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Research Article

A Formulation of L-Isothermic Surfaces in Three-Dimensional Minkowski Space

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The Cartan structure equations are used to study space-like and time-like isothermic surfaces in three-dimensional Minkowski space in a unified framework. When the lines of curvature of a surface constitute an isothermal system, the surface is called isothermic. This condition serves to define a system of one-forms such that, by means of the structure equations, the Gauss-Codazzi equations for the surface are determined explicitly. A Lax pair can also be obtained from these one-forms for both cases, and, moreover, a nonhomogeneous Schrödinger equation can be associated with the set of space-like surfaces.

1. Introduction

The study of isothermic surfaces can be traced back to the work of Bianchi and Bour [**1, 2**], as well as to Darboux [**3**]. These surfaces seem to have their origin in work by Lamé motivated by problems in heat conduction. An important subclass of isothermic surfaces was subsequently investigated by Bonnet. The study of these surfaces has seen renewed interest recently with the work of Rogers and Schief [**4–6**]. Rogers established that a Bäcklund transformation for isothermic surfaces is associated with a nonhomogeneous linear Schrödinger equation. This is largely due to the fact that the classical Gauss-Mainardi-Codazzi equations which are associated with surfaces in general are integrable in the sense they possess soliton solutions. Thus, these surfaces can be put in correspondence with solitonic solutions of certain nonlinear partial differential equations. Thus they have a strong appeal to those with interests that range from integrable equations to their associated Bäcklund transformations [**7–9**]. Thus, integrable systems theory can be applied to isothermic surfaces and used to study transformations of these surfaces as well. Consequently, isothermic surfaces constitute an important subclass of surfaces with a connection to solitons.

It is the purpose of this work to study the cases of both space-like and time-like surfaces as well as their immersion in three-dimensional Minkowski space $\mathbb{E}_1^3$ in a unified manner by basing the approach on the structure equations of Cartan and the associated moving frame [**10**]. Suppose that $\Sigma \subset \mathbb{E}_1^3$ is such a surface or manifold to which a first fundamental form is associated. With respect to the larger space $\mathbb{E}_1^3$, there exist both space-like and time-like surfaces residing in this larger space. Thus, a particular Σ could be either one of these two types of object. This is expressed by the fact that there are two ways in which the metric or first fundamental form can be specified intrinsically on the surface. In terms of two local coordinates, the metric may be written with two positive signs, hence a positive signature, or it may be written with a negative signature or alternating signs. In the former case, the surface Σ is referred to as space-like and in the latter case it is called time-like.

To start, let us outline the approach used here. Cartan's equations of structure are formulated in such a way that they are adapted to the signature of the flat metric of the ambient background space $\mathbb{E}_1^3$. These equations are defined in terms of a set of one-forms. By selecting these one-forms in a particular way along with the appropriate choice of signs, the system of structure equations can be restricted to study one of the classes of surface already described. In fact, a set of partial differential equations can be obtained which can be used to describe each of these types of surface. Therefore, the solutions of these equations can be used to describe a corresponding type of surface immersed in E_1^3.

In fact, it can be mentioned quite generally that new integrable equations have been obtained in the case of purely Euclidean space from the Cartan system by exploiting the one-to-one correspondence between the Ablowitz-Kaup-Newell-Segur (AKNS) program [**11, 12**] and the classical theory of surfaces in three dimensions. Its relationship to the problem of embedding surfaces in three-dimensional Euclidean space arises from the fact that the Gauss-Codazzi equations are in this case equivalent to Cartan's equations of structure for $SO(3)$. This correspondence suggests that the soliton connection can be given a deeper structure at the Riemannian level. In fact, in Euclidean space, much effort has been expended to exploit the equivalence between the AKNS systems and surface theory at the metric level in order to construct new nonlinear equations.

Once the structure equations have been formulated in this context, the one-forms can then be chosen for the case of isothermic surfaces. Classically, when the lines of curvature of a surface form an isothermal system, the surface is referred to as isothermic. The case studied here will be the one in which the third fundamental form is conformally flat in the manifold coordinates:

$$III = e^{2\vartheta}\left(du^2 + dv^2\right). \tag{1}$$

When the surface has a third fundamental form diagonal as in (1), it is often called L-isothermic. The prefix will be omitted here. This will lead to an equation whose solutions can be used to express the three fundamental forms which characterize surface. For each type of surface, the function in (1) turns out to be specified by a nonlinear second-order equation. It will be shown finally that Lax pairs can subsequently be formulated for each of the two second-order systems. Out of these results, further equations of physical interest can be developed which are relevant to both types of surface. One way in which these auxiliary equations arise is through compatibility conditions. Moreover, an interesting result is presented by showing that there is an important link between a nonhomogeneous Schrödinger equation and a combination of surface variables which are relevant to the more physical case of space-like isothermic surfaces. Investigations into this area have appeared [**13–15**]. Here the idea is to show how these geometries can be studied consistently by using the moving frame approach by simply altering some of the parameters in the metric.

2. Cartan Formulation of Structure Equations in Three-Space

A Darboux frame $(\mathbf{x}; e_1, e_2, e_3)$ is established on Σ such that the vectors (e_1, e_2) are tangent to surface Σ and e_3 is a normal vector to Σ; hence (e_1, e_2) determines an orientation for Σ. At a point $x \in M$, it is the case that

$$d\mathbf{x} = \omega_i e_i, \tag{2}$$

where $\mathbf{x}$ denotes a position vector in (2), ω_i constitute a basis of one-forms, and index i goes from 1 to 3. A surface Σ can be established by taking

$$\omega_3 = 0, \tag{3}$$

with e_3 a normal vector to Σ. The metric on the ambient background space is taken to have the following form [**13**]:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \eta \end{pmatrix} \tag{4}$$

so there is no loss of generality in defining the surface by means of (3). Moreover, the quantities ϵ and η in (4) can assume one of the two values $\epsilon = \pm 1$ and $\eta = \pm 1$. In the case in which $\epsilon = \eta = 1$, metric g specifies a three-dimensional Euclidean space $\mathbb{E}^3$ as usual. However, it is the intention here to study the cases in which only one of the quantities ϵ or η is taken to be negative. On the one hand, if $\epsilon = -1$ and $\eta = 1$ the surface metric is space-like, and if $\epsilon = 1$ and $\eta = -1$, the surface metric is time-like.

The surfaces immersed in $\mathbb{E}^3_1$ which are studied here have first fundamental form or metric on the surface defined by

$$I = \omega_1^2 + \epsilon\omega_2^2. \tag{5}$$

The two choices of sign for ϵ account for two classes of surface just introduced. For the basis vectors of the Darboux frame, Cartan's structure equations must hold and they are given as follows:

$$\begin{aligned} de_1 &= \omega_{12}e_2 + \omega_{13}e_3, \\ de_2 &= \omega_{21}e_1 + \omega_{23}e_3, \\ de_3 &= \omega_{31}e_1 + \omega_{32}e_2, \end{aligned} \tag{6}$$

$$\omega_{ij} + \omega_{ji} = 0. \tag{7}$$

As in Chern [**12**], we suppose the relative components of the frame field are ω_i and ω_{ij}. These are differential one-forms which depend on the two independent surface coordinates (u, v). To be able to discuss the embedding problem, the second fundamental form for Σ has to be defined as well. It is given by

$$II = \omega_1 \otimes \omega_{13} + \epsilon\omega_2 \otimes \omega_{23}. \tag{8}$$

To formulate and study the case of isothermic surfaces, the complete system of Cartan structure equations is required. Under the convention adopted for g given in (4), these equations can be presented in the notation of Chern [**12**] as

$$\begin{aligned} d\omega_1 &= \omega_2 \wedge \omega_{21}, \\ d\omega_2 &= \omega_1 \wedge \omega_{12}, \end{aligned} \tag{9}$$

$$\omega_1 \wedge \omega_{13} + \epsilon\omega_2 \wedge \omega_{23} = 0, \tag{10}$$

$$\begin{aligned} d\omega_{12} &= \epsilon\eta\omega_{13} \wedge \omega_{32}, \\ d\omega_{13} &= \omega_{12} \wedge \omega_{23}, \\ d\omega_{23} &= \epsilon\omega_{21} \wedge \omega_{13}. \end{aligned} \tag{11}$$

In the case of a space-like Σ, we take $\epsilon = 1$ and $\eta = -1$, so a space-like metric (5) results. For the time-like case $\epsilon = -1$

and $\eta = 1$, and a time-like metric (5) results. Each of these two cases will be studied by defining the one-forms which appear in (9)-(11) appropriately.

Theorem 1. *The Gauss-Codazzi equations (11) for embedding Σ can be expressed in the form of Cartan's structure equations for the group $SL(2,\mathbb{R})$ as*

$$d\sigma^i + \frac{1}{2}c^i_{jk}\sigma^j \wedge \sigma^k = 0. \tag{12}$$

The one-forms σ^i, where $i = 0, 1, 2$, are defined to be

$$\begin{aligned} \sigma^0 &= -\frac{\epsilon^{1/2}}{2i}\omega_{12}, \\ \sigma^1 &= \frac{1}{2\epsilon^{1/2}\eta^{1/2}}\left(i\epsilon^{1/2}\omega_{13} + \omega_{23}\right), \\ \sigma^2 &= -\frac{1}{2\epsilon^{1/2}\eta^{1/2}}\left(i\epsilon^{1/2}\omega_{13} - \omega_{23}\right). \end{aligned} \tag{13}$$

The structure constants of $SL(2,\mathbb{R})$ which appear in (13) are $c^0_{12} = 1$, $c^1_{01} = -c^2_{02} = -2$.

The proof is straightforward, simply substitute the forms (13) into (12), and solve for $d\omega_{12}$, $d\omega_{13}$ and $d\omega_{23}$. Upon carrying this out, system (11) appears directly.

3. Surfaces with Space-Like Metric

To obtain a metric which has a positive signature on surface Σ, the parameters which appear in g in (4) are set to the values $\epsilon = 1$ and $\eta = -1$. Metric g assumes the following form:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{14}$$

The structure equations (9)-(11) then take the following form:

$$\begin{aligned} d\omega_1 &= \omega_2 \wedge \omega_{21}, \\ d\omega_2 &= \omega_1 \wedge \omega_{12}, \end{aligned} \tag{15}$$

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0, \tag{16}$$

$$\begin{aligned} d\omega_{12} &= -\omega_{13} \wedge \omega_{32}, \\ d\omega_{13} &= \omega_{12} \wedge \omega_{23}, \\ d\omega_{23} &= \omega_{21} \wedge \omega_{13}, \end{aligned} \tag{17}$$

$$\omega_{ij} + \omega_{ji} = 0. \tag{18}$$

These equations constitute the Gauss-Codazzi system for the space-like surface.

In order to study isothermic surfaces, the one-forms which are to be used in (15)-(17) are defined in such a way that the third fundamental form is proportional to a flat metric on Σ (1). Given a coordinate chart (u, v) for Σ, the one-forms ω_1 and ω_2 are taken to depend on functions of the coordinate parameters as

$$\begin{aligned} \omega_1 &= A du, \\ \omega_2 &= B dv. \end{aligned} \tag{19}$$

The functions $A = A(u, v)$ and $B = B(u, v)$ depend on both (u, v) in general. Further, define the one-forms as follows:

$$\begin{aligned} \omega_{13} &= \kappa_1 A du, \\ \omega_{23} &= \kappa_2 B dv. \end{aligned} \tag{20}$$

Equations (19) and (20) imply that the first three fundamental forms of the surface can be constructed in the following way:

$$\begin{aligned} I &= A^2 du^2 + B^2 dv^2, \\ II &= \kappa_1 A^2 du^2 + \kappa_2 B^2 dv^2, \\ III &= \kappa_1^2 A^2 du^2 + \kappa_2^2 B^2 dv^2. \end{aligned} \tag{21}$$

From the fact that the mean and Gaussian curvatures are given by

$$\begin{aligned} H &= -\mathrm{tr}\left(II \cdot I^{-1}\right), \\ K &= \det\left(II \cdot I^{-1}\right), \end{aligned} \tag{22}$$

κ_1 and κ_2 in (20) can be interpreted as the principal curvatures of the surface. To obtain an expression for ω_{12} in terms of A and B from (19), let us suppose $\omega_{12} = \alpha du + \beta dv$, where functions α and β depend on both coordinates as do A and B. Substituting this ω_{12} into (15), we can solve for α and β and (15) reduces to a pair of identities provided that ω_{12} has the following form"

$$\omega_{12} = -\frac{A_v}{B}du + \frac{B_u}{A}dv. \tag{23}$$

Clearly, forms (19), (20) clearly satisfy (16) automatically. Finally, putting the set of forms into the remaining equations in (17) produces a system which can be used to determine the two functions A and B. In fact, doing so produces a coupled system of partial differential equations which must hold when expressed in terms of all the relevant functions κ_1, κ_2, A and B.

Differentiating ω_{12} in (23) gives

$$d\omega_{12} = \left(\left(\frac{A_v}{B}\right)_v + \left(\frac{B_u}{A}\right)_u\right) du \wedge dv, \tag{24}$$

Thus the first equation in (17) implies that A and B satisfy a second-order equation:

$$\left(\frac{A_v}{B}\right)_v + \left(\frac{B_u}{A}\right)_u - \kappa_1\kappa_2 AB = 0. \tag{25}$$

The next two equations of (17) yield the following pair:

$$\begin{aligned} \kappa_{1,v} + (\log A)_v (\kappa_1 - \kappa_2) &= 0, \\ \kappa_{2,u} + (\log B)_u (\kappa_2 - \kappa_1) &= 0. \end{aligned} \tag{26}$$

Therefore, (25) and (26) constitute the relevant system to be studied.

The condition that the space-like surface Σ be isothermic is that the third fundamental form be conformally flat in terms of the (u, v)-coordinate system like (1). In order to ensure this, it suffices to take

$$A\kappa_1 = B\kappa_2 = e^{\vartheta}. \tag{27}$$

This parameterization can now be used to transform the system of (25)-(26) into a set which depends on only variable $\vartheta = \vartheta(u, v)$.

To this end, differentiate $B\kappa_2$ in (27) with respect to u:

$$(B\kappa_2)_u = B_u\kappa_2 + B\kappa_{2,u} = \vartheta_u e^{\vartheta}. \tag{28}$$

Substituting (28) into the second equation of (26) simplifies to the following form:

$$\vartheta_u e^{\vartheta} - \kappa_1 B_u = 0. \tag{29}$$

By using (27) to eliminate e^{ϑ}, (29) becomes

$$B_u = A\vartheta_u. \tag{30}$$

Similarly, differentiate $A\kappa_1 = e^{\vartheta}$ with respect to v to obtain

$$(A\kappa_1)_v = A_v\kappa_1 + A\kappa_{1,v} = \vartheta_v e^{\vartheta}. \tag{31}$$

Substituting (31) into the first equation in (26) simplifies to

$$A_v = B\vartheta_v. \tag{32}$$

Using (30) and (32) in the second-order equation (25) as well as the fact that $\kappa_1\kappa_2 AB = e^{2\vartheta}$ becomes an equation in terms of only the ϑ variable:

$$\vartheta_{uu} + \vartheta_{vv} - e^{2\vartheta} = 0. \tag{33}$$

To summarize then, the Gauss-Mainardi-Codazzi equations reduce to the following form under (27):

$$\begin{aligned} A_v &= B\vartheta_v, \\ B_u &= A\vartheta_u, \\ \vartheta_{uu} + \vartheta_{vv} - e^{2\vartheta} &= 0. \end{aligned} \tag{34}$$

Based on the results in (34), it is possible to make further links to other types of equations which are of importance in mathematics and physics. These arise by working out the compatibility conditions between them. Suppose two independent functions F and G and related to A and B in the following way:

$$\begin{aligned} 2A &= F + G, \\ 2B &= G - F. \end{aligned} \tag{35}$$

Putting these in the first equation of (34) and collecting like functions on opposite sides and multiplying by $e^{-\vartheta}$ give

$$e^{-\vartheta}(G_u - G\vartheta_u) = (F_u + F\vartheta_u)e^{-\vartheta}. \tag{36}$$

Doing the same thing to the second equation gives

$$-\vartheta_v G e^{-\vartheta} + G_v e^{-\vartheta} = -e^{-2\vartheta}\left(F_v e^{\vartheta} + F\vartheta_v e^{\vartheta}\right). \tag{37}$$

Using the product rule on (36) and (37), the following pair of equations has been obtained:

$$\begin{aligned} \left(e^{-\vartheta}G\right)_u &= e^{-2\vartheta}\left(Fe^{\vartheta}\right)_u, \\ \left(e^{-\vartheta}G\right)_v &= -e^{-2\vartheta}\left(Fe^{\vartheta}\right)_v. \end{aligned} \tag{38}$$

Finally, the desired compatibility condition for F can be obtained by differentiating $(e^{-\vartheta}G)_u$ with respect to v, then $(e^{-\vartheta}G)_v$ with respect to u, and finally equating the results. After multiplying the result by $e^{-\vartheta}$, this simplifies to the following:

$$e^{-\vartheta}F_{uv} = \left(e^{-\vartheta}\right)_{uv} F. \tag{39}$$

To obtain an analogous equation for G, $(e^{\vartheta}F)_u$ is differentiated with respect to v and $(e^{\vartheta}F)_v$ with respect to u. Upon equating them, one obtains

$$e^{\vartheta}G_{uv} = \left(e^{\vartheta}\right)_{uv} G. \tag{40}$$

These steps have proved the following theorem.

Theorem 2. *The compatibility conditions for functions F and G defined in terms of A and B by (35) are specified in terms of the following Moutard equations:*

$$\begin{aligned} e^{-\vartheta}F_{uv} &= \left(e^{-\vartheta}\right)_{uv} F, \\ e^{\vartheta}G_{uv} &= \left(e^{\vartheta}\right)_{uv} G. \end{aligned} \tag{41}$$

Normally, there exists a close connection between the Moutard equation and a transformation called the fundamental transformation between surfaces. We show that a Lax pair exists for the second-order system in (34). Let $\mathbf{X}$, $\mathbf{Y}$ be unit space-like tangent vectors to Σ in $\mathbb{E}^3_1$.

Theorem 3. *Let $\mathbf{X}$, $\mathbf{Y}$ be unit tangent vectors and $\mathbf{N}$ a unit normal to the space-like surface Σ. Define the following matrix system which depends on function ϑ:*

$$\begin{aligned} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{N} \end{pmatrix}_u &= \begin{pmatrix} 0 & -\vartheta_v & \lambda e^{\vartheta} \\ \vartheta_v & 0 & 0 \\ \frac{1}{\lambda}e^{\vartheta} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{N} \end{pmatrix}, \\ \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{N} \end{pmatrix}_v &= \begin{pmatrix} 0 & \vartheta_u & 0 \\ -\vartheta_u & 0 & \lambda e^{\vartheta} \\ 0 & \frac{1}{\lambda}e^{\vartheta} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{N} \end{pmatrix}. \end{aligned} \tag{42}$$

In (42), λ is a spectral parameter. The zero curvature condition for system (42) is satisfied if and only if function ϑ satisfies the second-order equation of (34), namely,

$$\vartheta_{uu} + \vartheta_{vv} - e^{2\vartheta} = 0. \tag{43}$$

Proof. It suffices to differentiate the first matrix equation in (42) with respect to v, and the second with respect to u and require that the results agree identically. In other words, (42) is equivalent to the first-order system:

$$\begin{aligned}
\mathbf{X}_u &= -\vartheta_v \mathbf{Y} + \lambda e^{\vartheta}\mathbf{N},\\
\mathbf{Y}_u &= \vartheta_v \mathbf{X},\\
\mathbf{N}_u &= \frac{1}{\lambda} e^{\vartheta}\mathbf{X},\\
\mathbf{X}_v &= \vartheta_u \mathbf{Y},\\
\mathbf{Y}_v &= -\vartheta_u \mathbf{X} + \lambda e^{\vartheta}\mathbf{N},\\
\mathbf{N}_v &= \frac{1}{\lambda} e^{\vartheta}\mathbf{Y}.
\end{aligned} \tag{44}$$

Condition $\mathbf{X}_{uv} = \mathbf{X}_{vu}$ under (44) reduces to

$$\begin{aligned}
&-\vartheta_{vv}\mathbf{Y} - \vartheta_v\left(-\vartheta_u\mathbf{X} + \lambda e^{\vartheta}\mathbf{N}\right) + \lambda\vartheta_v e^{\vartheta}\mathbf{N} + e^{2\vartheta}\mathbf{Y}\\
&\quad = \vartheta_{uu}\mathbf{Y} + \vartheta_u\vartheta_v\mathbf{X}.
\end{aligned} \tag{45}$$

Simplifying this, the spectral parameter disappears and equality holds exactly when the function ϑ satisfies the second-order equation, $\vartheta_{uu} + \vartheta_{vv} - e^{2\vartheta} = 0$. Similarly, the condition $\mathbf{Y}_{uv} = \mathbf{Y}_{vu}$ is

$$\begin{aligned}
\vartheta_{vv}\mathbf{X} + \vartheta_v\vartheta_u\mathbf{Y} &= -\vartheta_{uu}\mathbf{X} - \vartheta_u\left(-\vartheta_v\mathbf{Y} + \lambda e^{\vartheta}\mathbf{N}\right)\\
&\quad + \lambda\vartheta_u e^{\vartheta}\mathbf{N} + e^{2\vartheta}\mathbf{X}.
\end{aligned} \tag{46}$$

This holds whenever ϑ satisfies this partial differential equation. Finally, $\mathbf{N}_{uv} = \mathbf{N}_{vu}$ simply reduces to

$$\frac{1}{\lambda}\left(\vartheta_v e^{\vartheta}\mathbf{X} + \vartheta_u e^{\vartheta}\mathbf{Y}\right) = \frac{1}{\lambda}\left(\vartheta_u e^{\vartheta}\mathbf{Y} + \vartheta_v e^{\vartheta}\mathbf{X}\right) \tag{47}$$

which is an identity. □

To write the position vector of the surface, it is useful to define the new variable $\mathbf{S}$ in terms of $\mathbf{X}$, $\mathbf{Y}$ as follows:

$$\mathbf{S} = \mathbf{X} + i\mathbf{Y}. \tag{48}$$

Taking $z = u + iv$ to be complex, the following complex derivatives are defined:

$$\begin{aligned}
\partial &= \frac{1}{2}\left(\partial_u - i\partial_v\right),\\
\bar{\partial} &= \frac{1}{2}\left(\partial_u + i\partial_v\right).
\end{aligned} \tag{49}$$

In terms of $\mathbf{S}$ and these derivatives, (44) can be abbreviated to the following form:

$$\begin{aligned}
\partial\mathbf{S} &= -(\partial\vartheta)\,\mathbf{S} + e^{\vartheta}\mathbf{N},\\
\bar{\partial}\mathbf{S} &= \left(\bar{\partial}\vartheta\right)\mathbf{S},\\
\partial\mathbf{N} &= \frac{1}{2}e^{\vartheta}\bar{\mathbf{S}},\\
\bar{\partial}\mathbf{N} &= \frac{1}{2}e^{\vartheta}\mathbf{S}.
\end{aligned} \tag{50}$$

The position vector $\mathbf{r}$ of the space-like surface will be obtained by integration of the following equation:

$$\mathbf{r}_z = \frac{1}{4}\left(P\mathbf{S} + R\bar{\mathbf{S}}\right). \tag{51}$$

To this end, introduce the scalar quantity:

$$\tau = \mathbf{r}\cdot\mathbf{N}, \tag{52}$$

which can be regarded as the distance from the origin to the tangent plane on the space-like surface at the point $\mathbf{r}$. Differentiating τ with respect to z and $\bar{z}$, we find that

$$\begin{aligned}
\partial\tau &= \frac{1}{2}e^{\vartheta}\mathbf{r}\cdot\bar{\mathbf{S}},\\
\bar{\partial}\tau &= \frac{1}{2}e^{\vartheta}\mathbf{r}\cdot\mathbf{S}.
\end{aligned} \tag{53}$$

The position vector $\mathbf{r}$ of the space-like surface therefore admits a decomposition of the following form:

$$\mathbf{r} = e^{-\vartheta}(\partial\tau)\,\mathbf{S} + e^{-\vartheta}\left(\bar{\partial}\tau\right)\bar{\mathbf{S}} - \tau\mathbf{N}. \tag{54}$$

To obtain $\partial\mathbf{r}$ in terms of τ and ϑ, differentiate $\mathbf{r}$ with respect to z and substitute (50):

$$\begin{aligned}
\mathbf{r}_z &= -\vartheta_z e^{-\vartheta}\tau_z\mathbf{S} + e^{-\vartheta}\tau_{zz}\mathbf{S} + e^{-\vartheta}\tau_z\mathbf{S}_z - \vartheta_z e^{-\vartheta}\tau_{\bar{z}}\bar{\mathbf{S}}\\
&\quad + e^{-\vartheta}\tau_{\bar{z}z}\bar{\mathbf{S}} + e^{-\vartheta}\tau_{\bar{z}}\bar{\mathbf{S}}_z - \tau_z\mathbf{N} - \tau\mathbf{N}_z.
\end{aligned} \tag{55}$$

Replacing the first derivatives from (44), this derivative simplifies to

$$\begin{aligned}
\mathbf{r}_z &= e^{-\vartheta}\left(-\vartheta_z\tau_z + \tau_{zz} - \vartheta_z\tau_z\right)\mathbf{S}\\
&\quad + e^{-\vartheta}\left(-\vartheta_z\tau_{\bar{z}} + \tau_{z\bar{z}} + \vartheta_z\tau_{\bar{z}} - \frac{1}{2}\tau e^{2\vartheta}\right)\bar{\mathbf{S}}\\
&\quad + \left(\tau_z - \tau_z\right)\mathbf{N}\\
&= e^{\vartheta}\left(e^{-2\vartheta}\tau_z\right)_z\mathbf{S} + \frac{1}{2}e^{-\vartheta}\left(2\tau_{z\bar{z}} - \tau e^{2\vartheta}\right)\bar{\mathbf{S}}.
\end{aligned} \tag{56}$$

Comparing this result with (51), this procedure allows us to write P and R in terms of ϑ and τ:

$$\begin{aligned}
P &= 4e^{\vartheta}\left(e^{-2\vartheta}\tau_z\right)_z,\\
R &= e^{-\vartheta}\left(4\tau_{z\bar{z}} - 2\tau e^{2\vartheta}\right).
\end{aligned} \tag{57}$$

It has been found that the position vector of the space-like surface is given by $\mathbf{r}$ where the real function τ is a solution of the following equation:

$$\tau_{zz} - 2\vartheta_z \tau_z = \frac{1}{4} e^{\vartheta} P. \tag{58}$$

Finally, it can be shown that (58) is equivalent to an inhomogeneous Schrödinger equation. To do so, a new variable $\Psi(z, \overline{z})$ is introduced and defined as

$$\Psi = e^{-\vartheta} \tau. \tag{59}$$

Differentiating both sides of Ψ with respect to z gives

$$e^{-2\vartheta} \tau_z = e^{-\vartheta} \left(\vartheta_z \Psi + \Psi_z\right), \tag{60}$$

and, after a second time, we have

$$\begin{aligned} \left(e^{-2\vartheta} \tau_z\right)_z &= -\vartheta_z e^{-\vartheta} \left(\vartheta_z \Psi + \Psi_z\right) \\ &\quad + e^{-\vartheta} \left(\vartheta_{zz} \Psi + \vartheta_z \Psi_z + \Psi_{zz}\right). \end{aligned} \tag{61}$$

Substituting this second derivative on the left of (58), the following second-order equation for Ψ results after dividing out $e^{-\vartheta}$ is

$$\Psi_{zz} + \left(\vartheta_{zz} - \vartheta_z^2\right) \Psi = \frac{1}{4} P. \tag{62}$$

Introducing the potential function which is defined in terms of ϑ as $V = \vartheta_{zz} - \vartheta_z^2$, (62) assumes the following form:

$$\Psi_{zz} + V \Psi = \frac{1}{4} P. \tag{63}$$

4. Surfaces with Time-like Metric

To obtain a metric for this case with a time-like structure on Σ it must be that $\epsilon = 1$ and $\eta = -1$ in (4). The metric g then assumes the following form:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{64}$$

Structure equations (9)-(11) then differ by signs and are given by

$$\begin{aligned} d\omega_1 &= \omega_2 \wedge \omega_{21}, \\ d\omega_2 &= \omega_1 \wedge \omega_{12}, \end{aligned} \tag{65}$$

$$\omega_1 \wedge \omega_{13} - \omega_2 \wedge \omega_{23} = 0, \tag{66}$$

$$\begin{aligned} d\omega_{12} &= -\omega_{13} \wedge \omega_{32}, \\ d\omega_{13} &= \omega_{12} \wedge \omega_{23}, \\ d\omega_{23} &= -\omega_{21} \wedge \omega_{13}, \end{aligned} \tag{67}$$

$$\omega_{ij} + \omega_{ji} = 0. \tag{68}$$

The one-forms ω_1 and ω_2 required to define the first fundamental form (5) are taken to be

$$\begin{aligned} \omega_1 &= -A du, \\ \omega_2 &= B dv, \end{aligned} \tag{69}$$

where A, B are functions of the coordinates (u, v). Furthermore, the one-forms ω_{13} and ω_{23} take the following form:

$$\begin{aligned} \omega_{13} &= -\kappa_1 A du, \\ \omega_{23} &= \kappa_2 B dv. \end{aligned} \tag{70}$$

Based on the one-forms (69)-(70), the three fundamental forms for Σ can be written as

$$\begin{aligned} I &= A^2 du^2 - B^2 dv^2, \\ II &= \kappa_1 A^2 du^2 - \kappa_2 B^2 dv^2, \\ III &= \kappa_1^2 A^2 du^2 - \kappa^2 B^2 dv^2. \end{aligned} \tag{71}$$

Since both ω_1 and ω_{13} differ from the previous case, ω_{12} has to be determined again, and it is given by

$$\omega_{12} = \frac{A_v}{B} du + \frac{B_u}{A} dv. \tag{72}$$

Equation (66) is satisfied automatically by this system of forms as well. The remaining three equations (67) can now be computed exactly as before. The conclusion is that a second-order equation results, namely,

$$\left(\frac{B_u}{A}\right)_u - \left(\frac{A_v}{B}\right)_v + \kappa_1 \kappa_2 AB = 0, \tag{73}$$

as in the previous case, and the pair

$$\begin{aligned} \kappa_{1,v} + \left(\log A\right)_v \left(\kappa_1 - \kappa_2\right) &= 0, \\ \kappa_{2,u} + \left(\log B\right)_u \left(\kappa_2 - \kappa_1\right) &= 0. \end{aligned} \tag{74}$$

Both equations in (74) are seen to be identical to their corresponding counterparts in (74). In this case as well, the equations in (73) and (74) can be written in such a way that the fundamental form III is conformally flat assuming the form (1), $A\kappa_1 = B\kappa_2 = e^{\vartheta}$. Since the steps are identical to the previous case, the results are summarized as follows:

$$\begin{aligned} A_v &= B\vartheta_v, \\ B_u &= A\vartheta_u, \\ \vartheta_{uu} - \vartheta_{vv} + e^{2\vartheta} &= 0. \end{aligned} \tag{75}$$

These are exactly analogous to (34), the first two being identical to those of the space-like case. The second-order equation differs by signs from the case (34). Since the first two equations are exactly the same, similar functions F and G can be introduced which are related to A and B as in (35). All the steps which lead to Theorem 2 are unchanged as they involve only the first two equations and are independent of the second-order equation. Thus, a version of Theorem 2 can be formulated here as well. The Lax pair however has to be different since the second-order equation is different.

Theorem 4. *Let* $\mathbf{X}$, $\mathbf{Y}$ *be unit tangent vectors to time-like surface* Σ *and* $\mathbf{N}$ *a unit normal vector to* Σ. *Define the following matrix system in terms of function* ϑ *as*

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\\ \mathbf{N}\end{pmatrix}_u = \begin{pmatrix}0 & -\vartheta_v & e^{\vartheta}\\ \vartheta_v & 0 & 0\\ e^{\vartheta} & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\\ \mathbf{N}\end{pmatrix},$$
$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\\ \mathbf{N}\end{pmatrix}_v = \begin{pmatrix}0 & \vartheta_u & 0\\ -\vartheta_u & 0 & e^{\vartheta}\\ 0 & e^{\vartheta} & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\\ \mathbf{N}\end{pmatrix} \tag{76}$$

The compatibility condition in (u, v) *for this system holds if and only if function* ϑ *satisfies the second-order equation in (75), namely,*

$$\vartheta_{uu} - \vartheta_{vv} + e^{2\vartheta} = 0. \tag{77}$$

The proof of Theorem 4 goes exactly as the proof of (42). To illustrate, the details for the $\mathbf{X}$ *equations will be given. Differentiating the first matrix equation by* v *and the second by* u *and substituting (76) for the first derivatives, it is found that*

$$\begin{aligned}\mathbf{X}_{uv} &= -\vartheta_{vv}\mathbf{Y} - \vartheta_v\left(-\vartheta_u\mathbf{X} + e^{\vartheta}\mathbf{N}\right) + \vartheta_v e^{\vartheta}\mathbf{N} + e^{2\vartheta}\mathbf{Y}\\ &= -\vartheta_{uu}\mathbf{Y} + \vartheta_u\vartheta_v\mathbf{X} = \mathbf{X}_{vu}.\end{aligned} \tag{78}$$

This will be satisfied provided that ϑ *satisfies the second-order equation in (75). A similar result is found to hold for the* $\mathbf{Y}$ *equation and* $\mathbf{N}_{uv} = \mathbf{N}_{vu}$ *holds as an identity.*

Again, if $\mathbf{S}$ is defined exactly as in the previous case, then, in terms of complex derivatives, and using the equations of (76), the system corresponding to (50) is

$$\begin{aligned}\bar{\partial}\mathbf{S} &= -\left(\bar{\partial}\vartheta\right)\bar{\mathbf{S}} + e^{\vartheta}\mathbf{N},\\ \bar{\partial}\mathbf{S} &= (\partial\vartheta)\,\bar{\mathbf{S}},\\ \partial\mathbf{N} &= -\frac{1}{2}e^{\vartheta}\mathbf{S},\\ \bar{\partial}\mathbf{N} &= -\frac{1}{2}e^{\vartheta}\bar{\mathbf{S}}.\end{aligned} \tag{79}$$

Taking $\mathbf{r}_z$ to have the same form (51) and $\mathbf{r}$ given by (54), then differentiating (54) with respect to z and comparing to (51), it is found that

$$\begin{aligned}b_{zz} - \vartheta_z b_z + \vartheta_{\bar{z}} b_{\bar{z}} + \frac{1}{2}e^{2\vartheta}b &= \frac{1}{4}Pe^{\vartheta},\\ b_{z\bar{z}} - \vartheta_z b_{\bar{z}} - \vartheta_{\bar{z}} b_z &= \frac{1}{4}Re^{\vartheta}.\end{aligned} \tag{80}$$

Supposing τ has the form (35), then the first equation in (80) becomes a second-order partial differential equation for the function $\Psi(z, \bar{z})$, namely,

$$\Psi_{zz} + \vartheta_z\Psi_z + \vartheta_{\bar{z}}\Psi_{\bar{z}} + \left(\vartheta_{zz} + \vartheta_z^2 + \frac{1}{2}e^{2\vartheta}\right)\Psi = \frac{1}{4}P. \tag{81}$$

5. Conclusions and Summary

In the Cartan framework, we can discuss isothermic surfaces in Minkowski three-space for both space-like and time-like cases. As Theorems 2 and 4 show, the classical Gauss-Mainardi-Codazzi system associated with isothermic surfaces is integrable in the modern solitonic sense. Bäcklund transformations will exist for both types of surface. The appearance of the Moutard equations (41) in both cases is remarkable, and, subsequently, Sturm-Liouville or Schrödinger equation (62). This leads to the final proposition.

Proposition 5. *Let* V *and* P *satisfy the compatibility condition* $P_{\alpha\beta} = 2P\,\mathrm{Im}\,V$ *and let* Ψ *be a real solution of the inhomogeneous Schrödinger equation (63). Then with* $\tau = e^{\vartheta}\Psi$, *(54) provides a position vector for a space-like isothermic surface.*

Data Availability

This is a theoretical work; no data was involved.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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