}|_{Xe} \in G(Xe)$. Since $zef = zfe \in Xe$, we obtain $(zef)\alpha^{-1} = (zef)\sigma^{-1} = z$. It is clear that $\alpha^{-1}f\alpha$ is an idempotent in M . We see that

$$\begin{aligned} (zef)f(\alpha^{-1}f\alpha) &= (zef)(\alpha^{-1}f\alpha) = (zf)\alpha \\ &= [(zf)e]\sigma = (zef)\sigma = z, \\ (zef)(\alpha^{-1}f\alpha)f &= (zf)\alpha f = [(zfe)\sigma]f \\ &= [(zef)\sigma]f = zf. \end{aligned} \quad (12)$$

Since $z \neq zf$, this implies that $f(\alpha^{-1}f\alpha) \neq (\alpha^{-1}f\alpha)f$ which is a contradiction. Therefore, $Xe \subseteq Xf$. Hence $ef = e$ and so $e = ef = fe$. It is easy to see that $f|_{Xe} = id_{Xe}$, the identity map on Xe , which implies that $f|_{Xe} \in G(Xe)$. Thus, conditions (1) and (3) in Lemma 8 are satisfied for f .

In order to verify condition (2) in Lemma 8 for f to be an element of F_e , let $y \in Y$ be such that $yf \notin Xe$. We show that $|(yf)f^{-1}| = 1$. Let $u, v \in (yf)f^{-1}$. Then $uf = vf = yf \notin Xe$.

Suppose that $u \neq v$. Assume that $v \in Y$. We define $d \in T(X)$ by

$$xd = \begin{cases} x & \text{if } x \in Y \setminus \{u\}, \\ xe & \text{if } x \in \{u\} \cup (X \setminus Y). \end{cases} \quad (13)$$

Obviously, $Xd \subseteq Y$ and it is easy to verify that $(xe)d = xe = x(de)$ for all $x \in X$. Hence, $(Xe)d = Xe$ and $ed = de$. It is easy to show that $d|_{Xe}$ is injective, whence $d|_{Xe} \in G(Xe)$. We prove $Xd \subseteq Yd$. Let $xd \in Xd$ for some $x \in X$. If $x \in Y \setminus \{u\}$, then $xd \in Yd$. But if $x \in \{u\} \cup (X \setminus Y)$, then $xd = xe = (xe)d \in Yd$. Thus, $Xd \subseteq Yd$ and so $d \in F(X, Y)$. We prove d satisfies the condition in Lemma 9. By the definition of d , we have $(X \setminus Y)d \subseteq Xe$. Let $a, b \in Y \setminus (Xe)d^{-1}$. Then $a, b \in Y$ and $ad, bd \notin Xe$, which implies that $a, b \in Y \setminus \{u\}$. So, if $ad = bd$, then $a = b$. Hence, d is injective on $Y \setminus (Xe)d^{-1}$ and $d \in F_e$. Next, we prove d is an idempotent. For $x \in Y \setminus \{u\}$, $(xd)d = xd$. If $x \in \{u\} \cup (X \setminus Y)$, then $(xd)d = (xe)d = xe = xd$. Thus, $d^2 = d$. Since d and f are idempotents in M , $df = fd$. However,

$$\begin{aligned} udf &= (ue)f = ue \in Xe, \\ ufd &= (vf)d = vdf = vf \notin Xe, \end{aligned} \quad (14)$$

a contradiction. Hence, $v \in X \setminus Y$. Define $g \in T(X)$ by

$$xg = \begin{cases} x & \text{if } x \in Y, \\ xe & \text{if } x \in X \setminus Y. \end{cases} \quad (15)$$

Similarly, by using the same argument as in the proof above for d , we can show that g is an idempotent in F_e . Since $f, g \in E(M)$, we obtain $fg = gf$. We see that

$$\begin{aligned} vfg &= vf \notin Xe \\ \text{but } vgf &= (ve)f = ve \in Xe, \end{aligned} \quad (16)$$

and this is a contradiction. Thus, $u = v$ and so condition (2) in Lemma 8 is also satisfied. Whence $f \in F_e$, therefore, $E(M) \subseteq F_e$ as required. $\square$

Now, we are ready to prove our main result (Theorem 13).

Theorem 13. *Let e be an idempotent in $F(X, Y)$. Then F_e is a maximal inverse subsemigroup of $F(X, Y)$ with minimum idempotent e .*

Proof. From Lemma 11, F_e is an inverse subsemigroup of $F(X, Y)$ with minimum idempotent e . To prove the maximality, let M be any inverse semigroup of $F(X, Y)$ such that $F_e \subseteq M \subsetneq F(X, Y)$. Then by Lemma 12, we have $E(M) \subseteq F_e$, which implies that e is the minimum idempotent in M . We prove $M = F_e$ by letting $\alpha \in M$. Since $\alpha, \alpha^{-1} \in M$, we obtain $e\alpha = \alpha e$ and $e\alpha^{-1} = \alpha^{-1}e$ by Lemma 2. And since $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are idempotents in M , $e = \alpha\alpha^{-1}e$ and $e = e\alpha^{-1}\alpha$. Then by Lemma 10, we get that $\alpha|_{Xe} \in G(Xe)$. Thus, conditions (1) and (3) in Lemma 8 are satisfied for α to be an element of F_e . Next, we prove that α satisfies condition (2) in Lemma 8.

Let $y \in Y$ be such that $y\alpha \notin Xe$. Let $u, v \in (y\alpha)\alpha^{-1}$. Then $u\alpha = v\alpha = y\alpha \notin Xe$; this implies that $u\alpha\alpha^{-1} = v\alpha\alpha^{-1} = y\alpha\alpha^{-1} \notin Xe$. Now, we have $\alpha\alpha^{-1} \in E(F_e)$, $y \in Y$ and $y(\alpha\alpha^{-1}) \notin Xe$. Since $u, v \in [y(\alpha\alpha^{-1})](\alpha\alpha^{-1})^{-1}$, we obtain that $u = v$ by using condition (2) in Lemma 8 for $\alpha\alpha^{-1} \in F_e$. Hence, $\alpha \in F_e$ and so $M \subseteq F_e$. Therefore, $M = F_e$ and F_e is a maximal inverse subsemigroup of $F(X, Y)$. $\square$

Corollary 14. *Let e be an idempotent in $F(X, Y)$. Then F_e is a maximal inverse subsemigroup of $T(X, Y)$.*

Proof. It follows directly from Theorem 13 and Proposition 6. $\square$

Remark 15. $F_e \neq F_f$ for all distinct idempotents $e, f \in F(X, Y)$. In fact, if there exist idempotents e and f in $F(X, Y)$ such that $F_e = F_f$, then $f \in F_e$ and $e \in F_f$ such that e and f are minimum idempotents of F_e and F_f , respectively. Thus, we get $e \leq f$ and $f \leq e$; it follows that $e = f$.

Corollary 16. *Let e be an idempotent in $F(X, Y)$ such that $Xe = Y$. Then $F_e = H_e$ is a maximal inverse subsemigroup of $F(X, Y)$.*

Proof. Since F_e is a maximal inverse subsemigroup of $F(X, Y)$ with the minimum idempotent e , we obtain that $H_e \subseteq F_e$ by Lemma 1. To show that $F_e \subseteq H_e$, let $\alpha \in F_e$. Then $\alpha|_{Xe} \in G(Xe)$ and $\alpha e = e\alpha$. Since $(Xe)\alpha = Xe$, it follows that $X\alpha = Y\alpha = (Xe)\alpha = Xe$. We prove that $\pi_\alpha = \pi_e$. Let $x, y \in X$ such that $x\alpha = y\alpha$. Then $(xe)\alpha = (x\alpha)e = (y\alpha)e = (ye)\alpha$; this implies that $xe = ye$ since $\alpha|_{Xe}$ is injective. Hence $\pi_\alpha \subseteq \pi_e$. On the other hand, let $xe = ye$. Then $(x\alpha)e = (xe)\alpha = (ye)\alpha = (y\alpha)e$. Since $x\alpha, y\alpha \in Y = Xe$ and e is an idempotent, we obtain $x\alpha = y\alpha$. Hence $\pi_e \subseteq \pi_\alpha$ and $\pi_\alpha = \pi_e$. Then by Lemma 4, we get that $\alpha \in H_e$. Thus, $F_e \subseteq H_e$. Therefore, $H_e = F_e$ is the maximal inverse subsemigroup of $F(X, Y)$. $\square$

Notice that if e is a constant map with range $\{a\} \subseteq Y$, then $I_e = I_a$ and

$$\begin{aligned} F_e &= F(X, Y) \cap I_a = \{\alpha \in F(X, Y) : \alpha \\ &= a, \alpha \text{ is injective on } X \setminus a\alpha^{-1}\}; \end{aligned} \quad (17)$$

that is, $F_e = F_a$. Thus, we get the following corollary which appeared in [14, Theorem 4.3].

Corollary 17. *If $e \in F(X, Y)$ is a constant map with range $\{a\} \subseteq Y$, then $F_e = F_a$ is a maximal inverse subsemigroup of $F(X, Y)$ and $T(X, Y)$.*

Next, we count the number of elements in F_e . Let e be any idempotent in $F(X, Y)$. We know that $H_e \cong G(Xe)$; see [17, Section 4]. Thus, if $Xe = Y$, then $F_e = H_e \cong G(Xe) = G(Y)$ by Corollary 16. Hence, $|F_e| = |G(Y)|$ when $Xe = Y$. In what follows, we assume that $Xe \subsetneq Y$.

Recall that the number of ways that r objects can be chosen from n distinct objects written as $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}. \quad (18)$$

Let A and B be (possibly empty) sets and let $\text{PI}(A, B)$ be the set of all partial injective maps from A into B . We note that $\text{PI}(A, B)$ contains the empty map. Moreover, if $A = \emptyset$ or $B = \emptyset$, then $|\text{PI}(A, B)| = 1$. And if A and B are finite sets such that $|A| = m$ and $|B| = n$, then we shall write $\text{PI}(m, n)$ instead of $\text{PI}(A, B)$. By a consideration of the cardinality of a domain of partial injective maps, we can verify that

$$|\text{PI}(m, n)| = \sum_{j=0}^{\min\{m, n\}} \binom{m}{j} \binom{n}{j} j! \quad \text{for } m, n \geq 1. \quad (19)$$

Since $Xe \subsetneq Y$, there exists $x \in Y \setminus Xe$ such that $xe \in Xe$. We see that $x \in ((xe)e^{-1} \setminus \{xe\}) \cap Y$. For each $y \in Xe$, we define

$$\begin{aligned} A(y) &= (ye^{-1} \setminus \{y\}) \cap Y, \\ B(y) &= ye^{-1} \cap (X \setminus Y). \end{aligned} \quad (20)$$

Then $ye^{-1} = \{y\} \cup A(y) \cup B(y)$ is a disjoint union such that $ye^{-1} \cap Xe = \{y\}$ and $A(y) \cap Xe = \emptyset = B(y) \cap Xe$. Moreover, $A(y) \neq \emptyset$ for some $y \in Xe$ since $Xe \subsetneq Y$. Let

$$\begin{aligned} \mathcal{A} &= \{y_i \in Xe : A(y_i) \neq \emptyset, i \in I\}, \\ \mathcal{B} &= \{y_j \in Xe : B(y_j) \neq \emptyset, j \in J\}. \end{aligned} \quad (21)$$

Then $y_i e = y_i$ and $y_i e^{-1} \cap Xe = \{y_i\}$ for all $i \in I$. It is easy to see that

$$\begin{aligned} X \setminus Xe &= \left(\bigcup_{i \in I} A(y_i) \right) \\ &\cup \left(\bigcup_{j \in J} B(y_j) \right) \text{ is a disjoint union.} \end{aligned} \quad (22)$$

Let α be any element in F_e . Then $\alpha e = e\alpha$ and $\alpha|_{Xe} = \sigma$ for some $\sigma \in G(Xe)$. For each $x \in X \setminus Xe$, $x \in A(y_i) \cup B(y_j)$ for some $y_i \in \mathcal{A}$ and $y_j \in \mathcal{B}$. If $x \in A(y_i)$, then $xe = y_i \in Xe$. Since $\alpha e = e\alpha$, we obtain $(x\alpha)e = (xe)\alpha = y_i\alpha = y_i\sigma$; that is, $x\alpha \in (y_i\sigma)e^{-1} = \{y_i\sigma\} \cup A(y_i\sigma) \cup B(y_i\sigma)$. Since $x\alpha \in Y$ and $B(y_i\sigma) \subseteq X \setminus Y$, we obtain that $x\alpha \in \{y_i\sigma\} \cup A(y_i\sigma)$. If $x \in B(y_j) \subseteq X \setminus Y$, then $x\alpha \in (X \setminus Y)\alpha \subseteq Xe$ by Lemma 9. Since e is an idempotent, we get $x\alpha = (x\alpha)e = (xe)\alpha = y_j\alpha = y_j\sigma$. By condition (2) in Lemma 8, we see that α is injective on all elements of $A(y_i)$ which are not mapped into $\{y_i\sigma\}$ for all $i \in I$. Thus for each $i \in I$, $\alpha|_{A(y_i)}$ corresponds with an element θ_i of $\text{PI}(A(y_i), A(y_i\sigma))$; that is, for $u \in A(y_i)$,

$$u\alpha = \begin{cases} u\theta_i & \text{if } u \in \text{dom}(\theta_i), \\ y_i\sigma & \text{if } u \in A(y_i) \setminus \text{dom}(\theta_i). \end{cases} \quad (23)$$

We have shown the following. Given $\alpha \in F_e$. Then α is a union of a permutation $\sigma \in G(Xe)$, a union of partial injections $\theta_i : A(y_i) \rightarrow A(y_i\sigma)$ for each $y_i \in \mathcal{A}$, and a union of functions from $B(y_j)$ into $\{y_j\sigma\}$ for each $y_j \in \mathcal{B}$.

Theorem 18. *Let e be any idempotent in $F(X, Y)$ with $Xe \subsetneq Y$. Let $G(Xe) = \{\sigma_k : k \in \Lambda\}$ and $\mathcal{A} = \{y_i \in Xe : A(y_i) \neq \emptyset, i \in I\}$. Then there is a one-to-one correspondence between F_e and $\bigcup_{k \in \Lambda} \{\sigma_k\} \times \prod_{i \in I} \text{PI}(A(y_i), A(y_i\sigma_k))$.*

Proof. For each $\alpha \in F_e$, we have $\alpha|_{Xe} = \sigma_k$ for some $k \in \Lambda$ and partial injections $\theta_i : A(y_i) \rightarrow A(y_i\sigma_k)$ for all $y_i \in \mathcal{A}$. We define

$$\Psi : F_e \longrightarrow \bigcup_{k \in \Lambda} \left[\{\sigma_k\} \times \prod_{i \in I} \text{PI}(A(y_i), A(y_i\sigma_k)) \right], \quad (24)$$

by $\alpha\Psi = (\sigma_k, (\theta_i)_{i \in I})$ for all $\alpha \in F_e$. We verify that Ψ is injective. Let $\alpha, \beta \in F_e$ be such that $\alpha\Psi = \beta\Psi$. Then $(\sigma_k, (\theta_i)_{i \in I}) = (\sigma_l, (\tau_i)_{i \in I})$ for some $\sigma_k, \sigma_l \in G(Xe)$, $(\theta_i)_{i \in I} \in$

$\prod_{i \in I} \text{PI}(A(y_i), A(y_i\sigma_k))$ and $(\tau_i)_{i \in I} \in \prod_{i \in I} \text{PI}(A(y_i), A(y_i\sigma_l))$. Thus, $\sigma_k = \sigma_l$ and $\theta_i = \tau_i$ for all $i \in I$; it follows that $x\alpha = x\beta$ for all $x \in Xe \cup (\bigcup_{i \in I} A(y_i)) = Y$. For $x \in X \setminus Y$, $x \in B(y_j)$ for some $y_j \in \mathcal{B} = \{y_j \in Xe : B(y_j) \neq \emptyset, j \in J\}$. We get that $x\alpha = y_j\sigma_k = y_j\sigma_l = x\beta$. Whence $\alpha = \beta$, next, we prove Ψ is surjective. Let $\sigma_k \in G(Xe)$ and $(\theta_i)_{i \in I} \in \prod_{i \in I} \text{PI}(A(y_i), A(y_i\sigma_k))$. We define $\alpha \in T(X, Y)$ which is determined by σ_k and θ_i for all $i \in I$ as follows. For $x \in X$,

$$x\alpha = \begin{cases} x\sigma_k & \text{if } x \in Xe, \\ x\theta_i & \text{if } x \in \text{dom}(\theta_i) \neq \emptyset, \\ y_i\sigma_k & \text{if } x \in A(y_i), (x \notin \text{dom}(\theta_i) \text{ or } \theta_i \text{ is the empty map}), \\ y_j\sigma_k & \text{if } x \in B(y_j). \end{cases} \quad (25)$$

We see that, for each $x \in X \setminus Y$, $x \in B(y_j)$ for some $y_j \in \mathcal{B}$. Then $y_j \in Xe \subseteq Y$ and $x\alpha = y_j\sigma_k = y_j\alpha$. So, $X\alpha \subseteq Y\alpha$, that is $\alpha \in F(X, Y)$. By the construction of α , conditions (1) and (2) in Lemma 8 are satisfied for α to be an element of F_e . To prove $\alpha e = e\alpha$, we let $x \in X$. If $x \in Xe$, then $(x\alpha)e = (x\sigma_k)e = x\sigma_k = x\alpha = (xe)\alpha$, since $x, x\sigma_k \in Xe$ and e is an idempotent. If $x \in \text{dom}(\theta_i) \neq \emptyset$, then $x \in A(y_i)$; it follows that $xe = y_i$ and $x\theta_i \in A(y_i)\theta_i \subseteq A(y_i\sigma_k)$. Thus, $(x\theta_i)e = y_i\sigma_k$ and $(x\alpha)e = (x\theta_i)e = y_i\sigma_k = y_i\alpha = (xe)\alpha$ since $y_i \in Xe$. If $x \in A(y_i)$ and $(x \notin \text{dom}(\theta_i) \text{ or } \theta_i \text{ is the empty map})$, then $xe = y_i$ and $(x\alpha)e = (y_i\sigma_k)e = y_i\sigma_k = y_i\alpha = (xe)\alpha$. And if $x \in B(y_j)$, then $xe = y_j$ and $(x\alpha)e = (y_j\sigma_k)e = y_j\sigma_k = y_j\alpha = (xe)\alpha$, since $y_j \in Xe$. Hence, $\alpha e = e\alpha$. Whence $\alpha \in F_e$ and $\alpha\Psi = (\sigma_k, (\theta_i)_{i \in I})$, therefore, Ψ is a bijection. $\square$

The following corollary is a straightforward consequence of Theorem 18.

Corollary 19. Let e be an idempotent in $F(X, Y)$ such that $Xe \subsetneq Y$ and $|Xe| = p \in \mathbb{Z}^+$. Let $\mathcal{A} = \{y_i \in Xe : A(y_i) \neq \emptyset, 1 \leq i \leq t \leq p\}$. For each $y_i \in \mathcal{A}$, we let $|A(y_i)| = m_i \geq 1$ and $|A(y_i\sigma_k)| = n_{i,k} \geq 0$ ($1 \leq i \leq t, 1 \leq k \leq p!$). Then

$$|F_e| = \sum_{k=1}^{p!} \left(\prod_{i=1}^t |PI(m_i, n_{i,k})| \right). \quad (26)$$

It is known that if $X = Y$, then $F(X, Y) = T(X)$ and $F_e = I_e$. Then by Corollary 19, we have the following corollary.

Corollary 20. Let e be an idempotent in $T(X)$ such that $Xe \subsetneq X$ and $|Xe| = p \in \mathbb{Z}^+$. Let $\mathcal{A} = \{y_i \in Xe : A(y_i) \neq \emptyset, 1 \leq i \leq t \leq p\}$. For each $y_i \in \mathcal{A}$, we let $|A(y_i)| = m_i \geq 1$ and $|A(y_i\sigma_k)| = n_{i,k} \geq 0$ ($1 \leq i \leq t, 1 \leq k \leq p!$). Then

$$|I_e| = \sum_{k=1}^{p!} \left(\prod_{i=1}^t |PI(m_i, n_{i,k})| \right). \quad (27)$$

If e is a constant map with range $\{a\} \subseteq Y$, then we get the following corollary which appeared in [14, Theorem 4.5].

Corollary 21. Let Y be a finite subset of X with $|Y| = r \geq 2$ and e a constant map in $F(X, Y)$ with range $\{a\} \subseteq Y$; then $|F_a| = \sum_{j=0}^{r-1} j! \binom{r-1}{j}^2$.

Proof. We note that $F_e = F_a$ by Corollary 17. Since $Xe = \{a\}$, we obtain $|Xe| = 1$ and there is a unique $A(a) = Y \setminus \{a\}$; that is, $|A(a)| = r - 1$. By Corollary 19, we have

$$\begin{aligned} |F_a| &= |F_e| = \sum_{k=1}^{1!} \left(\prod_{i=1}^1 |PI(A(a), A(a))| \right) \\ &= |PI(A(a), A(a))| = |PI(r-1, r-1)|, \end{aligned} \quad (28)$$

where

$$\begin{aligned} |PI(r-1, r-1)| &= \sum_{j=0}^{\min\{r-1, r-1\}} \binom{r-1}{j} \binom{r-1}{j} j! \\ &= \sum_{j=0}^{r-1} \binom{r-1}{j}^2 j!. \end{aligned} \quad (29)$$

Therefore, $|F_a| = \sum_{j=0}^{r-1} j! \binom{r-1}{j}^2$. $\square$

4. Maximal Regular of Ideals

Throughout this section, X is a finite set with n elements and Y a nonempty subset of X with $r \geq 2$ elements. For convenience, we write $F_{n,r}$ for $F(X, Y)$.

For each $\alpha \in F_{n,r}$ with $|X\alpha| = k \in \mathbb{Z}^+$, it follows from Lemma 3 that we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}, \quad (30)$$

where $a_1, a_2, \dots, a_k \in Y$ and $\{A_1, A_2, \dots, A_k\}$ is a partition of X such that $A_i \cap Y \neq \emptyset$ for all $1 \leq i \leq k$.

For $1 \leq k \leq r$, we define $J(F; k) = \{\alpha \in F_{n,r} : |X\alpha| = k\}$. Then $J(F; k)$ is a $\mathcal{J}$ -class of the semigroup $F_{n,r}$. Let $Q(F; k) = J(F; 1) \cup J(F; 2) \cup \dots \cup J(F; k) = \{\alpha \in F_{n,r} : |X\alpha| \leq k\}$, where $1 \leq k \leq r$. Then $Q(F; k)$ is an ideal of $F_{n,r}$ and it is a regular subsemigroup of $F_{n,r}$; see [17, Lemma 7].

Recall that the principal factor of $F_{n,r}$ is the Rees quotient

$$P_k = Q(F; k) / Q(F; k - 1), \quad (31)$$

where $2 \leq k \leq r$. It is usually convenient to think of it as $J(F; k) \cup \{0\}$, and the product of two elements of P_k is taken to be zero if it falls in $Q(F; k - 1)$. Since P_k is finite and it is not a zero semigroup, we obtain that P_k is a completely 0-simple semigroup; see [18, p. 233] for more details.

We need the following lemmas for proof of our main result.

Lemma 22 (see [1, p. 98]). *Let S be a completely 0-simple semigroup. If $a, b \in S$ and $ab \neq 0$, then $ab \in R_a \cap L_b$.*

Lemma 23 (see [1, Proposition 2.3.7]). *Let a, b be elements in $\mathcal{D}$ -class D . Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

Lemma 24. *Let a be any element in a semigroup S and T a regular subsemigroup of S .*

- (1) *If $L_a \cap T \neq \emptyset$, then $L_a \cap T$ contains an idempotent.*
- (2) *If $R_a \cap T \neq \emptyset$, then $R_a \cap T$ contains an idempotent.*

Proof. (1) Assume that $L_a \cap T \neq \emptyset$. Then there exists $b \in L_a \cap T$. So, $b\mathcal{L}a$ and $b = bb'b$ for some $b' \in T$. We get that $b'b$ is an idempotent such that $b'b \in T$ and $b'b\mathcal{L}b\mathcal{L}a$. Therefore, $b'b$ is an idempotent in $L_a \cap T$.

(2) If there is $b \in R_a \cap T$, then we can write $b = bb'b$ for some $b' \in T$ and show that bb' is an idempotent in $R_a \cap T$. $\square$

Lemma 25 (see [18, Lemma 3.5]). *For $1 \leq k \leq r - 1$, $Q(F; k) = \langle J(F; k) \rangle$.*

Lemma 26 (see [18, Lemma 4.2]). *If $\alpha \in J(F; k)$, then α can be written as a finite product of idempotents in $J(F; k)$.*

The following fact can be obtained from Lemmas 25 and 26 immediately.

Lemma 27. *For $1 \leq k \leq r - 1$, $Q(F; k) = \langle E(J(F; k)) \rangle$.*

Lemma 28. *Let $1 \leq k \leq r - 1$ and $\alpha \in J(F; k)$. If $\beta \in L_\alpha$, then $\beta = \gamma\alpha$ for some $\gamma \in J(F; k)$ such that $L_\gamma \neq L_\alpha$.*

Proof. Suppose that $\beta \in L_\alpha$. Thus $X\alpha = X\beta$; we write

$$\begin{aligned} \alpha &= \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}, \\ \beta &= \begin{pmatrix} B_1 & B_2 & \cdots & B_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}, \end{aligned} \quad (32)$$

where $a_1, a_2, \dots, a_k \in Y$ and $A_i \cap Y \neq \emptyset \neq B_i \cap Y$ for all $1 \leq i \leq k$. Since $X\alpha \neq Y$, there is $y \in Y \setminus X\alpha$ such that $y \in A_j$ for some $j \in \{1, 2, \dots, k\}$. We choose $d_{a_i} \in A_i \cap Y$ for all $i \in \{1, 2, \dots, j - 1, j + 1, \dots, k\}$ and define

$$\gamma = \begin{pmatrix} B_1 & \cdots & B_{j-1} & B_j & B_{j+1} & \cdots & B_k \\ d_{a_1} & \cdots & d_{a_{j-1}} & y & d_{a_{j+1}} & \cdots & d_{a_k} \end{pmatrix}. \quad (33)$$

Then $\gamma \in J(F; k)$ such that $X\gamma \neq X\alpha$ and $\beta = \gamma\alpha$. $\square$

Lemma 29. *Let $2 \leq k \leq r - 1$ and $\alpha \in J(F; k)$. If $\beta \in R_\alpha$, then $\beta = \alpha\gamma$ for some $\gamma \in J(F; k)$ such that $R_\gamma \neq R_\alpha$.*

Proof. Assume that $\beta \in R_\alpha$. Thus $\pi_\alpha = \pi_\beta$; we write

$$\begin{aligned} \alpha &= \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}, \\ \beta &= \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}, \end{aligned} \quad (34)$$

where $a_i, b_i \in Y$ and $A_i \cap Y \neq \emptyset$ for all $1 \leq i \leq k$. Define

$$\gamma = \begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & \{a_k\} \cup (X \setminus X\alpha) \\ b_1 & b_2 & \cdots & b_{k-1} & b_k \end{pmatrix}. \quad (35)$$

It is clear that $\gamma \in J(F; k)$. If $\gamma \notin R_\alpha$, then $R_\gamma \neq R_\alpha$ and $\beta = \alpha\gamma$. If such $\gamma \in R_\alpha$, we define

$$\begin{aligned} \gamma' &= \begin{pmatrix} \{a_1, y\} & a_2 & \cdots & a_{k-1} & \{a_k\} \cup ((X \setminus X\alpha) \setminus \{y\}) \\ b_1 & b_2 & \cdots & b_{k-1} & b_k \end{pmatrix}, \end{aligned} \quad (36)$$

where $y \in Y \setminus X\alpha$. So, $\gamma' \in J(F; k)$ such that $R_{\gamma'} \neq R_\gamma = R_\alpha$ and $\beta = \alpha\gamma'$. $\square$

For the case $k = 1$, the ideal $Q(F; 1) = J(F; 1)$ consisting of all constant maps X_a with range $\{a\} \subseteq Y$. It is easy to verify that $J(F; 1) \setminus \{X_a\}$ is a maximal regular subsemigroup of $Q(F; 1)$ for all $a \in Y$.

Lemma 30. *For $2 \leq k \leq r - 1$, $M = Q(F; k - 1) \cup (J(F; k) \setminus L_\alpha)$ is a maximal regular subsemigroup of $Q(F; k)$ for all $\alpha \in J(F; k)$.*

Proof. Let α be any element in $J(F; k)$. Let $\beta, \gamma \in J(F; k) \setminus L_\alpha$. If $\beta\gamma \in Q(F; k - 1)$, then $\beta\gamma \in M$. If $\beta\gamma \notin Q(F; k - 1)$, then $\beta\gamma \neq 0$ in $P_k = Q(F; k)/Q(F; k - 1)$; that is, $\beta\gamma \in R_\beta \cap L_\gamma$ by Lemma 22. Thus, $\beta\gamma \in L_\gamma \neq L_\alpha$ and so $\beta\gamma \in M$. Hence, M is a subsemigroup of $Q(F; k)$. We prove M is regular. Let $\lambda \in M$. If $\lambda \in Q(F; k - 1)$, then λ is a regular element in $Q(F; k - 1)$. If $\lambda \in J(F; k) \setminus L_\alpha$, we write

$$\lambda = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}, \quad (37)$$

where $a_1, a_2, \dots, a_k \in Y$ and $A_i \cap Y \neq \emptyset$ for all $1 \leq i \leq k$. Since $X\alpha \neq Y$, there exists $y \in Y \setminus X\alpha$ such that $y \in A_j$

for some $j \in \{1, 2, \dots, k\}$. We choose $d_{a_i} \in A_i \cap Y$ for all $i \in \{1, 2, \dots, j-1, j+1, \dots, k\}$ and define

$$\mu = \begin{pmatrix} a_1 & \cdots & a_{j-1} & a_j & a_{j+1} & \cdots & a_{k-1} & \{a_k\} \cup (X \setminus X\lambda) \\ d_{a_1} & \cdots & d_{a_{j-1}} & y & d_{a_{j+1}} & \cdots & d_{a_{k-1}} & d_{a_k} \end{pmatrix}. \quad (38)$$

Then $\mu \in J(F; k)$, $X\mu \neq X\alpha$, and $\lambda = \lambda\mu\lambda$. Hence, M is a regular semigroup. Next, we prove M is maximal. Let S be a regular subsemigroup of $Q(F; k)$ such that $M \subsetneq S \subseteq Q(F; k)$. Then there exists $\xi \in S \setminus M$; that is, $\xi \in L_\alpha \cap S$. Let τ be any element in $L_\alpha = L_\xi$. Then there is $\gamma \in J(F; k)$ such that $\gamma \notin L_\xi$ and $\tau = \gamma\xi$ by Lemma 28; that is, $\gamma \in M \subseteq S$. Thus, $\tau = \gamma\xi \in S$. Hence, $L_\alpha \subseteq S$ and therefore $S = Q(F; k)$. $\square$

Lemma 31. For $2 \leq k \leq r-1$, $M = Q(F; k-1) \cup (J(F; k) \setminus R_\alpha)$ is a maximal regular subsemigroup of $Q(F; k)$ for all $\alpha \in J(F; k)$.

Proof. Let α be any element in $J(F; k)$. Let $\beta, \gamma \in J(F; k) \setminus R_\alpha$. If $\beta\gamma \in Q(F; k-1)$, then $\beta\gamma \in M$. If $\beta\gamma \notin Q(F; k-1)$, then $\beta\gamma \neq 0$ in P_k and so $\beta\gamma \in R_\beta \cap L_\gamma$ by Lemma 22. Thus, $\beta\gamma \in R_\beta \neq R_\alpha$. So, $\beta\gamma \in M$. Hence, M is a subsemigroup of $Q(F; k)$. We prove M is regular. Let $\lambda \in M$. If $\lambda \in Q(F; k-1)$, then λ is a regular element in $Q(F; k-1)$. If $\lambda \in J(F; k) \setminus R_\alpha$, we write

$$\lambda = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}, \quad (39)$$

where $a_1, a_2, \dots, a_k \in Y$ and $A_i \cap Y \neq \emptyset$ for all $1 \leq i \leq k$. We choose $d_{a_i} \in A_i \cap Y$ for all $1 \leq i \leq k$ and define

$$\lambda' = \begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & \{a_k\} \cup (X \setminus X\lambda) \\ d_{a_1} & d_{a_2} & \cdots & d_{a_{k-1}} & d_{a_k} \end{pmatrix}. \quad (40)$$

Then $\lambda' \in J(F; k)$ and $\lambda = \lambda\lambda'\lambda$. If $\lambda' \notin R_\alpha$, then λ' is a regular element in M . If $\lambda' \in R_\alpha$, we define

$$\lambda'' = \begin{pmatrix} \{a_1, y\} & a_2 & \cdots & a_{k-1} & \{a_k\} \cup ((X \setminus X\alpha) \setminus \{y\}) \\ d_{a_1} & d_{a_2} & \cdots & d_{a_{k-1}} & d_{a_k} \end{pmatrix}, \quad (41)$$

where $y \in Y \setminus X\lambda$ since $k < r$. We obtain $\lambda'' \notin R_{\lambda'} = R_\alpha$ and $\lambda = \lambda\lambda''\lambda$. Hence, M is a regular semigroup. For a maximality of M , we let S be a regular subsemigroup of $Q(F; k)$ such that $M \subsetneq S \subseteq Q(F; k)$. Then there exists $\xi \in S \setminus M$; that is, $\xi \in R_\alpha \cap S$. Let τ be any element in $R_\alpha = R_\xi$. Then there is $\gamma \in J(F; k)$ such that $\gamma \notin R_\xi = R_\alpha$ and $\tau = \xi\gamma$ by Lemma 29; that is, $\gamma \in M \subseteq S$. Thus, $\tau \in S$ and so $R_\alpha \subseteq S$. Therefore, $S = Q(F; k)$. $\square$

Theorem 32. Each maximal regular subsemigroup of $Q(F; k)$, $2 \leq k \leq r-1$ must be one of the following forms:

- (1) $Q(F; k-1) \cup (J(F; k) \setminus L_\alpha)$;
- (2) $Q(F; k-1) \cup (J(F; k) \setminus R_\beta)$,

where $\alpha, \beta \in J(F; k)$, L_α is the $\mathcal{L}$ -class containing α in $J(F; k)$, and R_β is the $\mathcal{R}$ -class containing β in $J(F; k)$.

Proof. By Lemmas 30 and 31, both (1) and (2) are maximal regular subsemigroups of $Q(F; k)$.

On the other hand, let M be an arbitrary maximal regular subsemigroup of $Q(F; k)$. It is easy to see that $M \cup Q(F; k-1)$ is a regular subsemigroup of $Q(F; k)$ such that $M \subseteq M \cup Q(F; k-1) \subseteq Q(F; k)$. Then $M = M \cup Q(F; k-1)$ or $M \cup Q(F; k-1) = Q(F; k)$ by the maximality of M . If $M \cup Q(F; k-1) = Q(F; k)$, then $J(F; k) = M \cap J(F; k) \subseteq M$. We obtain $M \subseteq Q(F; k) = \langle J(F; k) \rangle \subseteq M$ by Lemma 25; this implies that $M = Q(F; k)$, a contradiction. Whence $M = M \cup Q(F; k-1)$ and so $Q(F; k-1) \subseteq M$, assume that $L_\alpha \cap M \neq \emptyset$ and $R_\alpha \cap M \neq \emptyset$ for all $\alpha \in J(F; k)$. We first prove that $E(J(F; k)) \subseteq M$. Let e be an idempotent in $J(F; k)$. Thus, $L_e \cap M \neq \emptyset \neq R_e \cap M$. By Lemma 24, $L_e \cap M$ and $R_e \cap M$ contain idempotents, say f and g respectively. So, $f\mathcal{L}e$ and $g\mathcal{R}e$. Since $L_f \cap R_g = L_e \cap R_e$ contains the idempotent e , $fg \in R_f \cap L_g \subseteq J(F; k)$ by Lemma 23. Since $fg \in M$ and M is regular, $fg = (fg)h(fg)$ for some $h \in M$. We see that $f(ghf)g = fg \in J(F; k)$, which implies that $ghf \in J(F; k)$; that is, $(gh)f \neq 0$ in P_k . Then by Lemma 22, we get $(gh)f \in R_{gh} \cap L_f$. Also, we have $g(hf) \in R_g \cap L_{hf}$. Whence $ghf \in L_f \cap R_g = L_e \cap R_e = H_e$, the group $\mathcal{H}$ -class contains e . So, there exists a positive integer t such that $e = (ghf)^t \in M$ since $g, h, f \in M$. Therefore, $E(J(F; k)) \subseteq M$. It follows from Lemma 27 that $Q(F; k) = \langle E(J(F; k)) \rangle \subseteq M \subseteq Q(F; k)$; hence $M = Q(F; k)$, a contradiction. Therefore, $L_\alpha \cap M = \emptyset$ for some $\alpha \in J(F; k)$ or $R_\beta \cap M = \emptyset$ for some $\beta \in J(F; k)$. If $M \cap L_\alpha = \emptyset$ for some $\alpha \in J(F; k)$, then

$$M \subseteq Q(F; k-1) \cup (J(F; k) \setminus L_\alpha) \subsetneq Q(F; k). \quad (42)$$

Thus, $M = Q(F; k-1) \cup (J(F; k) \setminus L_\alpha)$ by the maximality of M . If $M \cap R_\beta = \emptyset$ for some $\beta \in J(F; k)$, then

$$M \subseteq Q(F; k-1) \cup (J(F; k) \setminus R_\beta) \subsetneq Q(F; k). \quad (43)$$

Thus, $M = Q(F; k-1) \cup (J(F; k) \setminus R_\beta)$ by the maximality of M . $\square$

Notice that the number of $\mathcal{L}$ -classes in $J(F; k)$ equals $\binom{r}{k}$ and the number of $\mathcal{R}$ -classes in $J(F; k)$ equals $S(r, k)k^{n-r}$ where $S(r, k)$ is the Stirling number of the second kind; see [18, Section 2] for details. Therefore, the number of maximal regular subsemigroups of $Q(F; k)$ is equal to $\binom{r}{k} + S(r, k)k^{n-r}$ when $2 \leq k \leq r-1$.

We shall normally write T_n instead of $T(X)$. For $1 \leq k \leq n$, define

$$J_k = \{\alpha \in T_n : |X\alpha| = k\}, \quad (44)$$

$$K(n, k) = \{\alpha \in T_n : |X\alpha| \leq k\}.$$

It is well-known that $K(n, k) = J_1 \cup J_2 \cup \cdots \cup J_k$ and $K(n, k)$ is an ideal of T_n for all $1 \leq k \leq n$. If $n = r$, then $F_{n,r} = F_{n,n} = T_n$, $J(F; k) = J_k$, and $Q(F; k) = K(n, k)$. Therefore, we establish the following corollary which first appeared in [8, Theorem 2].

Corollary 33. Each maximal regular subsemigroup of $K(n, k)$, $2 \leq k \leq n-1$ must be one of the following forms:

$$(1) K(n, k-1) \cup (J_k \setminus L_\alpha);$$

$$(2) K(n, k-1) \cup (J_k \setminus R_\beta),$$

where $\alpha, \beta \in J_k$, L_α is the $\mathcal{L}$ -class containing α in J_k of T_n , and R_β is the $\mathcal{R}$ -class containing β in J_k of T_n .

We note that if $n = r$, then we immediately obtain that the number of maximal regular subsemigroups of $K(n, k)$ is equal to $\binom{n}{k} + S(n, k)$ when $2 \leq k \leq n-1$.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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