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Research Article

Close-to-Convexity of Convolutions of Classes of Harmonic Functions

Raj Kumar Garg[1] and Jay M. Jahangiri[2]

[1] *DAV University, Jalandhar, Punjab 144012, India*
[2] *Mathematical Sciences, Kent State University, Burton, OH 44021-9500, USA*

Correspondence should be addressed to Jay M. Jahangiri; jjahangi@kent.edu

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For $j = 1, 2$ and for positive integers m and n, we consider classes of harmonic functions $f_j = h_j + \overline{g_j}$, where $g_1(z) = z^n h_1(z)$ and $g_2'(z) = z^n h_2'(z)$ or $g_1'(z) = z^n h_1'(z)$ and $g_2'(z) = z^m h_2'(z)$, and we prove that their convolution $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $|z| < 1$.

1. Introduction

Let $\mathscr{A}$ denote the class of functions that are analytic in the open unit disk $\mathbb{E} := \{z : |z| < 1\}$ and let $\mathscr{A}'$ be the subclass of $\mathscr{A}$ consisting of functions h with the normalization $h(0) = h'(0) - 1 = 0$. Consider the family of complex-valued harmonic functions $f = u + iv$, where u and v are real harmonic in $\mathbb{E}$. Such functions can be expressed as $f = h + \overline{g}$, where $h \in \mathscr{A}$ and $g \in \mathscr{A}$. By Lewy's Theorem (see [1, 2] or [3]), a necessary and sufficient condition for the harmonic function $f = h + \overline{g}$ to be locally one-to-one and sense-preserving in $\mathbb{E}$ is that its Jacobian $J_f = |h'|^2 - |g'|^2$ should be positive or equivalently if and only if $h' \neq 0$ in $\mathbb{E}$ and the second complex dilatation ω of f satisfies $|\omega| = |g'/h'| < 1$ in $\mathbb{E}$. In the sequel, without loss of generality, we consider those locally one-to-one and sense-preserving harmonic functions $f = h + \overline{g}$ that are normalized by $f(0) = h(0) = 0$ and $f_z(0) = 1$ and have the representation

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}; \quad z \in \mathbb{E}. \quad (1)$$

The Hadamard product or convolution of two power series $h_1(z) = \sum_{n=1}^{\infty} a_n z^n$ and $h_2(z) = \sum_{n=1}^{\infty} c_n z^n$ is given by $h_1(z) * h_2(z) = (h_1 * h_2)(z) = \sum_{n=1}^{\infty} a_n c_n z^n$. Similarly, the convolution of two harmonic functions $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ is given by $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$.

A simply connected proper subdomain $\mathbb{D}$ of the complex domain $\mathbb{C}$ is said to be convex if the linear segment joining any two points of $\mathbb{D}$ lies entirely in $\mathbb{D}$ and is said to be close-to-convex if its complement in $\mathbb{C}$ is the union of closed half-lines with pairwise disjoint interiors. Consequently, a univalent analytic or harmonic function $f : \mathbb{E} \to \mathbb{C}$ is said to be convex or close-to-convex in $\mathbb{E}$ if $f(\mathbb{E})$ is convex or close-to-convex there. For $-1/2 \leq \alpha < 1$, a function $h \in \mathscr{A}'$ is said to be in the class $\mathscr{K}(\alpha)$ if $\operatorname{Re}[1 + zh''(z)/h'(z)] > \alpha$, $z \in \mathbb{E}$. It can easily be verified that $\mathscr{K}(\beta) \subset \mathscr{K}(\alpha)$ if $-1/2 \leq \alpha < \beta < 1$. If $h \in (\alpha)$ for $0 \leq \alpha < 1$ in $\mathbb{E}$, then h is said to be convex of order α in $\mathbb{E}$ (e.g., see [3] or [4]). A function $h \in \mathscr{K}(0)$ is simply called a convex function in $\mathbb{E}$.

Ruscheweyh and Sheil-Small [5] proved that the Hadamard product or convolution of two analytic convex functions is also convex analytic and that the convolution of an analytic convex function and an analytic close-to-convex function is close-to-convex analytic in the unit disk $\mathbb{E}$. Ironically, these results can not be extended to the harmonic case, since the convolution of harmonic functions, unlike the analytic case, proved to be very challenging.

Recently, Ahuja and Jahangiri [6] proved the following theorem.

Theorem 1. *Let the functions $h_1 \in \mathscr{A}'$ and $h_2 \in \mathscr{A}'$ be in the class $\mathscr{K}(\alpha)$ in $\mathbb{E}$. If either $g_1 = zh_1$, $g_2' = zh_2'$, $\alpha \geq -1/2$ or $g_1' = z^n h_1'$, $g_2' = z^n h_2'$, $\alpha \geq 0$, then $F = h_1 * h_2 + \overline{g_1 * g_2}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.*

The following question is asked in [6].

Question 2. Is Theorem 1 true for $g_1 = z^n h_1$ and $g_2' = z^n h_2'$ if $n > 1$?

In Theorem 3, we address Question 2. Moreover, in Theorem 4, we allow variations in the powers of z for the dilatations of harmonic functions. Also note that the techniques presented here to prove our theorems are different from those used in [6].

Theorem 3. *Let the functions $h_1 \in \mathscr{A}'$ and $h_2 \in \mathscr{A}'$ be so that $h_1 * h_2$ is convex in $\mathbb{E}$. Set $g_1(z) = z^n h_1(z)$ and $g_2'(z) = z^n h_2'(z)$, where $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. Then the convolution function $F(z) = h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.*

Theorem 4. *Let the functions $h_1 \in \mathscr{A}'$ and $h_2 \in \mathscr{A}'$ be convex of order $1/2$ in $\mathbb{E}$. Set $g_1'(z) = z^n h_1'(z)$, $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, and $g_2'(z) = z^m h_2'(z)$, $m \in \mathbb{N}$. Then the convolution function $F(z) = h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.*

In the following example, we demonstrate a case of close-to-convexity of convolutions of two harmonic functions.

Example 5. Consider

$$\begin{aligned} f_1(z) &= h_1(z) + \overline{g_1(z)} = \frac{z}{1-z} + \overline{\frac{z}{1-z} + \log(1-z)}, \\ f_2(z) &= h_2(z) + \overline{g_2(z)} = \arctan z + \frac{1}{2}\overline{\log(1+z^2)}. \end{aligned} \quad (2)$$

For $j = 1, 2$, it is easy to verify that $g_j'(z) = zh_j'(z)$ and

$$\begin{aligned} F(z) &= H(z) + \overline{G(z)} = f_1(z) * f_2(z) \\ &= h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)} \\ &= \arctan z \\ &\quad + \overline{\frac{1}{2}\log(1+z^2) - \frac{1}{2}\int_0^z \frac{1}{t}\log(1+t^2)dt} \\ &= \arctan z + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n-1}{2n^2}\overline{z^{2n}}. \end{aligned} \quad (3)$$

Also,

$$\begin{aligned} \left|\frac{G'(z)}{H'(z)}\right| &= \left|z - \frac{(1+z^2)\log(1+z^2)}{2z}\right| \\ &= \frac{|z|}{2}\left|1 - \frac{z^2}{2} + \frac{z^4}{6} - \frac{z^6}{12} + \frac{z^8}{20} + \cdots\right| \\ &\leq \frac{|z|}{2}\left[1 + \frac{|z|^2}{2} + \frac{|z|^4}{6} + \frac{|z|^6}{12} + \frac{|z|^8}{20} + \cdots\right] \\ &< \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right] < \frac{1}{2}(2) \\ &= 1. \end{aligned} \quad (4)$$

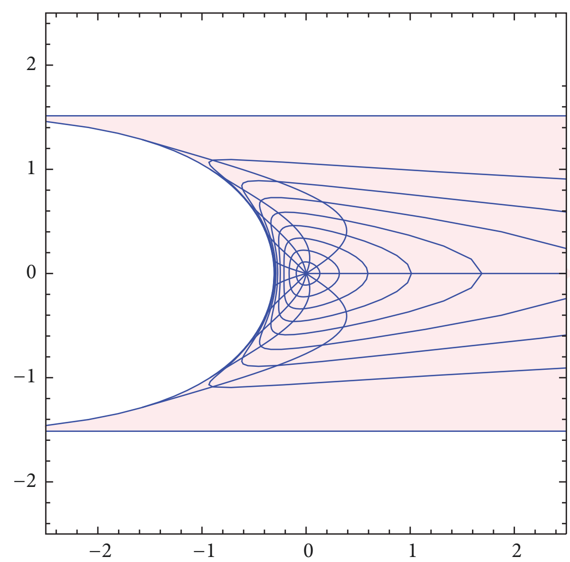

FIGURE 1: Image of E under f_1.

Therefore, $f_1 * f_2$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.

The images of E under f_1, f_2, and $f_1 * f_2$ are shown in Figures 1, 2, and 3, respectively.

2. Preliminary Lemmas and Proofs

To prove our theorems, we shall need the following four lemmas. Lemmas 6 and 9 are according to Clunie and Sheil-Small [2], Lemma 7 is a well-established result by Robinson [7], and Lemma 8 is a celebrated result by Ruscheweyh and Sheil-Small [5].

Lemma 6. *Let g and h be analytic in $\mathbb{E}$ so that $|g'(0)| < |h'(0)|$. If $h + \epsilon g$ is close-to-convex analytic in $\mathbb{E}$ for each ϵ; ($|\epsilon| = 1$), then $f = h + \overline{g}$ is close-to-convex harmonic in $\mathbb{E}$.*

Lemma 7. *If $P(z)$ and $Q(z)$ are analytic in $\mathbb{E}$, $|Q'(z)| \leq |P'(z)|$ for all $z \in \mathbb{E}$, $Q(0) = P(0) = 0$, and if $P(z)$ maps $\mathbb{E}$ onto a region which is starlike with respect to origin, then $|Q(z)| \leq |P(z)|$ for all $z \in \mathbb{E}$.*

A function ϕ analytic in $\mathbb{E}$ is convex of order α; $0 \leq \alpha < 1$ in $\mathbb{E}$ if and only if $z\phi'(z)$ is starlike of order α; $0 \leq \alpha < 1$ in $\mathbb{E}$ (e.g., see Duren [4]).

Lemma 8. *Let ϕ and ψ be analytic and starlike of order $1/2$ in $\mathbb{E}$. Then for each function F analytic in $\mathbb{E}$, the convolution $(\phi * F\psi)/(\phi * \psi)$ takes only values in the convex hull of $F(\mathbb{E})$.*

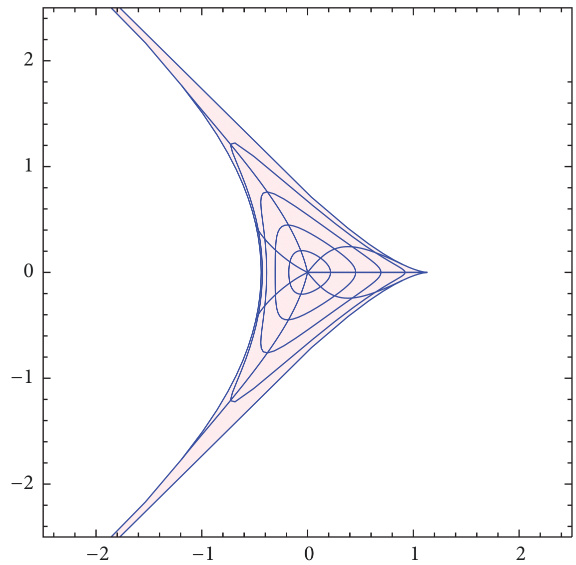

Figure 2: Image of E under f_2.

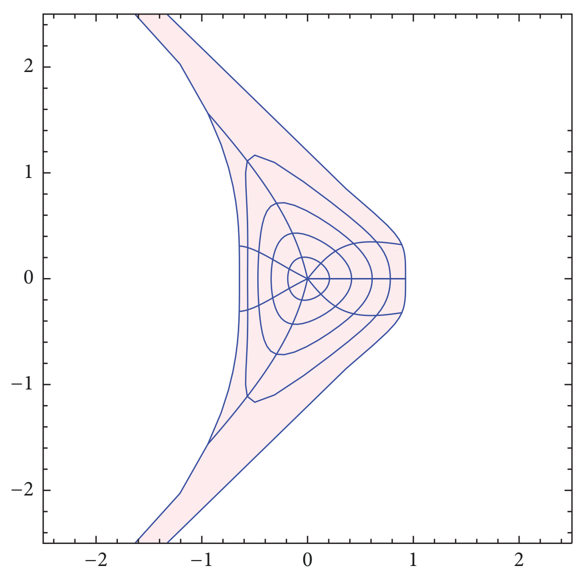

Figure 3: Image of E under $f_1 * f_2$.

Lemma 9. *If h and g are analytic in $\mathbb{E}$, h is convex in $\mathbb{E}$ and if $f = h + \overline{g}$ is locally univalent in $\mathbb{E}$, then the function $f = h + \overline{g}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$.*

Proof of Theorem 3. For the convolution function

$$\begin{aligned} F(z) &= h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)} \\ &= H(z) + \overline{G(z)} \end{aligned} \tag{5}$$

we note that $|G'(0)| = 0 < |H'(0)|$. To satisfy the condition of Lemma 6, we will show that $H(z) + \epsilon G(z)$ is close-to-convex analytic in $\mathbb{E}$ for each ϵ; ($|\epsilon| = 1$). To do so, it suffices to show that there exists a function ϕ analytic and convex in $\mathbb{E}$ so that

$$\operatorname{Re}\frac{[H(z) + \epsilon G(z)]'}{\phi'(z)} > 0; \quad |z| < 1. \tag{6}$$

We observe that

$$\begin{aligned} [H(z) + \epsilon G(z)]' &= H'(z) + \epsilon G'(z) \\ &= [h_1(z) * h_2(z)]' \\ &\quad + \epsilon [g_1(z) * g_2(z)]' \\ &= \left[\frac{1}{z} h_1(z) * h_2'(z)\right] \\ &\quad + \epsilon \left[\frac{1}{z} g_1(z) * g_2'(z)\right] \\ &= \left[\frac{1}{z} h_1(z) * h_2'(z)\right] \\ &\quad + \epsilon \left[\frac{1}{z} z^n h_1(z) * z^n h_2'(z)\right] \\ &= \left[\frac{1}{z} h_1(z) * h_2'(z)\right] \\ &\quad + \epsilon z^n \left[\frac{1}{z} h_1(z) * h_2'(z)\right] \\ &= (1 + \epsilon z^n)\left[\frac{1}{z} h_1(z) * h_2'(z)\right]. \end{aligned} \tag{7}$$

Now letting $\phi(z) = h_1(z) * h_2(z)$ yields

$$\begin{aligned} &\operatorname{Re}\frac{[H(z) + \epsilon G(z)]'}{\phi'(z)} \\ &\quad = \operatorname{Re}\frac{(1 + \epsilon z^n)\left[(1/z) h_1(z) * h_2'(z)\right]}{\phi'(z)} \\ &\quad = \operatorname{Re}\frac{(1 + \epsilon z^n)\left[(1/z) h_1(z) * h_2'(z)\right]}{(1/z) h_1(z) * h_2'(z)} \\ &\quad = \operatorname{Re}(1 + \epsilon z^n) > 0. \end{aligned} \tag{8}$$

Thus by Lemma 6, $F(z) = h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)}$ is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$. □

Proof of Theorem 4. We need to show that the convolution function

$$\begin{aligned} F(z) &= h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)} \\ &= H(z) + \overline{G(z)} \end{aligned} \tag{9}$$

is locally univalent and sense-preserving in $\mathbb{E}$.

Without loss of generality, we consider only the two cases of $m = n$ and $m > n$.

Case 1. If $m = n$, then we have to show that $|H'(z)| > |G'(z)|$ in $\mathbb{E}$. We observe that

$$\begin{aligned} \left|h_1'(z) * h_2'(z)\right| &> \left|z^n\right|\left|h_1'(z) * h_2'(z)\right| \\ &= \left|g_1'(z) * g_2'(z)\right|. \end{aligned} \tag{10}$$

Inequality (10) is equivalent to

$$\left|\left[h_1(z) * zh_2'(z)\right]'\right| > \left|\left[g_1(z) * zg_2'(z)\right]'\right| \tag{11}$$

or

$$\left|\left\{z\left[\frac{1}{z}h_1(z) * h_2'(z)\right]\right\}'\right| > \left|\left\{z\left[\frac{1}{z}g_1(z) * g_2'(z)\right]\right\}'\right| \tag{12}$$

or

$$\left|\left\{z\left[h_1(z) * h_2(z)\right]'\right\}'\right| > \left|\left\{z\left[g_1(z) * g_2(z)\right]'\right\}'\right|. \tag{13}$$

Since $h_1(z) * h_2(z)$ is convex in $\mathbb{E}$, $z[h_1(z) * h_2(z)]'$ is starlike in $\mathbb{E}$. Therefore, by Lemma 7, inequality (13) yields

$$\left|z\left[h_1(z) * h_2(z)\right]'\right| > \left|z\left[g_1(z) * g_2(z)\right]'\right| \tag{14}$$

or

$$\left|\left[h_1(z) * h_2(z)\right]'\right| > \left|\left[g_1(z) * g_2(z)\right]'\right| \tag{15}$$

or

$$\left|H'(z)\right| > \left|G'(z)\right|. \tag{16}$$

Case 2. If $m > n$, then we have

$$\begin{aligned} \left|h_1'(z) * h_2'(z)\right| &> \left|z^n\right|\left|h_1'(z) * h_2'(z)\right| \\ &= \left|z^n h_1'(z) * z^n h_2'(z)\right| \\ &= \left|g_1'(z) * z^{n-m} g_2'(z)\right| \\ &= \left|g_1'(z) * g_2'(z)\right| \\ &\quad \times \left|\frac{g_1'(z) * z^{n-m} g_2'(z)}{g_1'(z) * g_2'(z)}\right|. \end{aligned} \tag{17}$$

We note that

$$\begin{aligned} \frac{g_1'(z) * z^{n-m} g_2'(z)}{g_1'(z) * g_2'(z)} &= \frac{z^n h_1'(z) * z^{n-m} z^m h_2'(z)}{z^n h_1'(z) * z^m h_2'(z)} \\ &= \frac{z^n h_1'(z) * z^n h_2'(z)}{z^n h_1'(z) * z^m h_2'(z)} \\ &= \frac{z^{n-1}\left[zh_1'(z) * zh_2'(z)\right]}{z^{n-1}\left[zh_1'(z) * z^{m-n} zh_2'(z)\right]} \\ &= \frac{zh_1'(z) * zh_2'(z)}{zh_1'(z) * z^{m-n} zh_2'(z)}. \end{aligned} \tag{18}$$

Letting $\phi(z) = zh_1'(z)$, $\psi(z) = zh_2'(z)$, and $F(z) = z^{m-n}$ in Lemma 8 yields

$$\left|\frac{zh_1'(z) * zh_2'(z)}{zh_1'(z) * z^{m-n} zh_2'(z)}\right| > 1. \tag{19}$$

Therefore

$$\begin{aligned} \left|h_1'(z) * h_2'(z)\right| &> \left|g_1'(z) * g_2'(z)\right| \\ &\quad \times \left|\frac{g_1'(z) * z^{n-m} g_2'(z)}{g_1'(z) * g_2'(z)}\right| \\ &> \left|g_1'(z) * g_2'(z)\right|. \end{aligned} \tag{20}$$

This is exactly inequality (10). So a similar argument following inequality (10) will lead to the conclusion that

$$\left|H'(z)\right| > \left|G'(z)\right|. \tag{21}$$

Therefore, for either of cases $m = n$ or $m > n$,

$$\begin{aligned} F(z) &= h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)} \\ &= H(z) + \overline{G(z)} \end{aligned} \tag{22}$$

is locally univalent and sense-preserving in $\mathbb{E}$. Thus, by Lemma 9,

$$F(z) = h_1(z) * h_2(z) + \overline{g_1(z) * g_2(z)} \tag{23}$$

is locally one-to-one, sense-preserving, and close-to-convex harmonic in $\mathbb{E}$. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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