Research Article

Close-to-Convexity of Convolutions of Classes of Harmonic Functions

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1. Introduction

Let \( \mathcal{A} \) denote the class of functions that are analytic in the open unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) and let \( \mathcal{A}' \) be the subclass of \( \mathcal{A} \) consisting of functions \( h \) with the normalization \( h(0) = h'(0) - 1 = 0 \). Consider the family of complex-valued harmonic functions \( f = u + iv \), where \( u \) and \( v \) are real harmonic in \( \mathbb{E} \). Such functions can be expressed as \( f = h + \overline{g} \), where \( h \in \mathcal{A} \) and \( g \in \mathcal{A}' \). By Lewy’s Theorem (see [1, 2] or [3]), a necessary and sufficient condition for the harmonic function \( f = h + \overline{g} \) to be locally one-to-one and sense-preserving in \( \mathbb{E} \) is that its Jacobian \( J_f = |h'|^2 - |g'|^2 \) should be positive or equivalently if and only if \( h'(z) \neq 0 \) in \( \mathbb{E} \) and the second complex dilatation \( \omega \) of \( f \) satisfies \( |\omega| = |g'/h'| < 1 \) in \( \mathbb{E} \). In the sequel, without loss of generality, we consider those locally one-to-one and sense-preserving harmonic functions \( f = h + \overline{g} \) that are normalized by \( f(0) = h(0) = 0 \) and \( f_z(0) = 1 \) and have the representation

\[
 f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n; \quad z \in \mathbb{E}. \tag{1}
\]

The Hadamard product or convolution of two power series \( h_1(z) = \sum_{n=1}^{\infty} a_n z^n \) and \( h_2(z) = \sum_{n=1}^{\infty} c_n z^n \) is given by \( h_1(z) \ast h_2(z) = (h_1 \ast h_2)(z) = \sum_{n=1}^{\infty} a_n c_n z^n \). Similarly, the convolution of two harmonic functions \( f_1 = h_1 + \overline{g_1} \) and \( f_2 = h_2 + \overline{g_2} \) is given by \( f_1 \ast f_2 = h_1 + h_2 + \overline{g_1} \ast \overline{g_2} \).

A simply connected proper subdomain \( D \) of the complex domain \( \mathbb{C} \) is said to be convex if the linear segment joining any two points of \( D \) lies entirely in \( D \) and is said to be close-to-convex if its complement in \( \mathbb{C} \) is the union of closed half-lines with pairwise disjoint interiors. Consequently, a univalent analytic or harmonic function \( f : \mathbb{E} \to \mathbb{C} \) is said to be convex or close-to-convex in \( \mathbb{E} \) if \( f(\mathbb{E}) \) is convex or close-to-convex there. For \( -1/2 \leq \alpha < 1 \), a function \( h \in \mathcal{A}' \) is said to be in the class \( \mathcal{K}(\alpha) \) if \( \text{Re}[1 + zh''(z)/h'(z)] > \alpha, z \in \mathbb{E} \). It can easily be verified that \( \mathcal{K}(\beta) \subset \mathcal{K}(\alpha) \) if \( -1/2 \leq \alpha < \beta < 1 \). If \( h \in (\alpha) \) for \( 0 \leq \alpha < 1 \) in \( \mathbb{E} \), then \( h \) is said to be convex of order \( \alpha \) in \( \mathbb{E} \) (e.g., see [3] or [4]). A function \( h \in \mathcal{K}(0) \) is simply called a convex function in \( \mathbb{E} \).

Ruscheweyh and Sheil-Small [5] proved that the Hadamard product or convolution of two analytic convex functions is also convex analytic and that the convolution of an analytic convex function and an analytic close-to-convex function is close-to-convex analytic in the unit disk \( \mathbb{E} \). Ironically, these results cannot be extended to the harmonic case, since the convolution of harmonic functions, unlike the analytic case, proved to be very challenging.

Recently, Ahuja and Jahangiri [6] proved the following theorem.

\[ \text{Conclusion:} \quad \text{If } f \in \mathcal{K}(\alpha), \text{ then } f \ast f \in \mathcal{K}(\alpha). \]
Therorem 1. Let the functions $h_1 \in \mathcal{A}$ and $h_2 \in \mathcal{A}$ be in the class $\mathcal{H}(\alpha)$ in $\mathbb{E}$. If either $g_1 = zh_1$, $g_2 = zh_2$, $\alpha \geq -1/2$ or $g_1' = z^\alpha h_1$, $g_2' = z^\alpha h_2$, $\alpha \geq 0$, then $F = h_1 \ast h_2 + g_1 \ast g_2$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.

The following question is asked in [6].

Question 2. Is Theorem 1 true for $g_1 = z^n h_1$ and $g_2 = z^n h_2$ if $n > 1$?

In Theorem 3, we address Question 2. Moreover, in Theorem 4, we allow variations in the powers of $z$ for the dilatations of harmonic functions. Also note that the techniques presented here prove our theorems are different from those used in [6].

Theorem 3. Let the functions $h_1 \in \mathcal{A}$ and $h_2 \in \mathcal{A}$ be so that $h_1 \ast h_2$ is convex in $\mathbb{E}$. Set $g_1(z) = z^n h_1(z)$ and $g_2(z) = z^n h_2(z)$, where $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Then the convolution function $F(z) = h_1(z) \ast h_2(z) + g_1(z) \ast g_2(z)$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.

Theorem 4. Let the functions $h_1 \in \mathcal{A}$ and $h_2 \in \mathcal{A}$ be convex of order $1/2$ in $\mathbb{E}$. Set $g_1(z) = z^n h_1(z)$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and $g_2(z) = z^n h_2(z)$, $m \in \mathbb{N}$. Then the convolution function $F(z) = h_1(z) \ast h_2(z) + g_1(z) \ast g_2(z)$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.

In the following example, we demonstrate a case of close-to-convexity of convolutions of two harmonic functions.

Example 5. Consider

\begin{equation}
\begin{align*}
f_1(z) &= h_1(z) + g_1(z) = \frac{z}{1-z} + \frac{z}{1-z} + \log(1-z), \\
f_2(z) &= h_2(z) + g_2(z) = \arctan z + \frac{1}{2} \log(1 + z^2).
\end{align*}
\end{equation}

For $j = 1, 2$, it is easy to verify that $g_j'(z) = zh_j'(z)$ and

\begin{equation}
\begin{align*}
F(z) &= H(z) + \overline{G(z)} = f_1(z) \ast f_2(z) \\
&= h_1(z) \ast h_2(z) + g_1(z) \ast g_2(z) \\
&= \arctan z \\
&\quad + \frac{1}{2} \log(1 + z^2) - \frac{1}{2} \int_{\mathbb{R}} \frac{1}{t} \log(1 + t^2) dt \\
&= \arctan z + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n - 1}{2n} z^{2n-1}.
\end{align*}
\end{equation}

Also,

\begin{equation}
\begin{align*}
\left| \frac{G'(z)}{H'(z)} \right| &= \left| z - \frac{(1 + z^2) \log(1 + z^2)}{2z} \right| \\
&= |z| \left| \frac{1 - \frac{z^2}{2} + \frac{z^4}{6} - \frac{z^6}{12} + \frac{z^8}{20} + \cdots}{2} \right| \\
&\leq \frac{|z|}{2} \left| 1 + \frac{|z|^2}{6} + \frac{|z|^4}{12} + \frac{|z|^6}{20} + \cdots \right|
\end{align*}
\end{equation}

Therefore, $f_1 \ast f_2$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.

The images of $E$ under $f_1, f_2$, and $f_1 \ast f_2$ are shown in Figures 1, 2, and 3, respectively.

2. Preliminary Lemmas and Proofs

To prove our theorems, we shall need the following four lemmas. Lemmas 6 and 9 are according to Clunie and Sheil-Small [2], Lemma 7 is a well-established result by Robinson [7], and Lemma 8 is a celebrated result by Ruscheweyh and Sheil-Small [5].

Lemma 6. Let $g$ and $h$ be analytic in $\mathbb{E}$ so that $|g'(0)| < |h'(0)|$. If $h + eg$ is close-to-convex analytic in $\mathbb{E}$ for each $e; |e| = 1$, then $f = h + \bar{g}$ is close-to-convex harmonic in $\mathbb{E}$.

Lemma 7. If $P(z)$ and $Q(z)$ are analytic in $\mathbb{E}$, $|Q(z)| \leq |P(z)|$ for all $z \in \mathbb{E}$, $Q(0) = P(0) = 0$, and if $P(z)$ maps $\mathbb{E}$ onto a region which is starlike with respect to origin, then $|Q(z)| \leq |P(z)|$ for all $z \in \mathbb{E}$.

A function $\phi$ analytic in $\mathbb{E}$ is convex of order $\alpha; 0 \leq \alpha < 1$ in $\mathbb{E}$ if and only if $z \psi'(z)$ is starlike of order $\alpha; 0 \leq \alpha < 1$ in $\mathbb{E}$ (e.g., see Duren [4]).

Lemma 8. Let $\phi$ and $\psi$ be analytic and starlike of order $1/2$ in $\mathbb{E}$. Then for each function $F$ analytic in $\mathbb{E}$, the convolution $(\phi \ast F\psi)/(\phi \ast \psi)$ takes only values in the convex hull of $F(\mathbb{E})$. 

\( \text{Figure 1: Image of } E \text{ under } f_1. \)
Lemma 9. If $h$ and $g$ are analytic in $E$, $h$ is convex in $E$ and if $f = h + \frac{g}{z}$ is locally univalent in $E$, then the function $f = h + \frac{g}{z}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $E$.

Proof of Theorem 3. For the convolution function

$$F(z) = h_1(z) * h_2(z) + g_1(z) * g_2(z)$$

we note that $|G'(0)| = 0 < |H'(0)|$. To satisfy the condition of Lemma 6, we will show that $H(z) + eG(z)$ is close-to-convex analytic in $E$ for each $e; (|e| = 1)$. To do so, it suffices to show that there exists a function $\phi$ analytic and convex in $E$ so that

$$\text{Re} \left( \frac{H(z) + eG(z)}{\phi'(z)} \right) > 0; \quad |z| < 1. \quad (6)$$

We observe that

$$[H(z) + eG(z)]' = H'(z) + eG'(z)$$

$$= [h_1(z) * h_2(z)]' + e [g_1(z) * g_2(z)]'$$

$$= \frac{1}{z} h_1(z) * h_2'(z) + e \left[ \frac{1}{z} g_1(z) * g_2'(z) \right]$$

$$= \frac{1}{z} h_1(z) * h_2'(z) + e \left[ \frac{1}{z} g_1(z) * g_2(z) + z^n h_2'(z) \right]$$

$$= \frac{1}{z} h_1(z) * h_2'(z) + e z^n h_2'(z)$$

$$= (1 + e z^n) \left[ \frac{1}{z} h_1(z) * h_2'(z) \right]. \quad (7)$$

Now letting $\phi(z) = h_1(z) * h_2(z)$ yields

$$\text{Re} \left( \frac{[H(z) + eG(z)]'}{\phi'(z)} \right)$$

$$= \text{Re} \left( \frac{1 + e z^n}{{1/z} h_1(z) * h_2'(z)} \right) \quad \phi'(z)$$

$$= \text{Re} \left( \frac{1 + e z^n}{{1/z} h_1(z) * h_2'(z)} \right) \quad (1/z) h_1(z) * h_2'(z)$$

$$= \text{Re} \left( 1 + e z^n \right) > 0. \quad (8)$$

Thus by Lemma 6, $F(z) = h_1(z) * h_2(z) + \frac{g_1(z) * g_2(z)}{z}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $E$. \hfill \Box

Proof of Theorem 4. We need to show that the convolution function

$$F(z) = h_1(z) * h_2(z) + \frac{g_1(z) * g_2(z)}{z}$$

is locally univalent and sense-preserving in $E$.\hfill (9)
Without loss of generality, we consider only the two cases of \( m = n \) and \( m > n \).

**Case 1.** If \( m = n \), then we have to show that \(|H'(z)| > |G'(z)|\) in \( E \). We observe that

\[
\left| h'_1(z) + h'_2(z) \right| > \left| z^n \right| \left| h'_1(z) + h'_2(z) \right|
\]

or

\[
\left| z \left[ h'_1(z) + h'_2(z) \right] \right| > \left| \left[ z \left[ g'_1(z) + g'_2(z) \right] \right] \right|
\]

Since \( h'_1(z) + h'_2(z) \) is convex in \( E \), \( z[h'_1(z) + h'_2(z)] \) is starlike in \( E \). Therefore, by Lemma 7, inequality (13) yields

\[
\left| z \left[ h'_1(z) + h'_2(z) \right] \right| > \left| z \left[ g'_1(z) + g'_2(z) \right] \right|
\]

or

\[
\left| h'_1(z) + h'_2(z) \right| > \left| g'_1(z) + g'_2(z) \right|
\]

\[
H'(z) > G'(z).
\]

\[
(16)
\]

**Case 2.** If \( m > n \), then we have

\[
\left| h'_1(z) + h'_2(z) \right| > \left| z^n \right| \left| h'_1(z) + h'_2(z) \right|
\]

\[
= \left| z^n \left( h'_1(z) + z_h(z) \right) \right|
\]

\[
= \left| g'_1(z) + g'_2(z) \right|
\]

or

\[
\left| h'_1(z) + h'_2(z) \right| > \left| g'_1(z) + g'_2(z) \right|
\]

\[
H'(z) > G'(z).
\]

\[
(16)
\]

Letting \( \phi(z) = z h'_1(z), \psi(z) = z h'_2(z), \) and \( F(z) = z^{m-n} \) in Lemma 8 yields

\[
\frac{z h'_1(z) + z h'_2(z)}{z h'_1(z) + z^{m-n} z h'_2(z)} > 1.
\]

Therefore

\[
\left| h'_1(z) + h'_2(z) \right| > \left| g'_1(z) + g'_2(z) \right|
\]

or

\[
\left| h'_1(z) + h'_2(z) \right| > \left| g'_1(z) + g'_2(z) \right|
\]

\[
(20)
\]

This is exactly inequality (10). So a similar argument following inequality (10) will lead to the conclusion that

\[
\left| H'(z) \right| > \left| G'(z) \right|
\]

\[
(21)
\]

Therefore, for either of cases \( m = n \) or \( m > n \),

\[
F(z) = h'_1(z) + h'_2(z) + g'_1(z) + g'_2(z)
\]

\[
= H(z) + G(z)
\]

is locally univalent and sense-preserving in \( E \). Thus, by Lemma 9,

\[
F(z) = h'_1(z) + h'_2(z) + g'_1(z) + g'_2(z)
\]

\[
(23)
\]

is locally one-to-one, sense-preserving, and close-to-convex harmonic in \( E \).

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this article.

**References**


