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Research Article

Ordered Structures of Constructing Operators for Generalized Riesz Systems

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A sequence $\{\varphi_n\}$ in a Hilbert space $\mathscr{H}$ with inner product $< \cdot, \cdot >$ is called a generalized Riesz system if there exist an ONB $\boldsymbol{e} = \{e_n\}$ in $\mathscr{H}$ and a densely defined closed operator T in $\mathscr{H}$ with densely defined inverse such that $\{e_n\} \subset D(T) \cap D((T^{-1})^*)$ and $Te_n = \varphi_n$, $n = 0, 1, \cdots$, and $(\boldsymbol{e}, T)$ is called a constructing pair for $\{\varphi_n\}$ and T is called a constructing operator for $\{\varphi_n\}$. The main purpose of this paper is to investigate under what conditions the ordered set C_φ of all constructing operators for a generalized Riesz system $\{\varphi_n\}$ has maximal elements, minimal elements, the largest element, and the smallest element in order to find constructing operators fitting to each of the physical applications.

1. Introduction

Generalized Riesz systems can be used to construct some physical operators (non-self-adjoint Hamiltonian, generalized lowering operator, generalized raising operator, number operator, etc.) [1–3]. Then these operators provide a link to *quasi-Hermitian quantum mechanics*, and its relatives. Many researchers have investigated such operators both from the mathematical point of view and for their physical applications [4–9]. Let $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\boldsymbol{e}, T)$. Then, putting $\psi_n^T = (T^{-1})^* e_n$, $n = 0, 1, \cdots$, $\{\varphi_n\}$ and $\{\psi_n^T\}$ are biorthogonal sequences, that is, $< \varphi_n, \psi_m^T >= \delta_{nm}$, $n, m = 0, 1, \cdots$. For any $\{\alpha_n\} \subset \mathscr{C}$ we can define the operators: $H_\varphi^{\boldsymbol{\alpha}} := TH_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1}$, $A_\varphi^{\boldsymbol{\alpha}} := TA_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1}$, and $B_\varphi^{\boldsymbol{\alpha}} := TB_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1}$, where $H_{\boldsymbol{e}}^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_n e_n \otimes \overline{e}_n$, $A_{\boldsymbol{e}}^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_{n+1} e_{n+1} \otimes \overline{e}_n$, and $B_{\boldsymbol{e}}^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_{n+1} e_{n+1} \otimes \overline{e}_n$ are standard self-adjoint Hamiltonian, lowering operator, and raising operator for $\{e_n\}$, respectively, where for $x, y \in \mathscr{H}$, $(x \otimes y)\xi :=< \xi, y > x$, $\xi \in \mathscr{H}$. Since

$$H_\varphi^{\boldsymbol{\alpha}} \varphi_n = \alpha_n \varphi_n,$$

$$A_\varphi^{\boldsymbol{\alpha}} \varphi_n = \begin{cases} 0, & n = 0 \\ \alpha_n \varphi_{n-1}, & n = 1, 2, \cdots \end{cases} \tag{1}$$

$$B_\varphi^{\boldsymbol{\alpha}} \varphi_n = \alpha_{n+1} \varphi_{n+1}, \quad n = 0, 1, \cdots,$$

$H_\varphi^{\boldsymbol{\alpha}}$, $A_\varphi^{\boldsymbol{\alpha}}$, and $B_\varphi^{\boldsymbol{\alpha}}$ are called the non-self-adjoint Hamiltonian, the generalized lowering operator, and the generalized raising operator for $\{\varphi_n\}$, respectively. The physical operators of the extended quantum harmonic oscillator and the Swanson model are of this form (see Examples 9–11 in Section 3).

From this fact, it seems to be important to consider under what conditions biorthogonal sequences are generalized Riesz systems and in [1–3] we have investigated this problem. In this paper, we shall focus on the following facts: physical operators defined by a generalized Riesz system $\{\varphi_n\}$ depend on constructing pairs; for example, their operators may not be densely defined for some constructing pairs. On the other hand, if there exists a dense subspace $\mathscr{D}$ in $\mathscr{H}$ for a constructing pair $(\boldsymbol{e}, T)$ which is a core for T such that $H_{\boldsymbol{e}}^{\boldsymbol{\alpha}}\mathscr{D} \subset \mathscr{D}$, $A_{\boldsymbol{e}}^{\boldsymbol{\alpha}}\mathscr{D} \subset \mathscr{D}$, and $B_{\boldsymbol{e}}^{\boldsymbol{\alpha}}\mathscr{D} \subset \mathscr{D}$, then they have an algebraic structure; in detail, the O-algebra on $\mathscr{D}$ is defined by the restrictions of the operators $A_\varphi^{\boldsymbol{\alpha}}$ and $B_\varphi^{\boldsymbol{\alpha}}$ to $\mathscr{D}$ [10]. Thus it seems to be important to find a constructing pair fitting to each of the physical applications. From this reason, in this paper we shall investigate the properties of constructing pairs for a generalized Riesz system.

In Section 2, we shall investigate the basic properties of constructing operators. Let $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\boldsymbol{e}, T)$. The constructing operators for $\{\varphi_n\}$ are unique for the fixed ONB $\boldsymbol{e}$ in $\mathscr{H}$ if $\{\varphi_n\}$ is a Riesz basis; that is, T and T^{-1} are bounded, but they are not unique in general. So, we investigate the set $C_{\boldsymbol{e},\varphi}$ of all constructing operators for $\boldsymbol{e}$. In Proposition 1, we shall show that it is possible to fix an ONB $\boldsymbol{e} = \{e_n\}$ in $\mathscr{H}$ without loss of generality for our study in this paper. Hence, we fix an ONB $\boldsymbol{e}$ in $\mathscr{H}$ and denote $C_{\boldsymbol{e},\varphi}$ by C_φ for simplicity. We consider the following problem: *Is any sequence $\{\psi_n\}$ which is biorthogonal to $\{\varphi_n\}$ a generalized Riesz system?*

Here we put

$$C_\varphi^N := \left\{T \in C_\varphi; \left(T^{-1}\right)^* e_n = \psi_n,\ n = 0, 1, \cdots\right\}. \tag{2}$$

Then we shall show in Proposition 5 that if $C_\varphi^N \neq \emptyset$, then $\{\psi_n\}$ is a generalized Riesz system and $(\boldsymbol{e}, (T^{-1})^*)$ is a constructing pair for $\{\psi_n\}$ for every $T \in C_\varphi^N$, and the mapping

$$T \in C_\varphi^N \longmapsto \left(T^{-1}\right)^* \in C_\psi^N \tag{3}$$

is a bijection, where C_ψ is the set of all constructing operators for $\{\psi_n\}$ and

$$C_\psi^N := \left\{K \in C_\psi;\ \left(K^{-1}\right)^* e_n = \varphi_n,\ n = 0, 1, \cdots\right\}. \tag{4}$$

Furthermore, we shall show in Proposition 6 that if there exists a bounded operator T_0 in C_φ, then $C_\varphi = \{T_0\}$ and $C_\psi^N = \{(T_0^{-1})^*\}$.

In Section 3, we shall consider the ordered set C_φ with order $\subset$ and investigate under what conditions the ordered set C_φ has a maximal element, a minimal element, the smallest element, and the largest element. First we have shown that if $D_\varphi :=$ linear span $\{\varphi_n\}$ is dense in $\mathscr{H}$, then $C_\varphi = C_\varphi^N$ and there exists the smallest element of C_φ, and furthermore if D_φ and $D(\varphi) := \{x \in \mathscr{H};\ \sum_{k=0}^{\infty} | < x, \varphi_k > |^2 < \infty\}$ is dense in $\mathscr{H}$, there exist the smallest element of C_φ and the largest element of C_ψ^N, and in particular, if $\{\varphi_n\}$ and $\{\psi_n\}$ are regular biorthogonal sequences in $\mathscr{H}$, that is, both D_φ and D_ψ are dense in $\mathscr{H}$, then $C_\varphi = C_\varphi^N$, $C_\psi = C_\psi^N$, and C_φ has the smallest element and the largest element. Next we shall consider the case when D_φ is not necessarily dense in $\mathscr{H}$. In Theorem 14, we shall show that for a subset $\mathscr{F}$ of C_φ if there exists a closed operator A in $\mathscr{H}$ such that $T \subset A$ for all $T \in \mathscr{F}$, then $\mathscr{F}$ has a maximal element, and furthermore, if there exists a closed operator B in $\mathscr{H}$ such that $(T^{-1})^* \subset B$ for all $T \in \mathscr{F}$, then $\mathscr{F}$ have a maximal element and a minimal element.

For the existence of the smallest element of C_φ and of the largest element of C_φ, we shall show in Theorem 16 that if there exist closed operators A and B in $\mathscr{H}$ such that $T \subset A$ and $(T^{-1})^* \subset B$ for all $T \in C_\varphi$, then C_φ has the smallest element and the largest element. Furthermore, for a biorthogonal pair $(\{\varphi_n\}, \{\psi_n\})$ of generalized Riesz systems satisfying $C_\varphi = C_\varphi^N$ and $C_\psi = C_\psi^N$, we shall show in Theorem 18 that C_φ and C_ψ have the smallest element and the largest element, respectively, if and only if there exist closed operators A and B in $\mathscr{H}$ such that $T \subset A$ and $K \subset B$ for all $T \in C_\varphi$ and $K \in C_\psi$. These results seem to be useful to find fitting constructing operators for each physical model because every closed operator T in $\mathscr{H}$ satisfying $T_S \subset T \subset T_L$ belongs to C_φ, where T_S is the smallest element of C_φ and T_L is the largest element of C_φ.

2. The Basic Properties of Constructing Operators

In this section, we shall investigate the basic properties of constructing operators. Let $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\boldsymbol{e}, T)$. It is easily shown that if $\{\varphi_n\}$ is a Riesz basis, then the constructing operator T for $\{\varphi_n\}$ is unique for $\boldsymbol{e}$ (see Proposition 1 in detail). But, in general, the constructing operators for $\{\varphi_n\}$ are not unique, and so we put

$$C_{\boldsymbol{e},\varphi} := \{T; (\boldsymbol{e}, T) \text{ is a constructing pair for } \{\varphi_n\}\}. \tag{5}$$

First, we investigate the relationship between $C_{\boldsymbol{e},\varphi}$ and $C_{\boldsymbol{f},\varphi}$ for the other ONB $\boldsymbol{f} = \{f_n\}$ in $\mathscr{H}$.

Proposition 1. *Let $T \in C_{\boldsymbol{e},\varphi}$ and $\boldsymbol{f} = \{f_n\}$ be any ONB in $\mathscr{H}$. Then the following statements hold.*

(1) $(\boldsymbol{f}, TU^*)$ is a constructing pair for $\{\varphi_n\}$, where U is a unitary operator on $\mathscr{H}$ defined by $Ue_n = f_n$, $n = 0, 1, \cdots$, and

$$C_{\boldsymbol{f},\varphi} = \left\{TU^*;\ T \in C_{\boldsymbol{e},\varphi}\right\}. \tag{6}$$

(2) For the non-self-adjoint Hamiltonian, the generalized lowering operator, and the generalized raising operator for $\{\varphi_n\}$, we have

$$\begin{aligned} TH_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1} &= TU^*H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}UT^{-1}, \\ TA_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1} &= TU^*A_{\boldsymbol{f}}^{\boldsymbol{\alpha}}UT^{-1}, \\ TB_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1} &= TU^*B_{\boldsymbol{f}}^{\boldsymbol{\alpha}}UT^{-1}. \end{aligned} \tag{7}$$

Proof. (1) This is almost trivial.

(2) This follows from

$$\begin{aligned} D\left(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}\right) &= UD\left(H_{\boldsymbol{e}}^{\boldsymbol{\alpha}}\right), \\ H_{\boldsymbol{e}}^{\boldsymbol{\alpha}} &= U^*H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}U, \\ D\left(A_{\boldsymbol{f}}^{\boldsymbol{\alpha}}\right) &= UD\left(A_{\boldsymbol{e}}^{\boldsymbol{\alpha}}\right), \\ A_{\boldsymbol{e}}^{\boldsymbol{\alpha}} &= U^*A_{\boldsymbol{f}}^{\boldsymbol{\alpha}}U, \\ D\left(B_{\boldsymbol{f}}^{\boldsymbol{\alpha}}\right) &= UD\left(B_{\boldsymbol{e}}^{\boldsymbol{\alpha}}\right), \\ B_{\boldsymbol{e}}^{\boldsymbol{\alpha}} &= U^*B_{\boldsymbol{f}}^{\boldsymbol{\alpha}}U. \end{aligned} \tag{8}$$

□

By Proposition 1, we have the following.

Corollary 2. *Let $T \in C_{\boldsymbol{e},\varphi}$ and $T^* = U|T^*|$ be the polar decomposition of T^*. Then $\boldsymbol{f} := U^*\boldsymbol{e}$ is an ONB in $\mathscr{H}$ and $|T^*| = TU \in C_{\boldsymbol{f},\varphi}$. Furthermore, we have*

$$TH_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1} = \left|T^*\right| H_{\boldsymbol{f}}^{\boldsymbol{\alpha}} \left|T^*\right|^{-1},$$

$$TA_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1} = \left|T^*\right| A_f^{\boldsymbol{\alpha}} \left|T^*\right|^{-1},$$
$$TB_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1} = \left|T^*\right| B_f^{\boldsymbol{\alpha}} \left|T^*\right|^{-1}. \tag{9}$$

Thus we may fix an ONB $\boldsymbol{e} = \{e_n\}$ in $\mathscr{H}$ without loss of generality for investigating the properties of $C_{\boldsymbol{e},\varphi}$, and so throughout this paper, we fix an ONB $\boldsymbol{e}$ in $\mathscr{H}$ and denote $C_{\boldsymbol{e},\varphi}$ by C_φ for simplicity. Next we consider the following problem: *Suppose that $(\{\varphi_n\},\{\psi_n\})$ is a biorthogonal pair such that $\{\varphi_n\}$ is a generalized Riesz system. Then, is $\{\psi_n\}$ also a generalized Riesz system?*

Let $T \in C_\varphi$ and $\psi_n^T := (T^{-1})^* e_n$, $n = 0,1,\cdots$. Then $(\{\varphi_n\},\{\psi_n^T\})$ is a biorthogonal pair and $\{\psi_n^T\}$ is a generalized Riesz system with a constructing pair $(\boldsymbol{e},(T^{-1})^*)$. If $\psi_n^T = \psi_n$, $n = 0,1\cdots$, then $\{\psi_n\}$ is a generalized Riesz system with a constructing pair $(\boldsymbol{e},(T^{-1})^*)$. But, the equality $\psi_n^T := (T^{-1})^* e_n = \psi_n$, $n = 0,1,\cdots$ does not necessarily hold. To consider when this equality holds, we define the operators $T_{\varphi,\boldsymbol{e}}^0$, $T_{\varphi,\boldsymbol{e}}$, and $T_{\boldsymbol{e},\varphi}$ for any sequence $\{\varphi_n\}$ in $\mathscr{H}$ as follows:

$$T_{\varphi,\boldsymbol{e}}^0 := \text{the linear operator defined by } T_{\varphi,\boldsymbol{e}}^0 e_n = \varphi_n, \quad n = 0,1,\cdots,$$
$$T_{\varphi,\boldsymbol{e}} := \sum_{n=0}^{\infty} \varphi_n \otimes \overline{e}_n, \tag{10}$$
$$T_{\boldsymbol{e},\varphi} := \sum_{n=0}^{\infty} e_n \otimes \overline{\varphi}_n.$$

These operators have played an important role for our studies [3] and also in this paper. By Lemma 2.1, 2.2 in [3] we have the following.

Lemma 3. *(1) $T_{\varphi,\boldsymbol{e}}^0$ and $T_{\varphi,\boldsymbol{e}}$ are densely defined linear operators in $\mathscr{H}$ such that*

$$T_{\varphi,\boldsymbol{e}} \supset T_{\varphi,\boldsymbol{e}}^0,$$
$$T_{\varphi,\boldsymbol{e}}^0 e_n = T_{\varphi,\boldsymbol{e}} e_n = \varphi_n, \quad n = 0,1,\cdots. \tag{11}$$

(2) $D(T_{\boldsymbol{e},\varphi}) = D(\varphi) := \{x \in \mathscr{H};\ \sum_{n=0}^{\infty} | < x, \varphi_n > |^2 < \infty\}$ and $(T_{\varphi,\boldsymbol{e}}^0)^* = T_{\varphi,\boldsymbol{e}}^* = T_{\boldsymbol{e},\varphi}$.

(3) $T_{\varphi,\boldsymbol{e}}^0$ is closable if and only if $T_{\varphi,\boldsymbol{e}}$ is closable if and only if $D(\varphi)$ is dense in $\mathscr{H}$. If this holds, then

$$\overline{T}_{\varphi,\boldsymbol{e}}^0 = \overline{T}_{\varphi,\boldsymbol{e}} = \left(T_{\boldsymbol{e},\varphi}\right)^*. \tag{12}$$

From now on, let $(\{\varphi_n\},\{\psi_n\})$ be a biorthogonal pair.

Lemma 4. *Suppose that $\{\varphi_n\}$ is a generalized Riesz system and $T \in C_\varphi$. Then the following statements are equivalent.*

(i) $\psi_n^T := (T^{-1})^* e_n = \psi_n$, $n = 0,1,\cdots$.

(ii) $D_\psi \subset D(T^*)$.

If this holds true, then T is called natural.

Proof. (i)⟹(ii) This is trivial.

(ii)⟹(i) By definition of $T_{\varphi,\boldsymbol{e}}^0$, we have $T_{\varphi,\boldsymbol{e}}^0 \subset T$. Furthermore, by Lemma 3, (2), we have

$$T^* \subset \left(T_{\varphi,\boldsymbol{e}}^0\right)^* = T_{\boldsymbol{e},\varphi}. \tag{13}$$

Take an arbitrary $n \in \boldsymbol{N} \cup \{0\}$. Then, since

$$\left\langle T_{\varphi,\boldsymbol{e}}^0 e_k, \psi_n \right\rangle = \langle \varphi_k, \psi_n \rangle = \delta_{kn} = \langle e_k, e_n \rangle \tag{14}$$

for $k = 0,1,\cdots$, we have $\psi_n \in D((T_{\varphi,\boldsymbol{e}}^0)^*) = D(T_{\boldsymbol{e},\varphi})$ and $T_{\boldsymbol{e},\varphi}\psi_n = e_n$. Hence it follows from (13) that

$$T^* \psi_n = T_{\boldsymbol{e},\varphi}\psi_n = e_n. \tag{15}$$

Thus, we have

$$\psi_n = \left(T^*\right)^{-1} e_n = \psi_n^T. \tag{16}$$

This completes the proof. □

We denote the set of all natural constructing operators for $\{\varphi_n\}$ by C_φ^N; that is,

$$C_\varphi^N = \left\{T \in C_\varphi; \psi_n^T := \left(T^{-1}\right)^* e_n = \psi_n, n = 0,1,\cdots\right\}. \tag{17}$$

Then we have the following.

Proposition 5. *Suppose that $\{\varphi_n\}$ is a generalized Riesz system. Then the following statements hold.*

(1) If $C_\varphi^N \neq \emptyset$, then $\{\psi_n\}$ is a generalized Riesz system and $(\boldsymbol{e},(T^{-1})^*)$ is a constructing pair for $\{\psi_n\}$ for every $T \in C_\varphi^N$.

(2) Suppose that $\{\psi_n\}$ is also a generalized Riesz system and put

$$C_\psi = \{K;\ (\boldsymbol{e},K) \ \text{is a constructing pair for}\ \{\psi_n\}\},$$
$$C_\psi^N = \left\{K \in C_\psi;\ \varphi_n^K := \left(K^{-1}\right)^* e_n = \varphi_n,\ n = 0,1,\cdots\right\}. \tag{18}$$

Then the mapping

$$T \in C_\varphi^N \longmapsto \left(T^{-1}\right)^* \in C_\psi^N \tag{19}$$

is a bijection.

(3) Suppose that $T_0 \in C_\varphi^N$ and $T \in C_\varphi$ satisfying $T \subset T_0$ or $T_0 \subset T$. Then $T \in C_\varphi^N$. Similarly, suppose that $K_0 \in C_\psi^N$ and $K \in C_\psi$ satisfying $K \subset K_0$ or $K_0 \subset K$. Then $K \in C_\psi^N$.

Proof. The statements (1) and (2) are easily shown.

(3) Suppose that $T \subset T_0$. Then, since $(T_0^{-1})^* \subset (T^{-1})^*$, it follows that $\psi_n = (T_0^{-1})^* e_n = (T^{-1})^* e_n = \psi_n^T$, $n = 0,1,\cdots$, which implies that $T \in C_\varphi^N$. Similarly, we can show $T \in C_\varphi^N$ in case that $T_0 \subset T$ and can show $K \in C_\psi^N$ in case that $K \subset K_0$ or $K_0 \subset K$. This completes the proof. □

As for the uniqueness of constructing operators for a generalized Riesz system we have the following.

Proposition 6. *Let $\{\varphi_n\}$ be a generalized Riesz system. Then the following statements hold.*

(1) Suppose that $\{\varphi_n\}$ is a Riesz basis, then $\{\psi_n\}$ is also a Riesz basis, $C_\varphi = \{\overline{T}_{\varphi,\boldsymbol{e}}\} = \{T_{\boldsymbol{e},\psi}^{-1}\}$ and $C_\psi = \{\overline{T}_{\psi,\boldsymbol{e}}\} = \{T_{\boldsymbol{e},\varphi}^{-1}\}$.

(2) Suppose that D_φ and $D(\varphi)$ are dense in $\mathscr{H}$. Then we have the following.

(i) If there exists an element T_0 of C_φ such that T_0 is bounded, then $C_\varphi = \{T_0\} = \{\overline{T}_{\varphi,\boldsymbol{e}}\}$ and $C_\psi^N = \{(T_0^{-1})^*\} = \{T_{\boldsymbol{e},\varphi}^{-1}\}$.

(ii) If there exists an element T_0 of C_φ such that T_0^{-1} is bounded, then $C_\psi = \{(T_0^{-1})^*\} = \{\overline{T}_{\psi,\boldsymbol{e}}\}$ and $C_\varphi^N = \{T_0\} = \{T_{\boldsymbol{e},\psi}^{-1}\}$.

Proof. (1) Since $\{\varphi_n\}$ is a Riesz basis, there exists an element T_0 of C_φ such that T_0 and T_0^{-1} are bounded, which implies that $T = T_0 = \overline{T}_{\varphi,\boldsymbol{e}} = T_{\boldsymbol{e},\psi}^{-1}$ and $(T^{-1})^* = (T_0^{-1})^* = \overline{T}_{\psi,\boldsymbol{e}} = T_{\boldsymbol{e},\varphi}^{-1}$ for all $T \in C_\varphi$.

(2) (i) Since $\overline{T}_{\varphi,\boldsymbol{e}} \subset T_0$ and T_0 is bounded, we have $\overline{T}_{\varphi,\boldsymbol{e}} = T_0$. Take an arbitrary $T \in C_\varphi$. Then, since $\overline{T}_{\varphi,\boldsymbol{e}} \subset T$ and $\overline{T}_{\varphi,\boldsymbol{e}}$ is bounded, we have $\overline{T}_{\varphi,\boldsymbol{e}} = T$. Thus, $C_\varphi = \{\overline{T}_{\varphi,\boldsymbol{e}}\}$. We show $C_\psi^N = \{T_{\boldsymbol{e},\varphi}^{-1}\}$. Take an arbitrary $K \in C_\psi^N$. Since $(K^{-1})^* e_n = \varphi_n = \overline{T}_{\varphi,\boldsymbol{e}} e_n$ and $\overline{T}_{\varphi,\boldsymbol{e}}$ is bounded, it follows that $(K^{-1})^* = \overline{T}_{\varphi,\boldsymbol{e}}$, which implies $K = T_{\boldsymbol{e},\varphi}^{-1}$. Thus, $C_\psi^N = \{T_{\boldsymbol{e},\varphi}^{-1}\}$.

(ii) This is similarly shown. □

3. Ordered Structures of C_φ

In this section, we shall consider the ordered set C_φ of all constructing operators for a generalized Riesz system $\{\varphi_n\}$ with order $\subset$ and investigate when C_φ has maximal elements, minimal elements, the largest element, and the smallest element. The following result gives a motivation to study the ordered structures of C_φ

Lemma 7. *Suppose that $T, S \in C_\varphi$ and $T \subset S$. Then, for any linear operator A such that $T \subset A \subset S$, the closure $\overline{A}$ of A belongs to C_φ.*

Proof. This is trivial. □

For biorthogonal sequences satisfying density-conditions, we have the following.

Proposition 8. *The following statements hold.*

(1) Suppose that D_φ is dense in $\mathscr{H}$. Then, $\{\varphi_n\}$ is a generalized Riesz system and $C_\varphi = C_\varphi^N$, and $\overline{T}_{\varphi,\boldsymbol{e}}$ is the smallest element of C_φ. Furthermore, suppose that $D(\varphi)$ is dense in $\mathscr{H}$. Then, $T_{\boldsymbol{e},\varphi}^{-1}$ is the largest element of C_ψ^N.

(2) Suppose that D_ψ is dense in $\mathscr{H}$. Then, $\{\psi_n\}$ is a generalized Riesz system and $C_\psi = C_\psi^N$, and $\overline{T}_{\psi,\boldsymbol{e}}$ is the smallest element in C_ψ. Furthermore, suppose that $D(\psi)$ is dense in $\mathscr{H}$. Then, $T_{\boldsymbol{e},\psi}^{-1}$ is the largest element in C_φ.

(3) Suppose that $(\{\varphi_n\}, \{\psi_n\})$ is regular; that is, both D_φ and D_ψ are dense in $\mathscr{H}$. Then, $\{\varphi_n\}$ and $\{\psi_n\}$ are generalized Riesz systems and $C_\varphi = C_\varphi^N$ and $C_\psi = C_\psi^N$, and $\overline{T}_{\varphi,\boldsymbol{e}}$ is the smallest element in C_φ, $\overline{T}_{\psi,\boldsymbol{e}}$ is the smallest element in C_ψ, $T_{\boldsymbol{e},\varphi}^{-1}$ is the largest element in C_ψ, and $T_{\boldsymbol{e},\psi}^{-1}$ is the largest element in C_φ.

Proof. (1) We can show using Lemma 3 that $\{\varphi_n\}$ is a generalized Riesz system with a constructing pair $(\boldsymbol{e}, \overline{T}_{\varphi,\boldsymbol{e}})$ and the constructing operator $\overline{T}_{\varphi,\boldsymbol{e}}$ for $\{\varphi_n\}$ is the smallest element in C_φ. For more detail, refer to [3]. Furthermore, a sequence $\{\psi_n\}$ which is biorthogonal to $\{\varphi_n\}$ is unique. In fact, let $\{\psi_n\}$ and $\{\psi'_n\}$ be any sequences in $\mathscr{H}$ which are biorthogonal to $\{\varphi_n\}$. Then, since $<\psi_n, \varphi_m> = \delta_{nm} = <\psi'_n, \varphi_m>$ for $n, m = 0, 1, \cdots$ and D_φ is dense in $\mathscr{H}$, we have $\psi_n = \psi'_n$ for every $n = 0, 1, \cdots$. We show $C_\varphi = C_\varphi^N$. Take an arbitrary $T \in C_\varphi$. Then, $\{\psi_n^T\}$ is biorthogonal to $\{\varphi_n\}$. By the uniqueness of biorthogonal sequences to $\{\varphi_n\}$, we have $\psi_n^T = \psi_n$, $n = 0, 1, \cdots$, which implies that $T \in C_\varphi^N$ and $C_\varphi = C_\varphi^N$. Suppose that D_φ and $D(\varphi)$ are dense in $\mathscr{H}$. We show that $T_{\boldsymbol{e},\varphi}^{-1}$ is the largest element in C_ψ^N. Since $D(T_{\boldsymbol{e},\varphi}^{-1}) = T_{\boldsymbol{e},\varphi} D(T_{\boldsymbol{e},\varphi}) \supset T_{\boldsymbol{e},\varphi} D_\psi = D_{\boldsymbol{e}}$, $T_{\boldsymbol{e},\varphi}^{-1}$ is a densely defined closed operator in $\mathscr{H}$, and since $D(T_{\boldsymbol{e},\varphi}) = D(\varphi)$, it has a densely defined inverse $T_{\boldsymbol{e},\varphi}$. Furthermore, since $T_{\boldsymbol{e},\varphi}\psi_n = e_n$, $n = 0, 1, \cdots$, $\psi_n = T_{\boldsymbol{e},\varphi}^{-1} e_n$, $n = 0, 1, \cdots$. Thus we have $T_{\boldsymbol{e},\varphi}^{-1} \in C_\psi$. Since $((T_{\boldsymbol{e},\varphi}^{-1})^{-1})^* = T_{\boldsymbol{e},\varphi}^* = \overline{T}_{\varphi,\boldsymbol{e}} \in C_\varphi$, it follows that $T_{\boldsymbol{e},\varphi}^{-1} \in C_\psi^N$. Next we show that $T_{\boldsymbol{e},\varphi}^{-1}$ is the largest element in C_ψ^N. Take an arbitrary $K \in C_\psi^N$. Then $(K^{-1})^* \in C_\varphi$, and so $(K^{-1})^* e_n = \varphi_n$, $n = 0, 1, \cdots$. Hence we have $T_{\varphi,\boldsymbol{e}} \subset (K^{-1})^*$. Thus $K \subset T_{\boldsymbol{e},\varphi}^{-1}$, and so $T_{\boldsymbol{e},\varphi}^{-1}$ is the largest element in C_ψ^N.

(2) This is proved at the same way as (1).

(3) Since $D(\varphi) \supset D_\psi$ and $D(\psi) \supset D_\varphi$, it follows that $D(\varphi)$ and $D(\psi)$ are dense in $\mathscr{H}$, which implies by (1) and (2) that the statement (3) holds. □

Here we give some physical examples. Let $\{f_n\}$, $n = 0, 1, \cdots$, be an ONB in $L^2(\mathbb{R})$ consisting of the Hermite functions which is contained in the Schwartz space $\mathcal{S}(\mathbb{R})$ of all infinitely differential rapidly decreasing functions on $\mathbb{R}$. We define the moment operator p and the position operator q by

$$\begin{aligned} &D(p) \\ &\quad := \text{the set of all differentiable functions } f \text{ on } \mathbb{R} \text{ such that} \\ &\frac{df}{dx} \in L^2(\mathbb{R}), \\ &(pf)(x) := -i\frac{df}{dx}, \quad f \in D(p) \end{aligned} \tag{20}$$

and

$$\begin{aligned} &D(q) := \left\{ f \in L^2(\mathbb{R});\ \int_{-\infty}^{\infty} |xf(x)|^2\, dx < \infty \right\}, \\ &(qf)(x) := xf(x), \quad f \in D(q). \end{aligned} \tag{21}$$

Then p and q are self-adjoint operators in $L^2(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ is a core for p and q, and furthermore $p\mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and $q\mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, and $[p,q] := pq - qp = -i\mathbb{1}$ on $\mathcal{S}(\mathbb{R})$. Next we define the standard bosonic operators $a, a^\dagger$ by

$$a = \frac{1}{\sqrt{2}}(q + ip),\qquad a^\dagger = \frac{1}{\sqrt{2}}(q - ip). \tag{22}$$

Then,

$$af_n = \begin{cases} 0, & n = 0 \\ \sqrt{n} f_{n-1} & n = 1,2,\cdots, \end{cases} \tag{23}$$
$$a^\dagger f_n = \sqrt{n+1} f_{n+1}, \quad n = 0,1,\cdots$$

and $[a, a^\dagger] = \mathbb{1}$ on $\mathcal{S}(\mathbb{R})$.

Example 9 (the extended quantum harmonic oscillator). The Hamiltonian of this model is the non-self-adjoint operator, introduced in [11, 12],

$$H_\beta := \frac{\beta}{2}\left(p^2 + q^2\right) + i\sqrt{2}p = \beta a^\dagger a + \left(a - a^\dagger\right) + \frac{\beta}{2}\mathbb{1}, \quad \beta > 0. \tag{24}$$

We put

$$\varphi_0^{(\beta)} := e^{1/\beta^2} f_0\left(x - \frac{\sqrt{2}}{\beta}\right) = \frac{e^{1/\beta^2}}{\pi^{1/4}} e^{-(1/2)(x-\sqrt{2}/\beta)^2}. \tag{25}$$

Then, $\varphi_0^{(\beta)} \in \mathcal{S}(\mathbb{R})$ and $\varphi_0^{(\beta)} = e^{1/\beta^2} U(1/\beta) f_0 = e^{(a+a^\dagger)/\beta} f_0$, where $U(1/\beta)$ is a unitary operator defined by $U(1/\beta) := e^{(1/\beta)(a^\dagger - a)} = e^{-(\sqrt{2}/\beta)ip}$. Hence we can define a sequence $\boldsymbol{\varphi}_\beta := \{\varphi_n^{(\beta)}\}$ in $\mathcal{S}(\mathbb{R})$ by

$$\varphi_n^{(\beta)} := \frac{1}{\sqrt{n!}}\left(a^\dagger + \frac{1}{\beta}\right)^n \varphi_0^{(\beta)}, \quad n = 1,2,\cdots. \tag{26}$$

Similarly, we define a sequence $\boldsymbol{\psi}_\beta := \{\psi_n^{(\beta)}\}$ in $\mathcal{S}(\mathbb{R})$ as follows:

$$\psi_0^{(\beta)} := e^{1/\beta^2} f_0\left(x + \frac{\sqrt{2}}{\beta}\right),\qquad \psi_n^{(\beta)} := \frac{1}{\sqrt{n!}}\left(a^\dagger - \frac{1}{\beta}\right)^n \psi_0^{(\beta)}, \quad n = 1,2,\cdots. \tag{27}$$

Then $\{\varphi_n^{(\beta)}\}$ and $\{\psi_n^{(\beta)}\}$ are regular biorthogonal sequences in $L^2(\mathbb{R})$ which are generalized Riesz systems with constructing pairs $(\{f_n\}, e^{(a^\dagger+a)/\beta})$ and $(\{f_n\}, e^{-(a^\dagger+a)/\beta})$, respectively, and

$$\overline{A}_{\boldsymbol{\varphi}_\beta} := \overline{e^{(a^\dagger+a)/\beta} a e^{-(a^\dagger+a)/\beta}} = a - \frac{1}{\beta},\qquad \overline{B}_{\boldsymbol{\varphi}_\beta} := \overline{e^{(a^\dagger+a)/\beta} a^\dagger e^{-(a^\dagger+a)/\beta}} = a^\dagger + \frac{1}{\beta}. \tag{28}$$

By Proposition 8, $T_{\boldsymbol{\varphi}_\beta, \boldsymbol{f}}$ is the smallest constructing operator and $T_{\boldsymbol{f},\boldsymbol{\psi}_\beta}^{-1}$ is the largest constructing operator for $\{\varphi_n^{(\beta)}\}$ and $T_{\boldsymbol{\varphi}_\beta,\boldsymbol{f}} \subset e^{(a^\dagger+a)/\beta} \subset T_{\boldsymbol{f},\boldsymbol{\psi}_\beta}^{-1}$. Similarly, $T_{\boldsymbol{\psi}_\beta,\boldsymbol{f}}$ is the smallest constructing operator and $T_{\boldsymbol{f},\boldsymbol{\varphi}_\beta}^{-1}$ is the largest constructing operator for $\{\psi_n^{(\beta)}\}$ and $T_{\boldsymbol{\psi}_\beta,\boldsymbol{f}} \subset e^{-(a^\dagger+a)/\beta} \subset T_{\boldsymbol{f},\boldsymbol{\varphi}_\beta}^{-1}$.

The following example is a modification of the non-self-adjoint Hamiltonian H_β in Example 9 exchanging the momentum operator p with the position operator q.

Example 10. We introduce a non-self-adjoint Hamiltonian

$$H'_\beta := \frac{\beta}{2}\left(p^2 + q^2\right) + \sqrt{2}iq = \beta a^\dagger a + i\left(a + a^\dagger\right) + \frac{\beta}{2}, \quad \beta > 0. \tag{29}$$

We define sequences $\boldsymbol{\varphi}'_\beta := \{\varphi'_n\}$ and $\boldsymbol{\psi}'_\beta := \{\psi'_n\}$ in $\mathcal{S}(\mathbb{R})$ as follows:

$$\varphi'_0 := e^{-(i/\beta)a^\dagger} f_0,\qquad \varphi'_n := \frac{1}{\sqrt{n!}}\left(a^\dagger + \frac{i}{\beta}\right)^n \varphi'_0, \quad n = 1,2,\cdots \tag{30}$$

and

$$\psi'_0 := e^{1/\beta^2} e^{(i/\beta)a^\dagger} f_0,\qquad \psi'_n := \frac{1}{\sqrt{n!}}\left(a^\dagger - \frac{i}{\beta}\right)^n \psi'_0, \quad n = 1,2,\cdots. \tag{31}$$

Then $\boldsymbol{\varphi}'_\beta$ and $\boldsymbol{\psi}'_\beta$ are regular biorthogonal sequences in $L^2(\mathbb{R})$ which are generalized Riesz systems with constructing pairs $(\{f_n\}, T)$ and $(\{f_n\}, T^{-1})$, respectively, where $T := \overline{e^{-(i/\beta)a^\dagger} e^{(i/\beta)a}}$ and $T^{-1} = \overline{e^{-(i/\beta)a} e^{(i/\beta)a^\dagger}}$, and $\overline{TaT^{-1}} = a + i/\beta$, $\overline{Ta^\dagger T^{-1}} = a^\dagger + i/\beta$, $\overline{T^{-1}aT} = a - i/\beta$, and $\overline{T^{-1}a^\dagger T} = a^\dagger - i/\beta$.

Example 11 (the Swanson model). The Swanson Hamiltonian, introduced in [11, 13], is a non-self-adjoint Hamiltonian

$$\begin{aligned} H_\theta &:= \frac{1}{2}\left(p^2 + q^2\right) - \frac{i}{2}\tan 2\theta\left(p^2 - q^2\right) \\ &= a^\dagger a + \frac{i}{2}\tan 2\theta\left(a^2 + \left(a^\dagger\right)^2\right) + \frac{1}{2}\mathbb{1}, \\ &\qquad \theta \neq 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right). \end{aligned} \tag{32}$$

We define sequences $\boldsymbol{\varphi}_\theta := \{\varphi_n^{(\theta)}\}$ and $\boldsymbol{\psi}_\theta := \{\psi_n^{(\theta)}\}$ in $L^2(\mathbb{R})$ as follows:

$$\begin{aligned} \varphi_0^{(\theta)} &:= c_0 \sum_{k=0}^{\infty} e^{-i(\tan\theta/2)(a^\dagger)^2} f_0 \\ &= c_0 \sum_{k=0}^{\infty} (-i\tan\theta)^k \sqrt{\frac{(2k-1)!!}{(2k)!!}} f_{2k}, \\ \varphi_n^{(\theta)} &:= \frac{1}{\sqrt{n!}}\left(\cos\theta a^\dagger + i\sin\theta a\right)^n \varphi_0^{(\theta)} \end{aligned} \tag{33}$$

and

$$\psi_0^{(\theta)} := d_0 \sum_{k=0}^{\infty} e^{i(\tan\theta/2)(a^\dagger)^2} f_0$$
$$= d_0 \sum_{k=0}^{\infty} (i \tan\theta)^k \sqrt{\frac{(2k-1)!!}{(2k)!!}} f_{2k}, \quad (34)$$
$$\psi_n^{(\theta)} := \frac{1}{\sqrt{n!}} \left(\cos\theta a^\dagger - i \sin\theta a\right)^n \psi_0^{(\theta)},$$

where $(2k)!! = 2k(2k-2)\cdots 4 \cdot 2$, $(2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1$, and c_0 and d_0 are constants satisfying $< \varphi_0^{(\theta)}, \psi_0^{(\theta)} >= 1$. Then $\boldsymbol{\varphi}_\theta$ and $\boldsymbol{\psi}_\theta$ are regular biorthogonal sequences in $L^2(\mathbb{R})$ contained in $\mathcal{S}(\mathbb{R})$ which are generalized Riesz systems with constructing operators $T_\theta := e^{i(\theta/2)(a^2-(a^\dagger)^2)}$ and T_θ^{-1}, respectively. For the generalized lowering operator $A_\theta := T_\theta a T_\theta^{-1}$ and the raising operator $B_\theta := T_\theta a^\dagger T_\theta^{-1}$, we have

$$A_\theta = (\cos\theta)\, a + i (\sin\theta)\, a^\dagger,$$
$$B_\theta = (\cos\theta)\, a^\dagger + i (\sin\theta)\, a. \quad (35)$$

By Proposition 8, $T_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}}$ (resp., $T_{\boldsymbol{\psi}_\theta,\boldsymbol{f}}$) is the smallest constructing operator and $T_{\boldsymbol{\psi}_\theta,\boldsymbol{f}}^{-1}$ (resp., $T_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}}^{-1}$) is the largest constructing operator for $\boldsymbol{\varphi}_\theta$ (resp., $\boldsymbol{\psi}_\theta$) and every closed operator T (resp., K) in $L^2(\mathbb{R})$ satisfying $T_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}} \subset T \subset T_{\boldsymbol{\psi}_\theta,\boldsymbol{f}}^{-1}$ (resp., $T_{\boldsymbol{\psi}_\theta,\boldsymbol{f}} \subset K \subset T_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}}^{-1}$) is a constructing operator for $\boldsymbol{\varphi}_\theta$ (resp., $\boldsymbol{\psi}_\theta$).

All physical models discussed above are regular cases, but it seems to be mathematically meaningful to study nonregular cases and furthermore the studies may become useful for physical applications in future. Let $\{\varphi_n\}$ be a generalized Riesz system. First we investigate under what conditions C_φ has maximal elements and minimal elements.

Let C be a totally ordered subset of C_φ. Then, it is easily shown that $(T^{-1})^* e_n = (S^{-1})^* e_n, n = 0, 1, \cdots$, for any $T, S \in C$. Hence we may put

$$\psi_n^C = \left(T^{-1}\right)^* e_n, \quad T \in C,\ n = 0, 1, \cdots. \quad (36)$$

We have the following statements.

Lemma 12. *Let C be any totally ordered subset of C_φ. The following statements hold.*

(1) Suppose that $\cap_{T\in C} D(T^*)$ is dense in $\mathcal{H}$. Then there exists an upper bounded element G of C.

(2) Suppose that $\cap_{T\in C} R(T)$ is dense in $\mathcal{H}$. Then there exists a lower bounded element S of C.

(3) Suppose that $\cap_{T\in C} D(T^*)$ and $\cap_{T\in C} R(T)$ are dense in $\mathcal{H}$. Then for every linear operator A such that $S \subset A \subset G$, the closure $\overline{A}$ of A belongs to C_φ.

Proof. (1) We put

$$D(G) = \bigcup_{T\in C} D(T),$$
$$Gx = T_0 x, \quad x \in D(G), \quad (37)$$

where T_0 is an operator in C whose domain $D(T_0)$ contains x. Since C is totally ordered, it follows that $D(G)$ is a subspace in $\mathcal{H}$ and $Tx = T_0 x$ for any operators T, T_0 in C whose domains contain x. Hence, G does not depend on the method of choosing $T_0 \in C$ whose domain contains x. Thus G is a well-defined densely defined linear operator in $\mathcal{H}$ such that $T \subset G$ for all $T \in C$. We show that G is closable. Indeed, we may show

$$\bigcap_{T\in C} D(T^*) = D(G^*),$$
$$G^* y = T_0^* y, \quad (38)$$

where $T_0 \in C$ whose domain $D(T_0)$ contains x. Take an arbitrary $y \in \cap_{T\in C} D(T^*)$. Then, we have

$$\langle Gx, y\rangle = \langle T_0 x, y\rangle = \langle x, T_0^* y\rangle \quad (39)$$

for all $x \in D(G)$, where $T_0 \in C$ whose domain contains x. Hence, $y \in D(G^*)$ and $G^* y = T_0^* y$. Since $T \subset G$ for all $T \in C$, $D(G^*) \subset \cap_{T\in C} D(T^*)$ is trivial. Thus, (38) holds. By (38) and the assumption of (1), $D(G^*)$ is dense in $\mathcal{H}$, that is, G is closable. Next we show that G has a densely defined inverse. Suppose that $Gx = 0$, $x \in D(G)$. Then $T_0 x = 0$ for some $T_0 \in C$, and so $x = 0$ since T_0 has an inverse. Thus G has an inverse. Since $R(T) \subset R(G)$ and $R(T)$ is dense in $\mathcal{H}$ for all $T \in C$, it follows that the inverse of G is densely defined, which implies that the closure $\overline{G}$ of G has a densely defined closed operator in $\mathcal{H}$ such that $\overline{G} \supset T$ for all $T \in C$. Finally we show

$$\{e_n\} \subset D\left(\overline{G}\right) \cap D\left(\left(G^*\right)^{-1}\right),$$
$$\overline{G} e_n = \varphi_n, \quad n = 0, 1, \cdots. \quad (40)$$

Clearly, $\{e_n\} \subset D(T) \subset D(\overline{G})$ for all $T \in C$. Next we show that for any n there exists an element y_n of $\cap_{T\in C} D(T^*)$ such that

$$e_n = T^* y_n \quad (41)$$

for all $T \in C$. Indeed, take an arbitrary $T \in C$. Since $e_n \in D((T^*)^{-1}) = R(T^*)$, there exists an element y_n^T of $D(T^*)$ such that $e_n = T^* y_n^T$. Let any $T' \in C$. Since C is totally ordered, either $T' \subset T$ or $T \subset T'$ holds. Suppose that $T' \subset T$. Since $T^* \subset (T')^*$, it follows that $y_n^T, y_n^{T'} \in D((T')^*)$ and

$$\left(T'\right)^* y_n^T = T^* y_n^T = e_n = \left(T'\right)^* y_n^{T'}, \quad (42)$$

which implies that $y_n^T = y_n^{T'}$ since $(T')^*$ has inverse. The equality $y_n^T = y_n^{T'}$ is similarly shown in case that $T \subset T'$. Hence, we have that $y_n := y_n^T \in \cap_{T\in C} D(T^*)$ and $T^* y_n = e_n$ for all $T \in C$. Thus (41) holds. By (38) and (41) we have $e_n \in R(G^*) = D((G^*)^{-1})$. Furthermore, we have $\overline{G} e_n = T e_n = \varphi_n$, $n = 0, 1, \cdots$ for all $T \in C$. Thus we have $\overline{G} \in C_\varphi$ and $\overline{G}$ is an upper bounded element of C.

(2) We put

$$D(S) = \cap_{T\in C} D(T),$$
$$Sx = Tx, \quad x \in D(S), \quad (43)$$

where T is any element of C. Since $\{e_n\} \subset \cap_{T\in C} D(T)$ and $T_1 x = T_2 x$ for all $x \in \cap_{T\in C} D(T)$, S is a well-defined densely defined closed operator in $\mathscr{H}$ such that $S \subset T$ for all $T \in C$. Hence, it is sufficient to show $S \in C_\varphi$. Since $S \subset T$ for all $T \in C$ and T has the inverse, S has the inverse. Furthermore, we may show

$$R(S) = \cap_{T\in C} R(T). \tag{44}$$

In fact, take an arbitrary $y \in \cap_{T\in C} R(T)$. Since C is totally ordered, there exists an element x of $D(S) = \cap_{T\in C} D(T)$ such that $y = Tx$ for all $T \in C$. Hence, $y = Sx \in R(S)$. The inverse inclusion $R(S) \subset \cap_{T\in C} R(T)$ is clear. Hence (44) holds. By the assumption and (44), $R(S) = D(S^{-1})$ is dense in $\mathscr{H}$. Furthermore, since $\{e_n\} \subset D(T)$ for all $T \in C$, we have $\{e_n\} \subset D(S)$ and $Se_n = Te_n = \varphi_n$, $n = 0, 1, \cdots$. Since $S \subset T$, it follows that $\{e_n\} \subset D((T^*)^{-1}) \subset D((S^*)^{-1})$. Thus, $S \in C_\varphi$ and it is a lower bound of C.

(3) This follows from (1) and (2). □

For a subset $\mathscr{F}$ of C_φ, we put

$$\mathscr{F}_\psi := \left\{ \left(T^{-1}\right)^*;\ T \in \mathscr{F} \right\}. \tag{45}$$

Then we have the following.

Lemma 13. *Let $\mathscr{F}$ be a subset of C_φ. Then, T_0 is a maximal (resp., minimal, the largest and the smallest) element of $\mathscr{F}$ if and only if $(T_0^{-1})^*$ is a minimal (resp., maximal, the smallest and the largest) element of $\mathscr{F}_\psi$.*

Proof. Suppose that T_0 is a maximal element of $\mathscr{F}$. Take an arbitrary $K \in \mathscr{F}_\psi$ satisfying $K \subset (T_0^{-1})^*$. Then we have that $K = (T^{-1})^*$ for some $T \in \mathscr{F}$ and $T_0 \subset T$, which implies by the maximality of T_0 that $T = T_0$ and $K = (T_0^{-1})^*$. Thus, $(T_0^{-1})^*$ is a minimal element of $\mathscr{F}_\psi$. Furthermore, we can similarly show that if $(T_0^{-1})^*$ is a minimal element of $\mathscr{F}_\psi$, then T_0 is a maximal element $\mathscr{F}$. The other statements are similarly shown. □

Theorem 14. *Let $\mathscr{F}$ be a subset of C_φ. Then we have the following:*

(1) *The following statements are equivalent:*

(i) *$\mathscr{F}$ has a maximal element.*
(ii) *There exists a closed operator A in $\mathscr{H}$ such that $T \subset A$ for all $T \in \mathscr{F}$.*
(iii) *$\mathscr{F}_\psi$ has a minimal element.*

(2) *The following statements are equivalent:*

(i) *$\mathscr{F}$ has a minimal element.*
(ii) *There exists a closed operator B in $\mathscr{H}$ such that $(T^{-1})^* \subset B$ for all $T \in \mathscr{F}$.*
(iii) *$\mathscr{F}_\psi$ has a maximal element.*

(3) *The following statements are equivalent:*

(i) *$\mathscr{F}$ has a maximal element and a minimal element.*
(ii) *There exist closed operators A and B in $\mathscr{H}$ such that $T \subset A$ and $(T^{-1})^* \subset B$ for all $T \in \mathscr{F}$.*
(iii) *$\mathscr{F}_\psi$ has a maximal element and a minimal element.*

Proof. (1) (i)⟹(ii) This is trivial.

(ii)⟹(i) Suppose that there exists a closed operator A in $\mathscr{H}$ such that $T \subset A$ for all $T \in \mathscr{F}$. Then for any totally ordered subset C of $\mathscr{F}$ we have $D(A^*) \subset \bigcap_{T\in C} D(T^*)$. Hence, it follows that $\bigcap_{T\in C} D(T^*)$ is dense in $\mathscr{H}$, which implies by Lemma 12 that $\mathscr{F}$ has an upper bounded element. By Zorn's lemma, $\mathscr{F}$ has a maximal element.

(i)⟺(iii) This follows from Lemma 13.

(2) (i)⟹(ii) This is trivial.

(ii)⟹(i) Suppose that there exists a closed operator A in $\mathscr{H}$ such that $(T^{-1})^* \subset B$ for all $T \in \mathscr{F}$. Then we can similarly show that $\mathscr{F}_\psi$ has a maximal element, which implies by Lemma 13 that $\mathscr{F}$ has a minimal element.

(i)⟺(iii) This follows from Lemma 13.

(3) This follows from (1) and (2). This completes the proof. □

We remark that the closed operators A and B in Theorem 14 do not need any other conditions, for example, the existence of inverse.

By Theorem 14, we have the following.

Corollary 15. *Let $T_0 \in C_\varphi$ and put $\mathscr{F}_{T_0} = \{T \in C_\varphi;\ T_0 \subset T\}$. Then the following statements hold.*

(1) Suppose that there exists a closed operator A in $\mathscr{H}$ such that $T \subset A$ for all $T \in \mathscr{F}_{T_0}$. Then there exists a maximal element of C_φ which is an extension of T_0.

(2) Suppose that there exists a closed operator B in $\mathscr{H}$ such that $(T^{-1})^* \subset B$ for all $T \in \mathscr{F}_{T_0}$. Then there exists a minimal element of C_φ which is a restriction of T_0.

Proof. (1) By Theorem 14, $\mathscr{F}_{T_0}$ has a maximal element T_1. Here we show that T_1 is a maximal element of C_φ. Indeed, this follows since $T \in \mathscr{F}_{T_0}$ for any element T of C_φ satisfying $T_1 \subset T$. We can similarly show (2). □

Next we investigate the existence of the smallest element and of the largest element of C_φ.

Theorem 16. *C_φ has the smallest element and the largest element if and only if there exist closed operators A and B in $\mathscr{H}$ such that $T \subset A$ and $(T^{-1})^* \subset B$ for all $T \in C_\varphi$.*

Proof. Suppose that there exist closed operators A and B in $\mathscr{H}$ such that $T \subset A$ and $(T^{-1})^* \subset B$ for all $T \in C_\varphi$. We define an operator T_0 as follows:

$$\begin{aligned} D(T_0) &= \bigcap_{T\in C_\varphi} D(T), \\ T_0 x &= Tx, \quad x \in D(T_0), \end{aligned} \tag{46}$$

where T is an element of C_φ. Take an arbitrary $x \in D(T_0)$ and $T_1, T_2 \in C_\varphi$. Since $x \in D(T_1)$, $x \in D(T_2)$, $T_1 \subset A$ and $T_2 \subset A$, we have

$$T_1 x = T_2 x = Ax. \tag{47}$$

Thus, T_0 does not depend on the method of choosing $T \in C_\varphi$, and so T_0 is well defined. Since $\{e_n\} \subset \cap_{T\in C_\varphi} D(T) = D(T_0)$, T_0 is a densely defined closed operator in $\mathscr{H}$ such that $T_0 \subset T$ for all $T \in C_\varphi$. Since $(T^{-1})^* \subset B$ for all $T \in C_\varphi$, we have $D(B^*) \subset \bigcap_{T\in C_\varphi} R(T)$, which implies that $\bigcap_{T\in C_\varphi} R(T)$ is dense in $\mathscr{H}$. Hence, we can prove at the same way as the proof of Lemma 12 (2) that T_0 is the smallest element C_φ. Next we show that C_φ has the largest element. Take an arbitrary $T \in C_\varphi$. Then ψ_T is a generalized Riesz system with a constructing operator $(T^{-1})^*$ and $K \subset B$ and $(K^{-1})^* \subset A$ for all $K \in C^N_{\psi_T}$. Hence, as shown above there exists the smallest element K_1 of $C^N_{\psi_T}$, and so $K_1 = (T_1^{-1})^*$ for some $T_1 \in C_\varphi$ and $(T_1^{-1})^* \subset (T^{-1})^*$. Thus $T \subset T_1$ and T_1 is the largest element of C_φ. The converse is trivial. This completes the proof. □

As seen in Section 2, for a biorthogonal pair $(\{\varphi_n\}, \{\psi_n\})$, the equality $(T^{-1})^* e_n = \psi_n$, $n = 0, 1, \cdots$ does not necessarily hold for all $T \in C_\varphi$. From this fact we define the notion of *natural* pair of generalized Riesz systems.

Definition 17. A biorthogonal pair $(\{\varphi_n\}, \{\psi_n\})$ of generalized Riesz systems is said to be natural, if $C_\varphi = C^N_\varphi$ and $C_\psi = C^N_\psi$, that is, $(T^{-1})^* e_n = \psi_n$ for all n and $T \in C_\varphi$ and $(K^{-1})^* e_n = \varphi_n$ for all n and $K \in C_\psi$.

Theorem 18. *Let $(\{\varphi_n\}, \{\psi_n\})$ be a natural pair of generalized Riesz systems. Then C_φ and C_ψ have the smallest element and the largest element, respectively, if and only if there exist closed operators A and B in $\mathscr{H}$ such that $T \subset A$ and $K \subset B$ for all $T \in C_\varphi$ and $K \in C_\psi$.*

Proof. This is shown using Theorem 16 for the generalized Riesz systems for $\{\varphi_n\}$ and $\{\psi_n\}$. □

For a generalized Riesz system $\{\varphi_n\}$, suppose that there exist the largest element T_L of C_φ and the smallest element T_S of C_φ. Then every closed operator T in $\mathscr{H}$ satisfying $T_S \subset T \subset T_L$ is a constructing operator of $\{\varphi_n\}$, and so we can construct all kinds of non-self-adjoint Hamiltonians $TH^{\boldsymbol{\alpha}}_{\boldsymbol{e}}T^{-1}$, lowering operator $TA^{\boldsymbol{\alpha}}_{\boldsymbol{e}}T^{-1}$ and raising operator $TB^{\boldsymbol{\alpha}}_{\boldsymbol{e}}T^{-1}$ for $\{\varphi_n\}$. It may be possible to find constructing operators suitable for each of the physical models.

4. Conclusions

All the results presented in this paper are of pure mathematical nature, but we hope that they will be applied to more physical models in future. For example, we argue that cases like the CCR-algebras and their physical applications could probably studied by taking suitable constructing operators for convenient generalized Riesz systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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