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Research Article

On the Differentiability of Weak Solutions of an Abstract Evolution Equation with a Scalar Type Spectral Operator on the Real Axis

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Given the abstract evolution equation $y'(t) = Ay(t)$, $t \in \mathbb{R}$, with *scalar type spectral operator* A in a complex Banach space, found are conditions *necessary and sufficient* for all *weak solutions* of the equation, which a priori need not be strongly differentiable, to be strongly infinite differentiable on $\mathbb{R}$. The important case of the equation with a *normal operator* A in a complex Hilbert space is obtained immediately as a particular case. Also, proved is the following inherent smoothness improvement effect explaining why the case of the strong finite differentiability of the weak solutions is superfluous: if every weak solution of the equation is strongly differentiable at 0, then all of them are strongly infinite differentiable on $\mathbb{R}$.

"Curiosity is the lust of the mind."
Thomas Hobbes

1. Introduction

We find conditions on a *scalar type spectral* operator A in a complex Banach space necessary and sufficient for all *weak solutions* of the evolution equation

$$y'(t) = Ay(t), \quad t \in \mathbb{R}, \tag{1}$$

which a priori need not be strongly differentiable, to be strongly infinite differentiable on $\mathbb{R}$. The important case of the equation with a *normal operator* A in a complex Hilbert space is obtained immediately as a particular case. We also prove the following inherent smoothness improvement effect explaining why the case of the strong finite differentiability of the weak solutions is superfluous: if every weak solution of the equation is strongly differentiable at 0, then all of them are strongly infinite differentiable on $\mathbb{R}$.

The found results develop those of paper [1], where similar consideration is given to the strong differentiability of the weak solutions of the equation

$$y'(t) = Ay(t), \quad t \geq 0, \tag{2}$$

on $[0, \infty)$ and $(0, \infty)$.

Definition 1 (weak solution). Let A be a densely defined closed linear operator in a Banach space X and I be an interval of the real axis $\mathbb{R}$. A strongly continuous vector function $y : I \longrightarrow X$ is called a *weak solution* of the evolution equation

$$y'(t) = Ay(t), \quad t \in I, \tag{3}$$

if, for any $g^* \in D(A^*)$,

$$\frac{d}{dt}\langle y(t), g^* \rangle = \langle y(t), A^* g^* \rangle, \quad t \in I, \tag{4}$$

where $D(\cdot)$ is the *domain* of an operator, A^* is the operator *adjoint* to A, and $\langle \cdot, \cdot \rangle$ is the *pairing* between the space X and its dual X^* (cf. [2]).

Remarks 2.

(i) Due to the *closedness* of A, a weak solution of (3) can be equivalently defined to be a strongly continuous vector function $y : I \longmapsto X$ such that, for all $t \in I$,

$$\int_{t_0}^{t} y(s)\,ds \in D(A) \text{ and } \\ y(t) = y(t_0) + A\int_{t_0}^{t} y(s)\,ds, \tag{5}$$

where t_0 is an arbitrary fixed point of the interval I, and is also called a *mild solution* (cf. [3, Ch. II, Definition 6.3], see also [4, Preliminaries]).

(ii) Such a notion of *weak solution*, which need not be differentiable in the strong sense, generalizes that of *classical* one, strongly differentiable on I and satisfying the equation in the traditional plug-in sense, the classical solutions being precisely the weak ones strongly differentiable on I.

(iii) As is easily seen $y : \mathbb{R} \longrightarrow X$ is a weak solution of (1) *iff*

$$y_+(t) := y(t), \quad t \geq 0, \tag{6}$$

is a weak solution of (2) and

$$y_-(t) := y(-t), \quad t \geq 0, \tag{7}$$

is a weak solution of the equation

$$y'(t) = -Ay(t), \quad t \geq 0. \tag{8}$$

(iv) When a closed densely defined linear operator A in a complex Banach space X generates a strongly continuous group $\{T(t)\}_{t\in\mathbb{R}}$ of bounded linear operators (see, e.g., [3, 5]), i.e., the associated *abstract Cauchy problem* (*ACP*)

$$\begin{aligned} y'(t) &= Ay(t), \quad t \in \mathbb{R}, \\ y(0) &= f \end{aligned} \tag{9}$$

is *well-posed* (cf. [3, Ch. II, Definition 6.8]), the weak solutions of (1) are the orbits

$$y(t) = T(t)f, \quad t \in \mathbb{R}, \tag{10}$$

with $f \in X$ (cf. [3, Ch. II, Proposition 6.4], see also [2, Theorem]), whereas the classical ones are those with $f \in D(A)$ (see, e.g., [3, Ch. II, Proposition 6.3]).

(v) In our discourse, the associated *ACP* may be *ill-posed*, i.e., the scalar type spectral operator A need not generate a strongly continuous group of bounded linear operators (cf. [6]).

2. Preliminaries

Here, for the reader's convenience, we outline certain essential preliminaries.

Henceforth, unless specified otherwise, A is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ with strongly σ-additive *spectral measure* (the *resolution of the identity*) $E_A(\cdot)$ assigning to each Borel set δ of the complex plane $\mathbb{C}$ a projection operator $E_A(\delta)$ on X and having the operator's *spectrum* $\sigma(A)$ as its *support* [7, 8].

Observe that, in a complex finite-dimensional space, the scalar type spectral operators are all linear operators on the space, for which there is an *eigenbasis* (see, e.g., [7, 8]) and, in a complex Hilbert space, the scalar type spectral operators are precisely all those that are similar to the *normal* ones [9].

Associated with a scalar type spectral operator in a complex Banach space is the *Borel operational calculus* analogous to that for a *normal operator* in a complex Hilbert space [7, 8, 10, 11], which assigns to any Borel measurable function $F : \sigma(A) \longrightarrow \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda)\,dE_A(\lambda) \tag{11}$$

(see [7, 8]).

In particular,

$$A^n = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+, \tag{12}$$

($\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the set of *nonnegative integers*, $A^0 := I$, I is the *identity operator* on X), and

$$e^{zA} := \int_{\sigma(A)} e^{z\lambda} dE_A(\lambda), \quad z \in \mathbb{C}. \tag{13}$$

The properties of the *spectral measure* and *operational calculus*, exhaustively delineated in [7, 8], underlie the entire subsequent discourse. Here, we underline a few facts of particular importance.

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [8, 12], i.e., there is such an $M > 0$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \leq M. \tag{14}$$

Observe that the notation $\|\cdot\|$ is used here to designate the norm in the space $L(X)$ of all bounded linear operators on X. We adhere to this rather conventional economy of symbols in what follows also adopting the same notation for the norm in the dual space X^*.

For any $f \in X$ and $g^* \in X^*$, the *total variation measure* $v(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^*\rangle$ is a *finite* positive Borel measure with

$$v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \leq 4M\|f\|\|g^*\| \tag{15}$$

(see, e.g., [13, 14]).

Also (Ibid.), for a Borel measurable function $F : \mathbb{C} \longrightarrow \mathbb{C}$, $f \in D(F(A))$, $g^* \in X^*$, and a Borel set $\delta \subseteq \mathbb{C}$,

$$\int_{\delta} |F(\lambda)|\,dv(f, g^*, \lambda) \leq 4M\|E_A(\delta)F(A)f\|\|g^*\|. \tag{16}$$

In particular, for $\delta = \sigma(A)$, $E_A(\sigma(A)) = I$ and

$$\int_{\sigma(A)} |F(\lambda)|\, dv(f, g^*, \lambda) \le 4M \|F(A) f\| \|g^*\|. \quad (17)$$

Observe that the constant $M > 0$ in (15)–(17) is from (14).

Further, for a Borel measurable function $F : \mathbb{C} \longrightarrow [0, \infty)$, a Borel set $\delta \subseteq \mathbb{C}$, a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint Borel sets in $\mathbb{C}$, and $f \in X$, $g^* \in X^*$,

$$\begin{aligned} &\int_\delta F(\lambda)\, dv\left(E_A\left(\bigcup_{n=1}^{\infty}\Delta_n\right) f, g^*, \lambda\right) \\ &\quad = \sum_{n=1}^{\infty} \int_{\delta\cap\Delta_n} F(\lambda)\, dv\left(E_A(\Delta_n) f, g^*, \lambda\right). \end{aligned} \quad (18)$$

Indeed, since, for any Borel sets $\delta, \sigma \subseteq \mathbb{C}$,

$$E_A(\delta) E_A(\sigma) = E_A(\delta \cap \sigma) \quad (19)$$

[7, 8], for the total variation measure,

$$v\left(E_A(\delta) f, g^*, \sigma\right) = v\left(f, g^*, \delta \cap \sigma\right). \quad (20)$$

Whence, due to the *nonnegativity* of $F(\cdot)$ (see, e.g., [15]),

$$\begin{aligned} &\int_\delta F(\lambda)\, dv\left(E_A\left(\bigcup_{n=1}^{\infty}\Delta_n\right) f, g^*, \lambda\right) \\ &\quad = \int_{\delta\cap\bigcup_{n=1}^{\infty}\Delta_n} F(\lambda)\, dv(f, g^*, \lambda) \\ &\quad = \sum_{n=1}^{\infty} \int_{\delta\cap\Delta_n} F(\lambda)\, dv(f, g^*, \lambda) \\ &\quad = \sum_{n=1}^{\infty} \int_{\delta\cap\Delta_n} F(\lambda)\, dv\left(E_A(\Delta_n) f, g^*, \lambda\right). \end{aligned} \quad (21)$$

The following statement, allowing characterizing the domains of Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures, is fundamental for our discourse.

Proposition 3 ([16, Proposition 3.1]). *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $F : \sigma(A) \longrightarrow \mathbb{C}$ be a Borel measurable function. Then $f \in D(F(A))$ iff*

(i) *for each $g^* \in X^*$, $\int_{\sigma(A)} |F(\lambda)| dv(f, g^*, \lambda) < \infty$;*

(ii) $\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |F(\lambda)|>n\}} |F(\lambda)| dv(f, g^*, \lambda) \longrightarrow 0$, $n \longrightarrow \infty$,

where $v(f, g^*, \cdot)$ is the total variation measure of $\langle E_A(\cdot) f, g^*\rangle$.

The succeeding key theorem provides a description of the weak solutions of (2) with a scalar type spectral operator A in a complex Banach space.

Theorem 4 ([16, Theorem 4.2] with $T = \infty$). *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. A vector function $y : [0, \infty) \longrightarrow X$ is a weak solution of (2) iff there is an $f \in \bigcap_{t\ge0} D(e^{tA})$ such that*

$$y(t) = e^{tA} f, \quad t \ge 0, \quad (22)$$

the operator exponentials understood in the sense of the Borel operational calculus (see (13)).

Remark 5. Theorem 4 generalizes [17, Theorem 3.1], its counterpart for a normal operator A in a complex Hilbert space.

We also need the following characterizations of a particular weak solution's of (2) with a scalar type spectral operator A in a complex Banach space being strongly differentiable on a subinterval I of $[0, \infty)$.

Proposition 6 ([1, Proposition 3.1] with $T = \infty$). *Let $n \in \mathbb{N}$ and I be a subinterval of $[0, \infty)$. A weak solution $y(\cdot)$ of (2) is n times strongly differentiable on I iff*

$$y(t) \in D\left(A^n\right), \quad t \in I, \quad (23)$$

in which case

$$y^{(k)}(t) = A^k y(t), \quad k = 1, \ldots, n, \ t \in I. \quad (24)$$

Subsequently, the frequent terms *"spectral measure"* and *"operational calculus"* are abbreviated to *s.m.* and *o.c.*, respectively.

3. General Weak Solution

Theorem 7 (general weak solution). *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. A vector function $y : \mathbb{R} \longrightarrow X$ is a weak solution of (1) iff there is an $f \in \bigcap_{t\in\mathbb{R}} D(e^{tA})$ such that*

$$y(t) = e^{tA} f, \quad t \in \mathbb{R}, \quad (25)$$

the operator exponentials understood in the sense of the Borel operational calculus (see (13)).

Proof. As is noted in the Introduction, $y : \mathbb{R} \longrightarrow X$ is a weak solution of (1) *iff*

$$y_+(t) := y(t), \quad t \ge 0, \quad (26)$$

is a weak solution of (2) and

$$y_-(t) := y(-t), \quad t \ge 0, \quad (27)$$

is a weak solution of (8).

Applying Theorem 4, to $y_+(\cdot)$ and $y_-(\cdot)$, we infer that this is equivalent to the fact

$$y(t) = e^{tA} f, \quad t \in \mathbb{R}, \ \text{with some } f \in \bigcap_{t\in\mathbb{R}} D\left(e^{tA}\right). \quad (28)$$

□

Remarks 8.

(i) More generally, Theorem 4 and its proof can be easily modified to describe in the same manner all weak solution of (3) for an arbitrary interval I of the real axis $\mathbb{R}$.

(ii) Theorem 7 implies, in particular,

(a) that the subspace $\bigcap_{t\in\mathbb{R}} D(e^{tA})$ of all possible initial values of the weak solutions of (1) is the largest permissible for the exponential form given by (25), which highlights the naturalness of the notion of weak solution;

(b) that associated *ACP* (9), whenever solvable, is solvable *uniquely*.

(iii) Observe that the initial-value subspace $\bigcap_{t\in\mathbb{R}} D(e^{tA})$ of (1), containing the dense in X subspace $\bigcup_{\alpha>0} E_A(\Delta_\alpha)X$, where

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \le \alpha\}, \quad \alpha > 0, \tag{29}$$

which coincides with the class $\mathscr{E}^{\{0\}}(A)$ of *entire* vectors of A of *exponential type* [18], is *dense* in X as well.

(iv) When a scalar type spectral operator A in a complex Banach space generates a strongly continuous group $\{T(t)\}_{t\in\mathbb{R}}$ of bounded linear operators,

$$T(t) = e^{tA} \text{ and } D\left(e^{tA}\right) = X, \quad t \in \mathbb{R}, \tag{30}$$

[6], and hence, Theorem 7 is consistent with the well-known description of the weak solutions for this setup (see (10)).

(v) Clearly, the initial-value subspace $\bigcap_{t\in\mathbb{R}} D(e^{tA})$ of (1) is narrower than the initial-value subspace $\bigcap_{t\ge0} D(e^{tA})$ of (2) and the initial-value subspace $\bigcap_{t\ge0} D(e^{t(-A)}) = \bigcap_{t\le0} D(e^{tA})$ of (8); in fact it is the intersection of the latter two.

4. Differentiability of a Particular Weak Solution

Here, we characterize a particular weak solution's of (1) with a scalar type spectral operator A in a complex Banach space being strongly differentiable on a subinterval I of $\mathbb{R}$.

Proposition 9 (differentiability of a particular weak solution). *Let $n \in \mathbb{N}$ and I be a subinterval of $\mathbb{R}$. A weak solution $y(\cdot)$ of (1) is n times strongly differentiable on I iff*

$$y(t) \in D(A^n), \quad t \in I, \tag{31}$$

in which case,

$$y^{(k)}(t) = A^k y(t), \quad k = 1, \dots, n, \ t \in I. \tag{32}$$

Proof. The statement immediately follows from the prior theorem and Proposition 6 applied to

$$y_+(t) := y(t) \text{ and } y_-(t) := y(-t), \quad t \ge 0, \tag{33}$$

for an arbitrary weak solution $y(\cdot)$ of (1). □

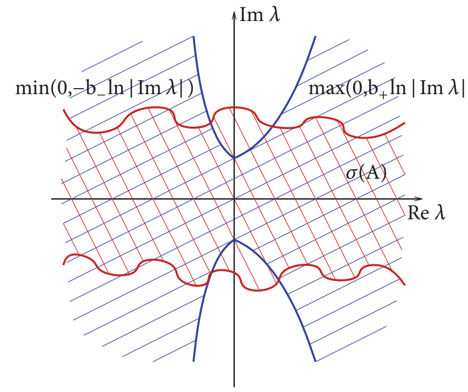

Figure 1

Remark 10. Observe that, as well as for Proposition 6, for $n = 1$, the subinterval I can degenerate into a singleton.

Inductively, we immediately obtain the following analog of [1, Corollary 3.2].

Corollary 11 (infinite differentiability of a particular weak solution). *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ and I be a subinterval of $\mathbb{R}$. A weak solution $y(\cdot)$ of (1) is strongly infinite differentiable on I ($y(\cdot) \in C^\infty(I, X)$) iff, for each $t \in I$,*

$$y(t) \in C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n), \tag{34}$$

in which case

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, \ t \in I. \tag{35}$$

5. Infinite Differentiability of Weak Solutions

In this section, we characterize the strong infinite differentiability on $\mathbb{R}$ of all weak solutions of (1) with a scalar type spectral operator A in a complex Banach space.

Theorem 12 (infinite differentiability of weak solutions). *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$. Every weak solution of (1) is strongly infinite differentiable on $\mathbb{R}$ iff there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A)\setminus\mathscr{L}_{b_-,b_+}$, where*

$$\mathscr{L}_{b_-,b_+} := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le \min(0, -b_- \ln|\operatorname{Im}\lambda|) \text{ or } \operatorname{Re}\lambda \ge \max(0, b_+ \ln|\operatorname{Im}\lambda|)\}, \tag{36}$$

is bounded (see Figure 1).

Proof. *"If" part*: suppose that there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A)\backslash\mathscr{L}_{b_-,b_+}$ is *bounded* and let $y(\cdot)$ be an arbitrary weak solution of (1).

By Theorem 7,

$$y(t) = e^{tA} f, \quad t \in \mathbb{R}, \text{ with some } f \in \bigcap_{t\in\mathbb{R}} D\left(e^{tA}\right). \tag{37}$$

Our purpose is to show that $y(\cdot) \in C^{\infty}(\mathbb{R}, X)$, which, by Corollary 11, is attained by showing that, for each $t \in \mathbb{R}$,

$$y(t) \in C^{\infty}(A) := \bigcap_{n=1}^{\infty} D(A^n). \tag{38}$$

Let us proceed by proving that, for any $t \in \mathbb{R}$ and $m \in \mathbb{N}$

$$y(t) \in D(A^m) \tag{39}$$

via Proposition 3.

For any $t \in \mathbb{R}$, $m \in \mathbb{N}$, and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&= \int_{\sigma(A)\backslash\mathscr{L}_{b_-,b_+}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\quad + \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid -1<\operatorname{Re}\lambda<1\}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\quad + \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 1\}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\quad + \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\le -1\}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&< \infty.
\end{aligned} \tag{40}$$

Indeed,

$$\int_{\sigma(A)\backslash\mathscr{L}_{b_-,b_+}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) < \infty \tag{41}$$

and

$$\int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid -1<\operatorname{Re}\lambda<1\}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) < \infty \tag{42}$$

due to the boundedness of the sets

$$\sigma(A)\backslash\mathscr{L}_{b_-,b_+} \text{ and } \left\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid -1 < \operatorname{Re}\lambda < 1\right\}, \tag{43}$$

the continuity of the integrated function on $\mathbb{C}$, and the finiteness of the measure $v(f, g^*, \cdot)$.

Further, for any $t \in \mathbb{R}$, $m \in \mathbb{N}$, and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 1\}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\le \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 1\}} [|\operatorname{Re}\lambda| + |\operatorname{Im}\lambda|]^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\qquad \text{since, for } \lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \text{ with } \operatorname{Re}\lambda \ge 1,\ e^{b_+^{-1}\operatorname{Re}\lambda} \ge |\operatorname{Im}\lambda|; \\
&\le \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 1\}} \left[\operatorname{Re}\lambda + e^{b_+^{-1}\operatorname{Re}\lambda}\right]^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\qquad \text{since, in view of } \operatorname{Re}\lambda \ge 1,\ b_+ e^{b_+^{-1}\operatorname{Re}\lambda} \ge \operatorname{Re}\lambda; \\
&\le \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 1\}} \left[b_+ e^{b_+^{-1}\operatorname{Re}\lambda} + e^{b_+^{-1}\operatorname{Re}\lambda}\right]^m \\
&\qquad \cdot e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&= [b_+ + 1]^m \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 1\}} e^{[mb_+^{-1}+t]\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\qquad \text{since } f \in \bigcap_{t\in\mathbb{R}} D\left(e^{tA}\right), \text{ by Proposition 3;} \\
&< \infty.
\end{aligned} \tag{44}$$

Finally, for any $t \in \mathbb{R}$, $m \in \mathbb{N}$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\le -1\}} |\lambda|^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\le \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\le -1\}} [|\operatorname{Re}\lambda| + |\operatorname{Im}\lambda|]^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\qquad \text{since, for } \lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \text{ with } \operatorname{Re}\lambda \le -1,\ e^{b_-^{-1}(-\operatorname{Re}\lambda)} \ge |\operatorname{Im}\lambda|; \\
&\le \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\le -1\}} \left[-\operatorname{Re}\lambda + e^{b_-^{-1}(-\operatorname{Re}\lambda)}\right]^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \quad \text{since, in view of } -\operatorname{Re}\lambda \ge 1,\ b_- e^{b_-^{-1}(-\operatorname{Re}\lambda)} \ge -\operatorname{Re}\lambda; \\
&\le \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\le -1\}} \left[b_- e^{b_-^{-1}(-\operatorname{Re}\lambda)} + e^{b_-^{-1}(-\operatorname{Re}\lambda)}\right]^m e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&= [b_- + 1]^m \int_{\{\lambda\in\sigma(A)\cap\mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\le -1\}} e^{[t-mb_-^{-1}]\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \quad \text{since } f \in \bigcap_{t\in\mathbb{R}} D\left(e^{tA}\right), \text{ by Proposition 3;} \\
&< \infty.
\end{aligned} \tag{45}$$

Also, for any $t \in \mathbb{R}$, $m \in \mathbb{N}$, and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \setminus \mathscr{L}_{b_-,b_+} \mid |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid -1 < \operatorname{Re}\lambda < 1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \geq 1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \leq -1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&\longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{46}$$

Indeed, since, due to the boundedness of the sets

$$\sigma(A) \setminus \mathscr{L}_{b_-,b_+} \text{ and } \left\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid -1 < \operatorname{Re}\lambda < 1\right\} \tag{47}$$

and the continuity of the integrated function on $\mathbb{C}$, the sets

$$\left\{\lambda \in \sigma(A) \setminus \mathscr{L}_{b_-,b_+} \mid |\lambda|^m e^{t \operatorname{Re}\lambda} > n\right\} \tag{48}$$

and

$$\left\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid -1 < \operatorname{Re}\lambda < 1, \ |\lambda|^m e^{t \operatorname{Re}\lambda} > n\right\} \tag{49}$$

are *empty* for all sufficiently large $n \in \mathbb{N}$, we immediately infer that, for any $t \in \mathbb{R}$ and $m \in \mathbb{N}$,

$$\begin{aligned}
&\lim_{n \longrightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \setminus \mathscr{L}_{b_-,b_+} \mid |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) = 0
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&\lim_{n \longrightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid -1 < \operatorname{Re}\lambda < 1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) = 0.
\end{aligned} \tag{51}$$

Further, for any $t \in \mathbb{R}$, $m \in \mathbb{N}$, and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \geq 1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \quad \text{as in } (44); \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} [b_+ + 1]^m \\
&\quad \cdot \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \geq 1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} e^{[mb_+^{-1} + t] \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&\qquad \text{since } f \in \bigcap_{t \in \mathbb{R}} D\left(e^{tA}\right), \text{ by } (16); \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} [b_+ + 1]^m \\
&\quad \cdot 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \geq 1, \ |\lambda|^m e^{t \operatorname{Re}\lambda} > n\right\}\right) \right. \\
&\quad \left. \cdot e^{[mb_+^{-1} + t]A} f \right\| \|g^*\| \\
&\leq [b_+ + 1]^m \\
&\quad \cdot 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \geq 1, \ |\lambda|^m e^{t \operatorname{Re}\lambda} > n\right\}\right) \right. \\
&\quad \left. \cdot e^{[mb_+^{-1} + t]A} f \right\| \quad \text{by the strong continuity of the } \textit{s.m.}; \\
&\longrightarrow [b_+ + 1]^m \, 4M \left\| E_A(\emptyset) \, e^{[mb_+^{-1} + t]A} f \right\| = 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{52}$$

Finally, for any $t \in \mathbb{R}$, $m \in \mathbb{N}$, and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \leq -1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} |\lambda|^m \\
&\quad \cdot e^{t \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \quad \text{as in } (45); \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} [b_- + 1]^m \\
&\quad \cdot \int_{\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \leq -1, |\lambda|^m e^{t \operatorname{Re}\lambda} > n\}} e^{[t - mb_-^{-1}] \operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&\qquad \text{since } f \in \bigcap_{t \in \mathbb{R}} D\left(e^{tA}\right), \text{ by } (16); \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} [b_- + 1]^m \\
&\quad \cdot 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \leq -1, \ |\lambda|^m e^{t \operatorname{Re}\lambda} > n\right\}\right) \right. \\
&\quad \left. \cdot e^{[t - mb_-^{-1}]A} f \right\| \|g^*\| \\
&\leq [b_- + 1]^m \\
&\quad \cdot 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \cap \mathscr{L}_{b_-,b_+} \mid \operatorname{Re}\lambda \leq -1, \ |\lambda|^m e^{t \operatorname{Re}\lambda} > n\right\}\right) \right. \\
&\quad \left. \cdot e^{[t - mb_-^{-1}]A} f \right\| \quad \text{by the strong continuity of the } \textit{s.m.}; \\
&\longrightarrow [b_+ + 1]^m \, 4M \left\| E_A(\emptyset) \, e^{[t - mb_-^{-1}]A} f \right\| = 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{53}$$

By Proposition 3 and the properties of the *o.c.* (see [8, Theorem XVIII.2.11 (f)]), (40) and (46) jointly imply that, for any $t \in \mathbb{R}$ and $m \in \mathbb{N}$,

$$f \in D\left(A^m e^{tA}\right), \tag{54}$$

which further implies that, for each $t \in \mathbb{R}$,

$$y(t) = e^{tA} f \in \bigcap_{n=1}^{\infty} D(A^n) =: C^{\infty}(A). \quad (55)$$

Whence, by Corollary 11, we infer that

$$y(\cdot) \in C^{\infty}(\mathbb{R}, X), \quad (56)$$

which completes the proof of the *"if"* part.

"Only if" part: let us prove this part *by contrapositive* assuming that, for any $b_+ > 0$ and $b_- > 0$, the set $\sigma(A)\backslash\mathscr{L}_{b_-,b_+}$ is *unbounded*. In particular, this means that, for any $n \in \mathbb{N}$, unbounded is the set

$$\sigma(A)\backslash\mathscr{L}_{(2n)^{-1},(2n)^{-1}} = \{\lambda \in \sigma(A) \mid -(2n)^{-1}\ln|\operatorname{Im}\lambda| < \operatorname{Re}\lambda < (2n)^{-1}\ln|\operatorname{Im}\lambda|\}. \quad (57)$$

Hence, we can choose a sequence of points $\{\lambda_n\}_{n=1}^{\infty}$ in the complex plane as follows:

$$\lambda_n \in \sigma(A), \quad n \in \mathbb{N},$$
$$-(2n)^{-1}\ln|\operatorname{Im}\lambda_n| < \operatorname{Re}\lambda_n < (2n)^{-1}\ln|\operatorname{Im}\lambda_n|, \quad n \in \mathbb{N}, \quad (58)$$
$$\lambda_0 := 0,$$
$$|\lambda_n| > \max[n^4, |\lambda_{n-1}|], \quad n \in \mathbb{N}.$$

The latter implies, in particular, that the points λ_n, $n \in \mathbb{N}$, are *distinct* ($\lambda_i \neq \lambda_j$, $i \neq j$).

Since, for each $n \in \mathbb{N}$, the set

$$\{\lambda \in \mathbb{C} \mid -(2n)^{-1}\ln|\operatorname{Im}\lambda| < \operatorname{Re}\lambda < (2n)^{-1}\ln|\operatorname{Im}\lambda|, \ |\lambda| > \max[n^4, |\lambda_{n-1}|]\} \quad (59)$$

is *open* in $\mathbb{C}$; along with the point λ_n, it contains an *open disk*

$$\Delta_n := \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n\} \quad (60)$$

centered at λ_n of some radius $\varepsilon_n > 0$, i.e., for each $\lambda \in \Delta_n$,

$$-(2n)^{-1}\ln|\operatorname{Im}\lambda| < \operatorname{Re}\lambda < (2n)^{-1}\ln|\operatorname{Im}\lambda| \text{ and } |\lambda| > \max[n^4, |\lambda_{n-1}|]. \quad (61)$$

Furthermore, we can regard the radii of the disks to be small enough so that

$$0 < \varepsilon_n < \frac{1}{n}, \quad n \in \mathbb{N}, \text{ and}$$
$$\Delta_i \cap \Delta_j = \emptyset, \quad (62)$$
$$i \neq j \ (\text{i.e., the disks are } \textit{pairwise disjoint}).$$

Whence, by the properties of the *s.m.*,

$$E_A(\Delta_i) E_A(\Delta_j) = 0, \quad i \neq j, \quad (63)$$

where 0 stands for the *zero operator* on X.

Observe also, that the subspaces $E_A(\Delta_n)X$, $n \in \mathbb{N}$, are *nontrivial* since

$$\Delta_n \cap \sigma(A) \neq \emptyset, \quad n \in \mathbb{N}, \quad (64)$$

with Δ_n being an *open set* in $\mathbb{C}$.

By choosing a unit vector $e_n \in E_A(\Delta_n)X$ for each $n \in \mathbb{N}$, we obtain a sequence $\{e_n\}_{n=1}^{\infty}$ in X such that

$$\|e_n\| = 1, \quad n \in \mathbb{N}, \text{ and}$$
$$E_A(\Delta_i) e_j = \delta_{ij} e_j, \quad i, j \in \mathbb{N}, \quad (65)$$

where δ_{ij} is the *Kronecker delta*.

As is easily seen, (65) implies that the vectors e_n, $n \in \mathbb{N}$, are *linearly independent*.

Furthermore, there is an $\varepsilon > 0$ such that

$$d_n := \operatorname{dist}(e_n, \operatorname{span}(\{e_i \mid i \in \mathbb{N}, \ i \neq n\})) \geq \varepsilon, \quad n \in \mathbb{N}. \quad (66)$$

Indeed, the opposite implies the existence of a subsequence $\{d_{n(k)}\}_{k=1}^{\infty}$ such that

$$d_{n(k)} \longrightarrow 0, \quad k \longrightarrow \infty. \quad (67)$$

Then, by selecting a vector

$$f_{n(k)} \in \operatorname{span}(\{e_i \mid i \in \mathbb{N}, \ i \neq n(k)\}), \quad k \in \mathbb{N}, \quad (68)$$

such that

$$\|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + \frac{1}{k}, \quad k \in \mathbb{N}, \quad (69)$$

we arrive at

$$\begin{aligned} 1 &= \|e_{n(k)}\| \quad \text{since, by (65)}, E_A(\Delta_{n(k)}) f_{n(k)} = 0; \\ &= \|E_A(\Delta_{n(k)})(e_{n(k)} - f_{n(k)})\| \\ &\leq \|E_A(\Delta_{n(k)})\| \|e_{n(k)} - f_{n(k)}\| \quad \text{by (14)}; \\ &\leq M\|e_{n(k)} - f_{n(k)}\| \leq M\left[d_{n(k)} + \frac{1}{k}\right] \longrightarrow 0, \\ &\qquad k \longrightarrow \infty, \end{aligned} \quad (70)$$

which is a *contradiction* proving (66).

As follows from the *Hahn-Banach Theorem*, for any $n \in \mathbb{N}$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \quad n \in \mathbb{N}, \text{ and}$$
$$\langle e_i, e_j^* \rangle = \delta_{ij} d_i, \quad i, j \in \mathbb{N}. \quad (71)$$

Let us consider separately the two possibilities concerning the sequence of the real parts $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$: its being *bounded* or *unbounded*.

First, suppose that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is *bounded*, i.e., there is such an $\omega > 0$ that

$$\left|\operatorname{Re}\lambda_n\right| \le \omega, \quad n \in \mathbb{N}, \tag{72}$$

and consider the element

$$f \coloneqq \sum_{k=1}^{\infty} k^{-2} e_k \in X, \tag{73}$$

which is well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ (l_1 is the space of absolutely summable sequences) and $\|e_k\| = 1$, $k \in \mathbb{N}$ (see (65)).

In view of (65), by the properties of the *s.m.*,

$$E_A\left(\bigcup_{k=1}^{\infty}\Delta_k\right) f = f \text{ and } E_A(\Delta_k) f = k^{-2} e_k, \quad k \in \mathbb{N}. \tag{74}$$

For any $t \ge 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \quad \text{by (74);} \\
&= \int_{\sigma(A)} e^{t\operatorname{Re}\lambda}\, dv\left(E_A\left(\bigcup_{k=1}^{\infty}\Delta_k\right) f, g^*, \lambda\right) \quad \text{by (18);} \\
&= \sum_{k=1}^{\infty} \int_{\sigma(A)\cap\Delta_k} e^{t\operatorname{Re}\lambda}\, dv(E_A(\Delta_k) f, g^*, \lambda) \quad \text{by (74);} \\
&= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A)\cap\Delta_k} e^{t\operatorname{Re}\lambda}\, dv(e_k, g^*, \lambda) \\
&\text{since, for } \lambda \in \Delta_k, \text{ by (72) and (62)}, \operatorname{Re}\lambda = \operatorname{Re}\lambda_k + (\operatorname{Re}\lambda - \operatorname{Re}\lambda_k) \le \operatorname{Re}\lambda_k + \left|\lambda - \lambda_k\right| \le \omega + \varepsilon_k \le \omega + 1; \\
&\le e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A)\cap\Delta_k} 1\, dv(e_k, g^*, \lambda) = e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} v(e_k, g^*, \Delta_k) \quad \text{by (15);} \\
&\le e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} 4M \|e_k\| \|g^*\| = 4Me^{t(\omega+1)} \|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty.
\end{aligned} \tag{75}$$

Also, for any $t < 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{t\operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \quad \text{by (74);} \\
&= \int_{\sigma(A)} e^{t\operatorname{Re}\lambda}\, dv\left(E_A\left(\bigcup_{k=1}^{\infty}\Delta_k\right) f, g^*, \lambda\right) \quad \text{by (18);} \\
&= \sum_{k=1}^{\infty} \int_{\sigma(A)\cap\Delta_k} e^{t\operatorname{Re}\lambda}\, dv(E_A(\Delta_k) f, g^*, \lambda) \quad \text{by (74);} \\
&= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A)\cap\Delta_k} e^{t\operatorname{Re}\lambda}\, dv(e_k, g^*, \lambda) \\
&\text{since, for } \lambda \in \Delta_k, \text{ by (72) and (62)}, \operatorname{Re}\lambda = \operatorname{Re}\lambda_k - (\operatorname{Re}\lambda_k - \operatorname{Re}\lambda) \ge \operatorname{Re}\lambda_k - \left|\operatorname{Re}\lambda_k - \operatorname{Re}\lambda\right| \ge -\omega - \varepsilon_k \ge -\omega - 1; \\
&\le e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A)\cap\Delta_k} 1\, dv(e_k, g^*, \lambda) = e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} v(e_k, g^*, \Delta_k) \quad \text{by (15);} \\
&\le e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} 4M \|e_k\| \|g^*\| = 4Me^{-t(\omega+1)} \|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty.
\end{aligned} \tag{76}$$

Similarly to (75), for any $t \geq 0$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\}} e^{t \operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\} \cap \Delta_k} 1\, dv(e_k, g^*, \lambda) \\
&\qquad \text{by (74);} \\
&= e^{t(\omega+1)} \\
&\quad \cdot \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\} \cap \Delta_k} 1\, dv(E_A(\Delta_k) f, g^*, \lambda) \\
&\qquad \text{by (18);} \\
&= e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\}} 1\, dv\left(E_A\left(\bigcup_{k=1}^{\infty} \Delta_k\right) f, g^*, \lambda\right) \quad \text{by (74);} \\
&= e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\}} 1\, dv(f, g^*, \lambda) \\
&\qquad \text{by (16);} \\
&\leq e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\|E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\right\}\right) f\right\| \cdot \|g^*\| \\
&\leq 4M e^{t(\omega+1)} \left\|E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\right\}\right) f\right\| \\
&\qquad \text{by the strong continuity of the } s.m.; \\
&\longrightarrow 4M e^{t(\omega+1)} \|E_A(\emptyset) f\| = 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{77}$$

Similarly to (76), for any $t < 0$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\}} e^{t \operatorname{Re}\lambda}\, dv(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\} \cap \Delta_k} 1\, dv(e_k, g^*, \lambda) \quad \text{by (74);} \\
&= e^{-t(\omega+1)} \\
&\quad \cdot \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\} \cap \Delta_k} 1\, dv(E_A(\Delta_k) f, g^*, \lambda) \\
&\qquad \text{by (18);} \\
&= e^{-t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\}} 1\, dv\left(E_A\left(\bigcup_{k=1}^{\infty} \Delta_k\right) f, g^*, \lambda\right) \quad \text{by (74);} \\
&= e^{-t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\}} 1\, dv(f, g^*, \lambda) \\
&\qquad \text{by (16);} \\
&\leq e^{-t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\|E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\right\}\right) f\right\| \cdot \|g^*\| \\
&\leq 4M e^{-t(\omega+1)} \left\|E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re}\lambda} > n\right\}\right) f\right\| \\
&\qquad \text{by the strong continuity of the } s.m.; \\
&\longrightarrow 4M e^{-t(\omega+1)} \|E_A(\emptyset) f\| = 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{78}$$

By Proposition 3, (75), (76), (77), and (78) jointly imply that

$$f \in \bigcap_{t \in \mathbb{R}} D\left(e^{tA}\right), \tag{79}$$

and hence, by Theorem 7,

$$y(t) := e^{tA} f, \quad t \in \mathbb{R}, \tag{80}$$

is a weak solution of (1).

Let

$$h^* := \sum_{k=1}^{\infty} k^{-2} e_k^* \in X^*, \tag{81}$$

the functional being well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_k^*\| = 1$, $k \in \mathbb{N}$ (see (71)).

In view of (71) and (66), we have

$$\left\langle e_n, h^* \right\rangle = \left\langle e_k, k^{-2} e_k^* \right\rangle = d_k k^{-2} \geq \varepsilon k^{-2}, \quad k \in \mathbb{N}. \tag{82}$$

Hence,

$$\begin{aligned}
&\int_{\sigma(A)} |\lambda|\, dv(f, h^*, \lambda) \quad \text{by (18) as in (75);} \\
&= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} |\lambda|\, dv(e_k, h^*, \lambda) \\
&\qquad \text{since, for } \lambda \in \Delta_k, \text{ by (61), } |\lambda| \geq k^4; \\
&\geq \sum_{k=1}^{\infty} k^{-2} k^4 v(e_k, h^*, \Delta_k) \geq \sum_{k=1}^{\infty} k^2 \left|\left\langle E_A(\Delta_k) e_k, h^* \right\rangle\right| \quad \text{by (65) and (82);} \\
&\geq \sum_{k=1}^{\infty} k^2 \varepsilon k^{-2} = \infty.
\end{aligned} \tag{83}$$

By Proposition 3, (83) implies that

$$y(0) = f \notin D(A), \tag{84}$$

which, by Proposition 9 ($n = 1$, $I = \{0\}$) further implies that the weak solution $y(t) = e^{tA} f$, $t \in \mathbb{R}$, of (1) is not strongly differentiable at 0.

Now, suppose that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is *unbounded*.

Therefore, there is a subsequence $\{\operatorname{Re}\lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\begin{gathered}
\operatorname{Re}\lambda_{n(k)} \longrightarrow \infty \text{ or} \\
\operatorname{Re}\lambda_{n(k)} \longrightarrow -\infty, \\
k \longrightarrow \infty.
\end{gathered} \tag{85}$$

Let us consider separately each of the two cases.

First, suppose that

$$\operatorname{Re}\lambda_{n(k)} \longrightarrow \infty, \quad k \longrightarrow \infty \tag{86}$$

Then, without loss of generality, we can regard that

$$\operatorname{Re}\lambda_{n(k)} \geq k, \quad k \in \mathbb{N}. \tag{87}$$

Consider the elements

$$\begin{aligned} f &:= \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e_{n(k)} \in X \text{ and} \\ h &:= \sum_{k=1}^{\infty} e^{-(n(k)/2)\operatorname{Re}\lambda_{n(k)}} e_{n(k)} \in X, \end{aligned} \tag{88}$$

well defined since, by (87),

$$\left\{e^{-n(k)\operatorname{Re}\lambda_{n(k)}}\right\}_{k=1}^{\infty}, \left\{e^{-(n(k)/2)\operatorname{Re}\lambda_{n(k)}}\right\}_{k=1}^{\infty} \in l_1 \tag{89}$$

and $\|e_{n(k)}\| = 1$, $k \in \mathbb{N}$ (see (65)).

By (65),

$$\begin{aligned} &E_A\left(\bigcup_{k=1}^{\infty} \Delta_{n(k)}\right) f = f \text{ and} \\ &E_A\left(\Delta_{n(k)}\right) f = e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e_{n(k)}, \\ &k \in \mathbb{N}, \end{aligned} \tag{90}$$

and

$$\begin{aligned} &E_A\left(\bigcup_{k=1}^{\infty} \Delta_{n(k)}\right) h = h \text{ and} \\ &E_A\left(\Delta_{n(k)}\right) h = e^{-(n(k)/2)\operatorname{Re}\lambda_{n(k)}} e_{n(k)}, \\ &k \in \mathbb{N}. \end{aligned} \tag{91}$$

For any $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned} &\int_{\sigma(A)} e^{t\operatorname{Re}\lambda}\, dv\left(f, g^*, \lambda\right) \quad \text{by (18) as in (75);} \\ &= \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} \int_{\sigma(A)\cap\Delta_{n(k)}} e^{t\operatorname{Re}\lambda}\, dv\left(e_{n(k)}, g^*, \lambda\right) \\ &\text{since, for } \lambda \in \Delta_{n(k)}, \text{ by (62), } \operatorname{Re}\lambda = \operatorname{Re}\lambda_{n(k)} + \left(\operatorname{Re}\lambda - \operatorname{Re}\lambda_{n(k)}\right) \leq \operatorname{Re}\lambda_{n(k)} + \left|\lambda - \lambda_{n(k)}\right| \leq \operatorname{Re}\lambda_{n(k)} + 1; \\ &\leq \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e^{t(\operatorname{Re}\lambda_{n(k)}+1)} \int_{\sigma(A)\cap\Delta_{n(k)}} 1\, dv\left(e_{n(k)}, g^*, \lambda\right) \\ &= e^t \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} v\left(e_{n(k)}, g^*, \Delta_{n(k)}\right) \quad \text{by (15);} \\ &\leq e^t \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} 4M \left\|e_{n(k)}\right\| \left\|g^*\right\| = 4Me^t \left\|g^*\right\| \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} \\ &< \infty. \end{aligned} \tag{92}$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$n(k) \geq t + 1, \tag{93}$$

in view of (87),

$$e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} \leq e^{-k}. \tag{94}$$

For any $t < 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned} &\int_{\sigma(A)} e^{t\operatorname{Re}\lambda}\, dv\left(f, g^*, \lambda\right) \quad \text{by (18) as in (75);} \\ &= \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} \int_{\sigma(A)\cap\Delta_{n(k)}} e^{t\operatorname{Re}\lambda}\, dv\left(e_{n(k)}, g^*, \lambda\right) \end{aligned}$$

$$\text{since, for } \lambda \in \Delta_{n(k)}, \text{ by } (62), \ \operatorname{Re}\lambda = \operatorname{Re}\lambda_{n(k)} - \left(\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right) \geq \operatorname{Re}\lambda_{n(k)} - \left|\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right| \geq \operatorname{Re}\lambda_{n(k)} - 1;$$
$$\leq \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e^{t(\operatorname{Re}\lambda_{n(k)}-1)} \int_{\sigma(A)\cap\Delta_{n(k)}} 1\, dv\left(e_{n(k)}, g^*, \lambda\right)$$
$$= e^{-t}\sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} v\left(e_{n(k)}, g^*, \Delta_{n(k)}\right) \quad \text{by } (15);$$
$$\leq e^{-t}\sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} 4M \left\|e_{n(k)}\right\| \left\|g^*\right\| = 4Me^{-t}\left\|g^*\right\| \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}}$$
$$< \infty. \tag{95}$$

Indeed, for all $k \in \mathbb{N}$, in view of $t < 0$,

$$n(k) - t \geq n(k) \geq 1, \tag{96}$$

and hence, in view of (87),

$$e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} \leq e^{-k}. \tag{97}$$

Similarly to (92), for any $t \geq 0$ and an arbitrary $n \in \mathbb{N}$,

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} e^{t\operatorname{Re}\lambda}\, dv\left(f, g^*, \lambda\right)$$
$$\leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} e^{t}\sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}} 1\, dv\left(e_{n(k)}, g^*, \lambda\right)$$
$$= e^{t}\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-[(n(k)/2)-t]\operatorname{Re}\lambda_{n(k)}} e^{-(n(k)/2)\operatorname{Re}\lambda_{(k)}}$$
$$\int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}} 1\, dv\left(e_{n(k)}, g^*, \lambda\right)$$
$$\text{since, by } (87), \text{ there is an } L > 0 \text{ such that } e^{-[(n(k)/2)-t]\operatorname{Re}\lambda_{n(k)}} \leq L, \ k \in \mathbb{N};$$
$$\leq Le^{t}\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-(n(k)/2)\operatorname{Re}\lambda_{n(k)}} \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}} 1\, dv\left(e_{n(k)}, g^*, \lambda\right) \quad \text{by } (91);$$
$$= Le^{t}\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}} 1\, dv\left(E_A\left(\Delta_{n(k)}\right)h, g^*, \lambda\right) \quad \text{by } (18);$$
$$= Le^{t}\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} 1\, dv\left(E_A\left(\bigcup_{k=1}^{\infty}\Delta_{n(k)}\right)h, g^*, \lambda\right) \quad \text{by } (91);$$
$$= Le^{t}\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} 1\, dv\left(h, g^*, \lambda\right) \quad \text{by } (16);$$
$$\leq Le^{t}\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M\left\|E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\right\}\right)h\right\|\left\|g^*\right\|$$

$$\begin{aligned}
&\le 4LMe^{t}\left\|E_A\left(\left\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\right\}\right)h\right\| \\
&\qquad \text{by the strong continuity of the } s.m.; \\
&\longrightarrow 4LMe^{t}\left\|E_A(\emptyset)h\right\|=0,\quad n\longrightarrow\infty.
\end{aligned} \tag{98}$$

Similar to 5, for any $t<0$ and an arbitrary $n\in\mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^*\in X^*\mid\|g^*\|=1\}}\int_{\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \\
&\le \sup_{\{g^*\in X^*\mid\|g^*\|=1\}} e^{-t}\sum_{k=1}^{\infty}e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}}\int_{\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}}1\,dv(e_{n(k)},g^*,\lambda) \\
&= e^{-t}\sup_{\{g^*\in X^*\mid\|g^*\|=1\}}\sum_{k=1}^{\infty}e^{-[(n(k)/2)-t]\operatorname{Re}\lambda_{n(k)}}e^{-(n(k)/2)\operatorname{Re}\lambda_{(k)}} \\
&\int_{\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}}1\,dv(e_{n(k)},g^*,\lambda) \\
&\text{since, by (87), there is an } L>0 \text{ such that } e^{-[(n(k)/2)-t]\operatorname{Re}\lambda_{n(k)}}\le L,\ k\in\mathbb{N}; \\
&\le Le^{-t}\sup_{\{g^*\in X^*\mid\|g^*\|=1\}}\sum_{k=1}^{\infty}e^{-(n(k)/2)\operatorname{Re}\lambda_{n(k)}}\int_{\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}}1\,dv(e_{n(k)},g^*,\lambda) \\
&\qquad \text{by (91);} \\
&= Le^{-t}\sup_{\{g^*\in X^*\mid\|g^*\|=1\}}\sum_{k=1}^{\infty}\int_{\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_{n(k)}}1\,dv(E_A(\Delta_{n(k)})h,g^*,\lambda) \\
&\qquad \text{by (18);} \\
&= Le^{-t}\sup_{\{g^*\in X^*\mid\|g^*\|=1\}}\int_{\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\}}1\,dv\left(E_A\left(\bigcup_{k=1}^{\infty}\Delta_{n(k)}\right)h,g^*,\lambda\right) \\
&\qquad \text{by (91);} \\
&= Le^{-t}\sup_{\{g^*\in X^*\mid\|g^*\|=1\}}\int_{\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\}}1\,dv(h,g^*,\lambda)\quad\text{by (16);} \\
&\le Le^{-t}\sup_{\{g^*\in X^*\mid\|g^*\|=1\}}4M\left\|E_A\left(\left\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\right\}\right)h\right\|\|g^*\| \\
&\le 4LMe^{-t}\left\|E_A\left(\left\{\lambda\in\sigma(A)\mid e^{t\operatorname{Re}\lambda}>n\right\}\right)h\right\| \\
&\qquad \text{by the strong continuity of the } s.m.; \\
&\longrightarrow 4LMe^{-t}\left\|E_A(\emptyset)h\right\|=0,\quad n\longrightarrow\infty.
\end{aligned} \tag{99}$$

By Proposition 3, (92), 5, 5, and (99) jointly imply that

$$f\in\bigcap_{t\in\mathbb{R}}D\left(e^{tA}\right), \tag{100}$$

and hence, by Theorem 7,

$$y(t):=e^{tA}f,\quad t\in\mathbb{R}, \tag{101}$$

is a weak solution of (1).

Since, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$, by (62), (87),

$$\begin{aligned} \operatorname{Re}\lambda &= \operatorname{Re}\lambda_{n(k)} - \left(\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right) \\ &\geq \operatorname{Re}\lambda_{n(k)} - \left|\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right| \geq \operatorname{Re}\lambda_{n(k)} - \varepsilon_{n(k)} \\ &\geq \operatorname{Re}\lambda_{n(k)} - \frac{1}{n(k)} \geq k - 1 \geq 0 \end{aligned} \tag{102}$$

and, by (61),

$$\operatorname{Re}\lambda < (2n(k))^{-1} \ln |\operatorname{Im}\lambda|, \tag{103}$$

we infer that, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$,

$$|\lambda| \geq |\operatorname{Im}\lambda| \geq e^{2n(k)\operatorname{Re}\lambda} \geq e^{2n(k)(\operatorname{Re}\lambda_{n(k)} - 1/n(k))}. \tag{104}$$

Using this estimate, for the functional $h^* \in X^*$ defined by (81), we have

$$\begin{aligned} &\int_{\sigma(A)} |\lambda| \, dv\left(f, h^*, \lambda\right) \quad \text{by (18) as in (75);} \\ &= \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} \int_{\Delta_{n(k)}} |\lambda| \, dv\left(e_{n(k)}, h^*, \lambda\right) \\ &\geq \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e^{2n(k)(\operatorname{Re}\lambda_{n(k)} - 1/n(k))} v\left(e_{n(k)}, h^*, \Delta_{n(k)}\right) \\ &= \sum_{k=1}^{\infty} e^{-2} e^{n(k)\operatorname{Re}\lambda_{n(k)}} \left|\left\langle E_A\left(\Delta_{n(k)}\right) e_{n(k)}, h^*\right\rangle\right| \\ &\qquad \text{by (87), (65), and (82);} \\ &\geq \sum_{k=1}^{\infty} e^{-2} \varepsilon \frac{e^{n(k)}}{n(k)^2} = \infty. \end{aligned} \tag{105}$$

By Proposition 3, (83) implies that

$$y(0) = f \notin D(A), \tag{106}$$

which, by Proposition 9 ($n = 1$, $I = \{0\}$), further implies that the weak solution $y(t) = e^{tA} f$, $t \in \mathbb{R}$, of (1) is not strongly differentiable at 0.

The remaining case of

$$\operatorname{Re}\lambda_{n(k)} \longrightarrow -\infty, \quad k \longrightarrow \infty, \tag{107}$$

is symmetric to the case of

$$\operatorname{Re}\lambda_{n(k)} \longrightarrow \infty, \quad k \longrightarrow \infty, \tag{108}$$

and is considered in absolutely the same manner, which furnishes a weak solution $y(\cdot)$ of (1) such that

$$y(0) \notin D(A), \tag{109}$$

and hence, by Proposition 9 ($n = 1$, $I = \{0\}$), not strongly differentiable at 0.

With every possibility concerning $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ considered, we infer that assuming the opposite to the *"if"* part's premise allows to find a weak solution of (1) on $[0, \infty)$ that is not strongly differentiable at 0, much less strongly infinite differentiable on $\mathbb{R}$.

Thus, the proof by contrapositive of the *"only if" part* is complete and so is the proof of the entire statement. □

From Theorem 12 and [1, Theorem 4.2], the latter characterizing the strong infinite differentiability of all weak solution of (2) on $(0, \infty)$, we also obtain the following.

Corollary 13. *Let A be a scalar type spectral operator in a complex Banach space. If all weak solutions of (2) are strongly infinite differentiable on $(0, \infty)$, then all weak solutions of (1) are strongly infinite differentiable on $\mathbb{R}$.*

Remark 14. As follows from Theorem 12, all weak solutions of (1) with a scalar type spectral operator A in a complex Banach space can be *strongly infinite differentiable* while the operator A is *unbounded*, e.g., when A is an unbounded *self-adjoint* operator in a complex Hilbert space (cf. [17, Theorem 7.1]). This fact contrasts the situation when a closed densely defined linear operator A in a complex Banach space generates a strongly continuous group $\{T(t)\}_{t\in\mathbb{R}}$ of bounded linear operators, i.e., the associated abstract Cauchy problem is *well-posed* (see Remarks 1.1), in which case even the (left or right) strong differentiability of all weak solutions of (1) at 0 immediately implies *boundedness* for A (cf. [3]).

6. The Cases of Normal and Self-Adjoint Operators

As an important particular case of Theorem 12, we obtain

Corollary 15 (the case of a normal operator). *Let A be a normal operator in a complex Hilbert space. Every weak solution of (1) is strongly infinite differentiable on $\mathbb{R}$ iff there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \backslash \mathscr{L}_{b_-, b_+}$, where*

$$\begin{aligned} \mathscr{L}_{b_-, b_+} := \{\lambda \in \mathbb{C} \mid &\operatorname{Re}\lambda \\ &\leq \min\left(0, -b_- \ln |\operatorname{Im}\lambda|\right) \text{ or } \operatorname{Re}\lambda \\ &\geq \max\left(0, b_+ \ln |\operatorname{Im}\lambda|\right)\}, \end{aligned} \tag{110}$$

is bounded (see Figure 1).

Remark 16. Corollary 15 develops the results of paper [17], where similar consideration is given to the strong differentiability of the weak solutions of (2) with a normal operator A in a complex Hilbert space on $[0, \infty)$ and $(0, \infty)$.

From Corollary 13, we immediately obtain the following.

Corollary 17. *Let A be a normal operator in a complex Hilbert space. If all weak solutions of (2) are strongly infinite differentiable on $(0, \infty)$ (cf. [17, Theorem 5.2]), then all weak solutions of (1) are strongly infinite differentiable on $\mathbb{R}$.*

Considering that, for a self-adjoint operator A in a complex Hilbert space X,

$$\sigma(A) \subseteq \mathbb{R} \tag{111}$$

(see, e.g., [10, 11]), we further arrive at the following.

Corollary 18 (the case of a self-adjoint operator). *Every weak solution of (1) with a self-adjoint operator A in a complex Hilbert space is strongly infinite differentiable on $\mathbb{R}$.*

Cf. [17, Theorem 7.1].

7. Inherent Smoothness Improvement Effect

As is observed in the proof of the *"only if"* part of Theorem 12, the opposite to the *"if"* part's premise implies that there is a weak solution of (1), which is not strongly differentiable at 0. This renders the case of finite strong differentiability of the weak solutions superfluous and we arrive at the following inherent effect of smoothness improvement.

Proposition 19. *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. If every weak solution of (1) is strongly differentiable at 0, then all of them are strongly infinite differentiable on $\mathbb{R}$.*

Cf. [1, Proposition 5.1].

8. Concluding Remark

Due to the *scalar type spectrality* of the operator A, Theorem 12 is stated exclusively in terms of the location of its *spectrum* in the complex plane, similarly to the celebrated *Lyapunov stability theorem* [19] (cf. [3, Ch. I, Theorem 2.10]), and thus is an intrinsically qualitative statement (cf. [1, 20]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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