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Research Article

Oscillatory and Asymptotic Behavior of a First-Order Neutral Equation of Discrete Type with Variable Several Delay under Δ Sign

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We obtain necessary and sufficient conditions so that every solution of neutral delay difference equation $\Delta(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}) + q_n G(y_{\sigma(n)}) = f_n$ oscillates or tends to zero as $n \longrightarrow \infty$, where $\{q_n\}$ and $\{f_n\}$ are real sequences and $G \in C(\mathbb{R}, \mathbb{R})$, $xG(x) > 0$, and $m_1, m_2, \ldots, m_k$ are positive integers. Here Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, and $\{\sigma_n\}$ is an increasing unbounded sequences with $\sigma_n \le n$. This paper complements, improves, and generalizes some past and recent results.

1. Introduction

Consider the neutral delay difference equation of first order

$$\Delta\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right) + q_n G\left(y_{\sigma(n)}\right) = f_n \tag{1}$$

where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, q_n and f_n are members of infinite real sequences, and m_j are positive integers. Further, assume $\{p_n^j\}$ are real sequences for each $j \in 1, 2, \ldots k$ and that $G \in C(\mathbb{R}, \mathbb{R})$ and $\sigma(n) \le n$ are monotonic increasing sequences which are unbounded.

We study the oscillatory behavior of solutions of neutral difference equation (1) under the following assumptions.

(H1) $xG(x) > 0$ for $x \neq 0$.

(H2) There exists a bounded sequence $\{F_n\}$ such that $\Delta F_n = f_n$.

(H3) The sequence $\{F_n\}$ in (H2) satisfies $\lim_{n\longrightarrow\infty} F_n = 0$.

(H4) $q_n > 0$, $\sum_{n=n_0}^{\infty} q_n = \infty$.

In addition to the above we assume some new conditions on p_n^j (see (12), (22), (26), and (30) in next section). It is important to note that our results hold good for the solutions of the neutral equation

$$\Delta\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right) + \sum_{j=1}^{l} q_n^j G\left(y_{\sigma_j(n)}\right) = f_n \tag{2}$$

under the assumption

$$\sum_{n=n_0}^{\infty}\left(\sum_{j=1}^{l} q_n^j\right) = \infty, \quad q_n^j > 0 \tag{3}$$

instead of (H4). The following neutral difference equations/delay difference equations are obtained as particular case of (2).

$$\Delta\left(y_n - p_n y_{n-m}\right) + \sum_{j=1}^{l} q_n^j G\left(y_{\sigma_j(n)}\right) = f_n, \tag{4}$$

$$\Delta\left(y_n - p_n y_{n-m}\right) + q_n G\left(y_{\sigma(n)}\right) = f_n, \tag{5}$$

$$\Delta\left(y_n\right) + \sum_{j=1}^{l} q_n^j G\left(y_{\sigma_j(n)}\right) = 0, \tag{6}$$

and

$$\Delta\left(y_n\right) + q_n G\left(y_{\sigma(n)}\right) = 0. \tag{7}$$

The neutral difference equations (5) are seen as the discrete analogue of the neutral differential equations

$$\left(y(t) - p(t)\, y(t-\tau)\right)' + q(t)\, G\left(y(t-\sigma)\right) = f(t). \tag{8}$$

The oscillatory and asymptotic behavior of delay difference equations and neutral difference equations have been intensively studied in recent years due to its various application in different field of science and technology [1]. It is observed that several articles (see [2–4]) exist in literature for the study of neutral difference equations/delay difference equations with several delay, i.e., for (4) or (6), respectively. However study of neutral equations with several delay term under Δ symbol, i.e., (1) or (2), seems to be relatively scarce in literature. Use of lemmas from [1, Lemma 1.5.1 and 1.5.2] or its discrete analogue (see [5]) plays an important role in studying (4) [6], (5) [7], and (8) [8]. In this context, one may note these lemmas cannot be applied to the study of (1) or (2). Hence study of (1) and (2) needs a different approach.

The work in this paper complements and generalizes the work in [3, 9]. This can be verified that the results in [3, 9] which are concerned with the study of (6) and (7) cannot be applied to the delay difference equation

$$\Delta\left(y_n\right) + \left(e^{-2} - e^{-3}\right) y_{n-2} = 0. \tag{9}$$

which has a solution $y_n = e^{-n}$ tending to zero. It is because the primary assumption,

$$\liminf_{k\longrightarrow\infty} \sum_{i=\sigma(k)}^{k-1} q_i > \frac{1}{e}, \tag{10}$$

is not satisfied. However, note that (10) implies (H4) and (H4) is satisfied in (9) and hence the results of this paper give an answer to the behavior of solutions of neutral equations like (9). While working on nonlinear neutral equations most of the authors [7, 8, 10–12] assume the condition that G is nondecreasing unlike this paper.

Let n_0 be a fixed nonnegative integer. Let $\max\{m_1, m_2, \dots, m_k\} = m$ and $\rho = \min\{\sigma(n_0), n_0 - m\}$. By a solution of (1) we mean a real sequence $\{y_n\}$ which is defined for all positive integers $n \ge \rho$ and satisfies (1) for $n \ge n_0$. Clearly if the initial condition

$$y_n = a_n \quad \text{for } \rho \le n \le n_0, \tag{11}$$

is given then (1) has a unique solution satisfying the given initial condition (11). A solution $\{y_n\}$ of (1) is said to be oscillatory if, for every positive integer $n_0 > 0$, there exists $n \ge n_0$ such that $y_n y_{n+1} \le 0$; otherwise $\{y_n\}$ is said to be nonoscillatory. In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all n sufficiently large. Here we assume the existence of solution of (1) and study its oscillatory and asymptotic behavior.

2. Sufficient Condition

In this section we present some results which prove that (H4) is sufficient for any solution of (2) to be oscillatory or tending to zero as $n \longrightarrow \infty$. Moreover we give some examples to illustrate and signify our results. Our first result and the subsequent ones are as follows.

Theorem 1. *Suppose that (H1)–(H4) hold. Assume that there exists a positive constant p such that the sequences $\{p_n^j\}$ for $j = 1, 2, \dots, k$ satisfy the condition*

$$\begin{gathered} p_n^j \ge 0, \quad \text{for every } j = 1, 2, \dots, k \\ \text{and } \sum_{j=1}^{k} \limsup_{n\longrightarrow\infty} p_n^j < p < 1. \end{gathered} \tag{12}$$

Then every solution of (1) oscillates or tends to zero as $n \longrightarrow \infty$.

Proof. Let $y = \{y_n\}$ be any solution of (1) for $n \ge n_0$, where n_0 is a fixed positive integer. If it oscillates then there is nothing to prove; otherwise, it leads to two distinct possibilities, either $y_n > 0$ or $y_n < 0$. for $n \ge n_1 > n_0$. Consider the first one, i.e., $y_n > 0$ eventually. There exits positive integer $n_2 \ge n_1$ such that $y_n > 0$, $y_{n-m_j} > 0$ for each j, and $y_{\sigma(n)} > 0$ for $n \ge n_2$. For $n \ge n_2$, let

$$z_n = y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}, \tag{13}$$

and

$$w_n = z_n - F_n. \tag{14}$$

From (1), (13), and (14), it follows due to (H1) that

$$\Delta w_n = -q_n G\left(y_{\sigma(n)}\right) \le 0. \tag{15}$$

Then there exists $n_3 \ge n_2$ such that w_n is monotonic and is of constant sign for $n \ge n_3$. For the sake of a contradiction assume that y_n is not bounded. Then there exists a subsequence $\{y_{n_r}\}$ such that

$$n_r \longrightarrow \infty, \quad y_{n_r} \longrightarrow \infty \quad \text{as } r \longrightarrow \infty, \tag{16}$$

and

$$y(n_r) = \max\{y_n : n_3 \le n \le n_r\}. \tag{17}$$

Since $\sigma(n) \longrightarrow \infty$ as $n \longrightarrow \infty$, we may choose r large enough so that $\sigma(n_r) \ge n_3$. For $0 < \epsilon$, because of (H3), we can find a positive integer n_4 such that $n \ge n_4 \ge n_3$ implies $|F_n| < \epsilon$. As (12) holds, then using (13), (14), and (17) we obtain

$$\begin{aligned} w_{n_r} &= y_{n_r} - \sum_{j=1}^{k} p_n^j y_{n_r-m_j} - F_{n_r} \ge \left(1 - \sum_{j=1}^{k} p_n^j\right) y_{n_r} - \epsilon \\ &> (1-p)\, y_{n_r} - \epsilon. \end{aligned} \tag{18}$$

Taking $r \longrightarrow \infty$, we find $\lim_{n\longrightarrow\infty} w_n = \infty$, a contradiction as w_n is monotonic decreasing. Hence y_n is bounded which implies w_n and z_n are bounded and $\lim_{n\longrightarrow\infty} w_n$ exists. Further it follows that $\liminf_{n\longrightarrow\infty} y_n$ and $\limsup_{n\longrightarrow\infty} y_n$ exist. We claim $\liminf_{n\longrightarrow\infty} y_n = 0$. Otherwise, let $y_n \ge \alpha > 0$. Next boundedness of y_n yields $y_n \le \beta$. Hence we have $0 \le \alpha < y_n \le \beta$, which will be used for bounding the G term in (1) from below.

From the continuity of G and assumption (H1) it follows that there exists a positive lower bound m for G on $[\alpha, \beta]$. Hence there exists n_5 such that $G(y_{\sigma(n)}) > m > 0$ for $n > n_5$. Then summing (15) from $n = n_5$ to $s - 1$ we obtain

$$w_{n_5} - w_s = \sum_{j=n_5}^{s-1} q_j G\left(y_{\sigma(j)}\right) \ge m \sum_{j=n_5}^{\infty} q_j. \tag{19}$$

Since the left hand side is the member of a bounded sequence, while the right hand side approaches $+\infty$, we have a contradiction. This yields $\liminf_{n\longrightarrow\infty} y_n = 0$. From (H3), monotonic nature of w_n and (14), it follows that $\lim_{n\longrightarrow\infty} z_n$ exists finitely. Let $\lim_{n\longrightarrow\infty} z_n = \delta$. If $\delta > 0$, then

$$\begin{aligned} 0 < \delta &= \liminf_{n\longrightarrow\infty} z_n = \liminf_{n\longrightarrow\infty}\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right) \\ &\le \liminf_{n\longrightarrow\infty} y_n + \limsup_{n\longrightarrow\infty}\left(-\sum_{j=1}^{k} p_n^j y_{n-m_j}\right) \\ &= -\liminf_{n\longrightarrow\infty}\left(\sum_{j=1}^{k} p_n^j y_{n-m_j}\right) \le -\sum_{j=1}^{k} \liminf_{n\longrightarrow\infty}\left(p_n^j y_{n-m_j}\right) \\ &\le -\sum_{j=1}^{k}\left(\liminf_{n\longrightarrow\infty} p_n^j\right)\left(\liminf_{n\longrightarrow\infty} y_{n-m_j}\right) \le 0, \end{aligned} \tag{20}$$

a contradiction. If $\delta \le 0$ then

$$\begin{aligned} 0 \ge \delta &= \limsup_{n\longrightarrow\infty} z_n = \limsup_{n\longrightarrow\infty}\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right) \\ &\ge \limsup_{n\longrightarrow\infty} y_n + \liminf_{n\longrightarrow\infty}\left(-\sum_{j=1}^{k} p_n^j y_{n-m_j}\right) \\ &\ge \limsup_{n\longrightarrow\infty} y_n - \limsup_{n\longrightarrow\infty}\left(\sum_{j=1}^{k} p_n^j y_{n-m_j}\right) \\ &\ge \limsup_{n\longrightarrow\infty} y_n - \sum_{j=1}^{k} \limsup_{n\longrightarrow\infty}\left(p_n^j y_{n-m_j}\right) \\ &\ge \limsup_{n\longrightarrow\infty} y_n - \sum_{j=1}^{k} \limsup_{n\longrightarrow\infty} p_n^j \limsup_{n\longrightarrow\infty} y_{n-m_j} \\ &\ge \limsup_{n\longrightarrow\infty} y_n \left(1 - \sum_{j=1}^{k} \limsup_{n\longrightarrow\infty} p_n^j\right) \\ &\ge \limsup_{n\longrightarrow\infty} y_n (1-p). \end{aligned} \tag{21}$$

Hence $\limsup_{n\longrightarrow\infty} y_n \le 0$, by (12), which implies the desired result $\lim_{n\longrightarrow\infty} y_n = 0$. If $y_n < 0$ for $n > n_1$ then proceeding as above we can arrive at $\lim_{n\longrightarrow\infty} y_n = 0$. Thus the theorem is proved. □

Theorem 2. *Suppose that (H1)–(H4) hold. Assume that there exists a positive constant p such that the sequences $\{p_n^j\}$ for $j = 1, 2, \ldots, k$ satisfy the condition*

$$\begin{gathered} p_n^j \le 0, \quad \textit{for every } j = 1, 2, \ldots, k \\ \textit{and } \sum_{j=1}^{k} \liminf_{n\longrightarrow\infty} p_n^j > -p > -1. \end{gathered} \tag{22}$$

Then every solution of (1) oscillates or tends to zero as $n \longrightarrow \infty$.

Proof. Proceeding as in the proof of Theorem 1 and setting z_n, w_n as in (13) and (14), respectively, we obtain (15) and further prove y_n is bounded with $\liminf_{n\longrightarrow\infty} y_n = 0$. From (H3) and the fact that w_n is monotonic it follows that $\lim_{n\longrightarrow\infty} w_n =$

$\lim_{n\longrightarrow\infty} z_n = \delta \in \mathbb{R}$. As $z_n \geq 0$, so $\delta \geq 0$. We claim $\delta = 0$; if not then $\delta > 0$, and this implies

$$\begin{aligned}\delta &= \liminf_{n\longrightarrow\infty} z_n = \liminf_{n\longrightarrow\infty}\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right)\\ &\leq \liminf_{n\longrightarrow\infty} y_n + \limsup_{n\longrightarrow\infty}\left(-\sum_{j=1}^{k} p_n^j y_{n-m_j}\right)\\ &\leq \sum_{j=1}^{k}\limsup_{n\longrightarrow\infty}\left(-p_n^j\right)\limsup_{n\longrightarrow\infty}\left(y_{n-m_j}\right)\\ &= \sum_{j=1}^{k} -\liminf_{n\longrightarrow\infty}\left(p_n^j\right)\limsup_{n\longrightarrow\infty}\left(y_{n-m_j}\right)\\ &\leq p\limsup_{n\longrightarrow\infty}\left(y_n\right) \leq p\alpha.\end{aligned} \tag{23}$$

Hence we get

$$\alpha \geq \frac{\delta}{p} > \delta. \tag{24}$$

Again

$$\begin{aligned}\delta &= \limsup_{n\longrightarrow\infty} z_n = \limsup_{n\longrightarrow\infty}\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right)\\ &\geq \limsup_{n\longrightarrow\infty} y_n + \liminf_{n\longrightarrow\infty}\left(-\sum_{j=1}^{k} p_n^j y_{n-m_j}\right)\\ &= \limsup_{n\longrightarrow\infty} y_n + \liminf_{n\longrightarrow\infty}\left(\sum_{j=1}^{k}\left(-p_n^j\right) y_{n-m_j}\right)\\ &\geq \limsup_{n\longrightarrow\infty} y_n + \sum_{j=1}^{k}\liminf_{n\longrightarrow\infty}\left(\left(-p_n^j\right) y_{n-m_j}\right)\\ &\geq \limsup_{n\longrightarrow\infty} y_n + \sum_{j=1}^{k}\liminf_{n\longrightarrow\infty}\left(-p_n^j\right)\liminf_{n\longrightarrow\infty} y_{n-m_j}\\ &= \limsup_{n\longrightarrow\infty} y_n = \alpha,\end{aligned} \tag{25}$$

a contradiction, due to inequality (24). Hence we conclude $\delta = 0$ and from $z_n > y_n$, it follows that $\lim_{n\longrightarrow\infty} y_n \leq 0$. Hence $\lim_{n\longrightarrow\infty} y_n = 0$.

The proof for the case $y_n < 0$ for large n is similar. Hence the theorem is proved. □

Remark 3. Theorems 1 and 2 hold good for $k = 0$ and $k = 1$. Hence these results could be compared with results concerned with the difference equations (4), (5), (6), and (7).

Theorem 4. *Suppose that (H1)–(H4) hold. Assume that there exists a positive constant p such that the sequences $\{p_n^j\}$ for $j = 1, 2, \ldots, k$ satisfy the condition*

$$p_n^j < 0 \quad \text{for every } j = 1,2,\ldots,k \text{ and there exists, } i \in \{1,2,3,\ldots,k\} \text{ such that } \limsup_{n\longrightarrow\infty} p_n^i - \sum_{j\neq i}\liminf_{n\longrightarrow\infty} p_n^j < -1. \tag{26}$$

Then every solution of (1) oscillates or tends to zero as $n \longrightarrow \infty$.

Proof. Proceeding as in the proof of Theorem 1 and setting z_n, w_n as in (13) and (14), respectively, we obtain (15) and further prove y_n is bounded with $\liminf_{n\longrightarrow\infty} y_n = 0$. From (H3) and that w_n is monotonic it follows that $\lim_{n\longrightarrow\infty} w_n = \lim_{n\longrightarrow\infty} z_n = \delta \in \mathbb{R}$. As $z_n \geq 0$, so $\delta \geq 0$. We claim $\delta = 0$. If not, then $\delta > 0$, and this implies

$$\begin{aligned}\delta &= \liminf_{n\longrightarrow\infty} z_n = \liminf_{n\longrightarrow\infty}\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right)\\ &\leq \limsup_{n\longrightarrow\infty}\left(y_n + \sum_{j\neq i} - p_n^j y_{n-m_j}\right)\\ &\quad + \liminf_{n\longrightarrow\infty}\left(-p_n^i y_{n-m_i}\right)\\ &\leq \limsup_{n\longrightarrow\infty} y_n + \limsup_{n\longrightarrow\infty}\sum_{j\neq i} - p_n^j y_{n-m_j}\\ &\quad + \limsup_{n\longrightarrow\infty}\left(-p_n^j\right)\liminf_{n\longrightarrow\infty}\left(y_{n-m_j}\right)\\ &\leq \limsup_{n\longrightarrow\infty} y_n + \sum_{j\neq i1}\limsup_{n\longrightarrow\infty}\left(-p_n^j\right)\limsup_{n\longrightarrow\infty}\left(y_{n-m_j}\right)\\ &\leq \limsup_{n\longrightarrow\infty}\left(y_n\right)\left[1 - \sum_{j\neq i}\liminf_{n\longrightarrow\infty} p_n^j\right].\end{aligned} \tag{27}$$

Again we have

$$\begin{aligned}\delta &= \limsup_{n\longrightarrow\infty} z_n = \limsup_{n\longrightarrow\infty}\left(y_n - \sum_{j=1}^{k} p_n^j y_{n-m_j}\right)\\ &\geq \liminf_{n\longrightarrow\infty} y_n + \limsup_{n\longrightarrow\infty}\left(-\sum_{j=1}^{k} p_n^j y_{n-m_j}\right)\\ &= \limsup_{n\longrightarrow\infty}\left(-p_n^i y_{n-mi}\right) + \liminf_{n\longrightarrow\infty}\sum_{j\neq i}\left(-p_n^j y_{n-m_j}\right)\\ &\geq \limsup_{n\longrightarrow\infty} y_{n-m_j}\liminf_{n\longrightarrow\infty}\left(-p_n^i\right)\\ &\quad + \sum_{j\neq i}\liminf_{n\longrightarrow\infty}\left(\left(-p_n^j\right) y_{n-m_j}\right)\end{aligned}$$

$$\begin{aligned}&\geq \limsup_{n\to\infty} y_n \left(-\limsup_{n\to\infty} p_n^i\right)\\&\quad+\sum_{j\neq i}\liminf_{n\to\infty}\left(-p_n^j\right)\liminf_{n\to\infty} y_{n-m_j}\\&\geq \limsup_{n\to\infty} y_n \left(-\limsup_{n\to\infty} p_n^i\right).\end{aligned}\tag{28}$$

From (27) and (28), it follows that

$$\limsup_{n\to\infty} y_n \left(\left(\sum_{j\neq i}\liminf_{n\to\infty} p_n^j\right)-1-\limsup_{n\to\infty} p_n^i\right)\leq 0\tag{29}$$

Using (26), we obtain $\lim_{n\to\infty} y_n = 0$. Thus the theorem is proved. □

Next, we intend to present a result where p_n^j, $j = 1, 2, 3, \ldots, k$, satisfy the following condition:

$$p_n^j > 0 \quad \text{for every } j = 1, 2, \ldots, k \text{ and there exists, } i \in \{1, 2, 3, \ldots, k\} \text{ such that } \liminf_{n\to\infty} p_n^i - \sum_{j\neq i}\limsup_{n\to\infty} p_n^j > 1.\tag{30}$$

For that purpose we give an example which would lead us to our next result.

Example 5. Consider the first-order neutral delay difference equation with several delays and variable coefficients

$$\begin{aligned}&\Delta\left[y_n-\left(1+2^{-n}\right)y_{n-1}-\left(4+2^{-n}\right)y_{n-2}\right]\\&\quad+2^{(2n+1)/3}y_{n-4}=0.\end{aligned}\tag{31}$$

Note that p_n^j satisfy (30) for the above neutral delay difference equation (31). This neutral delay difference equation has an unbounded solution $y_n = 2^n$ tending to ∞ as $n \longrightarrow \infty$ unlike other results presented so far.

The above example is the motivating point to the statement of our next result. Since the proof is almost similar to that of Theorem 4, it is omitted.

Theorem 6. *Suppose that (H1)–(H4) hold. Assume that there exists a positive constant p such that the sequences $\{p_n^j\}$ for $j = 1, 2, \ldots, k$ satisfy the condition (30). Then every bounded solution of (1) oscillates or tends to zero as $n \longrightarrow \infty$.*

Remark 7. The above Theorems 4 and 6 hold for $k = 1$ but not for $k = 0$. Hence these results can be compared with results concerned with neutral delay difference equations (4) and (5).

Few examples are noted below to illustrate our results and establish its significance.

Example 8. Consider the first-order neutral delay difference equation

$$\Delta\left(y_n-p_n^1y_{n-1}-p_n^2y_{n-4}\right)+\frac{97}{60y_{n-2}^3}=0,\quad n\geq 5,\tag{32}$$

where

$$p_n^1=\begin{cases}\dfrac{1}{4}, & \text{if } n \text{ is odd},\\ \dfrac{1}{5}, & \text{if } n \text{ is even},\end{cases}\tag{33}$$

and

$$p_n^2=\begin{cases}\dfrac{1}{3}, & \text{if } n \text{ is odd},\\ \dfrac{1}{2}, & \text{if } n \text{ is even}.\end{cases}\tag{34}$$

The neutral delay difference equation (32) satisfies all the conditions of Theorem 1. As such, it has an oscillatory solution $y_n = (-1)^n$.

Example 9. Consider the first-order inhomogeneous neutral delay difference equation

$$\begin{aligned}&\Delta\left(y_n+p_n^1y_{n-4}+p_n^2y_{n-5}\right)+\frac{3}{4}2^{n-4}y_{n-3}^3\\&\quad=-\frac{3}{2^{n+1}}-\frac{3}{2^{2n-2}},\quad n\geq 5,\end{aligned}\tag{35}$$

where $p_n^1 = 2^{-n}+1/16$ and $p_n^2 = 2^{-n}+1/32$. This neutral delay difference equation satisfies all the conditions of Theorem 2. As such, it has a bounded positive solution $y_n = 2^{-n}$ tending to zero as $n \longrightarrow \infty$. Note that, no result in the papers cited under reference can be applied to the neutral delay difference equations (32) and (35).

Remark 10. Results of [3, 9] cannot be applied to the delay difference equation (9), because the condition (10) is not satisfied. However, due to Remark 3, Theorem 1 can be applied to the delay equation (9) as all the conditions are satisfied and as such the delay equation has a positive bounded solution e^{-n} tending to zero as $n \longrightarrow \infty$. Thus our work complements the work in [3, 9]. Further, since we do not assume G is nondecreasing, our Theorems 1, 2, 4, and 6 improve and generalize the results in [7].

3. Necessary Conditions

In this section we show that (H4) is necessary for every solution of (1) to be oscillatory or tending to zero as $n \longrightarrow \infty$. For this, we need the following lemma.

Lemma 11 (Krasnoselskii's fixed point theorem [13]). *Let X be a Banach space and S be a bounded closed convex subset of*

X. Let A, B be operators from S to X such that $Ax + By \in S$ for every pair of $x, y \in S$. If A is a contraction and B is completely continuous then the equation

$$Ax + Bx = x \tag{36}$$

has a solution in S.

Theorem 12. *Assume that (H2) holds. Further, assume that one of the conditions of (12) and (22) hold. Then (H4) is a necessary condition for all solution of (1) to be oscillatory or tending to zero as $n \longrightarrow \infty$.*

Proof. Suppose the condition (12) holds. The proof for the case when (22) holds would follow on similar lines. Assume for the sake of contradiction that (H4) does not hold. Hence

$$\sum_{n=n_0}^{\infty} q_n < \infty. \tag{37}$$

Thus, all we need to show is the existence of a bounded solution y_n of (1) with $\liminf_{n\longrightarrow\infty} y_n > 0$. From (H2), we find a positive constant c and a positive integer $n_1 > n_0 > 0$ such that

$$|F_n| < c \quad \text{for } n \geq n_1. \tag{38}$$

Choose a positive constant L such that $L \geq 5c/1 - p$. Since $G \in C(\mathbb{R}, \mathbb{R})$, let

$$\mu = \max\{|G(x)| : c \leq x \leq L\}. \tag{39}$$

Let

$$\eta = \max\{m_1, m_2, \ldots, m_k\}. \tag{40}$$

Then using (37) one can fix $n_2 > n_1$ such that for $n \geq n_2$ it follows that

$$\mu \sum_{i=n}^{\infty} q_i < c. \tag{41}$$

Choose $N_1 > n_2$ such that

$$N_0 = \min\{\sigma(N_1), N_1 - \eta\}. \tag{42}$$

Let $X = \ell_\infty^{N_0}$, Banach space of real bounded sequences $x = \{x_n\}$ with $x_1 = x_2 = \cdots = x_{N_0}$ and supremum norm

$$\|x\| = \sup(|x_n| : n \geq N_0). \tag{43}$$

Define

$$S = \{y \in X : c \leq y_n \leq L, n \geq N_0\}. \tag{44}$$

Clearly S is a bounded closed and convex subset of X. Now we define two operators A and $B : S \longrightarrow X$ as follows. For $y \in S$, define

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_0 \leq n \leq N_1 \\ \sum_{j=1}^{k} p_n^j y_{n-m_j} + F_n + 3c, & n \geq N_1. \end{cases}$$
$$(By)_n = \begin{cases} (By)_{N_1}, & N_0 \leq n \leq N_1 \\ \sum_{i=n}^{\infty} q_i G(y_{\sigma(i)}), & n \geq N_1. \end{cases} \tag{45}$$

First we show that if $x, y \in S$ then $Ax + By \in S$. Hence, for $x = \{x_n\}$ and $y = \{y_n\} \in S$ and for $n \geq N_1$ we obtain

$$\begin{aligned}(Ax)_n + (By)_n &\leq \sum_{j=1}^{k} p_n^j x_{n-m_j} + 3c + \sum_{i=n}^{\infty} q_i |G(y_{\sigma(i)})| \\ &\quad + |F_n| \leq pL + 5c \leq L.\end{aligned} \tag{46}$$

On the other hand

$$\begin{aligned}(Ax)_n + (By)_n &\geq 3c - \sum_{i=n}^{\infty} q_i |G(y_{\sigma(i)})| - |F_n| \\ &\geq 3c - c - c \geq c.\end{aligned} \tag{47}$$

Hence

$$c \leq (Ax)_n + (By)_n \leq L \quad \text{for } n \geq N_1. \tag{48}$$

Thus, we proved that $Ax + By \in S$ for any $x, y \in S$. Next we show that A is a contraction on S. In fact for $x, y \in S$ and $n \geq N_1$ we have

$$\begin{aligned}\|(Ax)_n - (Ay)_n\| &\leq \sum_{j=1}^{k} |p_n^j| |x_{n-m_j} - y_{n-m_j}| \\ &\leq p\|x - y\|\end{aligned} \tag{49}$$

This implies A is a contraction because $0 < p < 1$. Next we show that B is completely continuous. For this as a first step we show that B is continuous. Suppose the sequence $x^l \equiv \{x_n^l\} \longrightarrow x \equiv \{x_n\}$ in S as $l \longrightarrow \infty$ (with l taken from the index set). Since S is closed then $x \in S$. For $n \geq N_1$ we have

$$|(Bx^l)_n - (Bx)_n| \leq \sum_{i=n}^{\infty} q_i |G(x_{\sigma(i)}^l) - G(x_{\sigma(i)})| \tag{50}$$

Since G is continuous, therefore $|G(x_{\sigma(i)}^l) - G(x_{\sigma(i)})| \longrightarrow 0$ as $l \longrightarrow \infty$. Hence B is continuous. Next what remained to show is BS is relatively compact. Using the result [14, Theorem 3.3], we need only show that BS is uniformly cauchy. Let $x \equiv \{x_n\}$ be a sequence in S. From (H2) and (37), it follows that, for $\epsilon > 0$, there exists $N^* \geq N_1$ such that, for $n \geq N^*$,

$$\sum_{i=n}^{\infty} q_i |G(x_{\sigma(i)})| < \frac{\epsilon}{2}. \tag{51}$$

Then for $n_3 > n_4 \geq N^*$ we have

$$|(Bx)_{n_3} - (Bx)_{n_4}| < \epsilon. \tag{52}$$

Thus BS is uniformly cauchy. Hence it is relatively compact. Then by Lemma 11, we can find x^0 in S such that $Ax^0 + Bx^0 = x^0$. Clearly, $(x^0)_n$ is a bounded, positive solution of (1) with limit infimum greater than or equal to $c > 0$. Thus the theorem is proved. □

Theorem 13. *Assume that (H2) holds. Further assume that one of the conditions of (26) and (30) holds. Then (H4) is a necessary condition for all solution of (1) to be oscillatory or tending to zero as $n \longrightarrow \infty$.*

Proof. Suppose that p_n satisfies (30). The proof for the case when (26) holds is similar. Assume for the sake of contradiction that (H4) does not hold. Hence (37) holds. Thus, all we need to show is the existence of a bounded solution y_n of (1) with $\liminf_{n\longrightarrow\infty} y_n > 0$. From (H2), we find a positive constant L and a positive integer $n_1 > n_0 > 0$ such that

$$|F_n| < L \quad for\ n \geq n_1. \tag{53}$$

By (30), we can find a small positive real ϵ, a lower bound c for p_n^i, and upper bounds d_j for p_n^j ($j \neq i$ and $1 \leq j \leq k$) such that $c - \sum d_j - 1 = \epsilon$. Let $\sum d_j = d$. Hence $c = d + 1 + \epsilon$. Next choose an upper bound b for p_n^i such that $b < (c^2 - c)/d$. The nonexistence of such an upper bound for p_n^i would lead to the fact that, for all $\delta > 0, b = c + \delta$ and $b \geq (c^2 - c)/d$. Taking $\delta = \epsilon$, we have $\epsilon^2 + \epsilon \leq 0$, a contradiction. Choose a real λ as follows:

$$0 < \lambda = \frac{(L+\epsilon)\,d + (c-1)\,(c+L+\epsilon)}{c^2 - (bd + c)}. \tag{54}$$

Let

$$H = \frac{b\lambda + L + \epsilon}{c - 1}. \tag{55}$$

From (54) and (55) it follows that

$$\lambda - \frac{Hd + L + \epsilon}{c} = 1. \tag{56}$$

Since $G \in C(\mathbb{R}, \mathbb{R})$, let

$$\mu = \max\{|G(x)| : 1 \leq x \leq H\}. \tag{57}$$

Let $\eta = \max\{m_1, m_2, \ldots, m_k\}$. Then using (37), one can fix $n_2 > n_1$ such that for $n \geq n_2$ it follows that

$$\mu \sum_{i=n}^{\infty} q_i < \epsilon. \tag{58}$$

Choose $N_1 > n_2$ such that

$$N_0 = \min\{\sigma(N_1), N_1 - \eta\}. \tag{59}$$

Let $X = \ell_\infty^{N_0}$, Banach space of real bounded sequences $x = \{x_n\}$ with $x_1 = x_2 = \cdots = x_{N_0}$ and supremum norm

$$\|x\| = \sup(|x_n| : n \geq N_0). \tag{60}$$

Define

$$S = \{y \in X : 1 \leq y_n \leq H, n \geq N_0\}. \tag{61}$$

Clearly S is a bounded closed and convex subset of X. Now we define two operators A and $B : S \longrightarrow X$ as follows. For $y \in S$, define

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_0 \leq n \leq N_1 \\ \dfrac{y_{n+m_i}}{p_{n+m_i}^i} - \dfrac{\sum_{j\neq i} p_{n+m_i}^j y_{n-m_j+m_i}}{p_{n+m_i}^i} + \dfrac{b\lambda}{p_{n+m_i}^i} - \dfrac{F_{n+m_i}}{p_{n+m_i}^i}, & n \geq N_1. \end{cases}$$
$$(By)_n = \begin{cases} (By)_{N_1}, & N_0 \leq n \leq N_1 \\ \dfrac{-\sum_{j=n+m_i}^{\infty} q_j G(y_{\sigma(j)})}{p_{n+m_i}^i}, & n \geq N_1. \end{cases} \tag{62}$$

Proceeding as in the proof of above theorem we show that (i) if $x, y \in S$ then $Ax + By > 1$ by (56) and $Ax + By < H$ by (55), so that $Ax + By \in S$, (ii) $\|Ax_n - Ay_n\| < [(d+1)/c]\|x_n - y_n\|$, hence A is a contraction on S, and (iii) B is completely continuous. This completes the proof of the theorem. □

Remark 14. For the results in this section, we assume none of conditions (H3), G is nondecreasing, and $xG(x) > 0$, whereas the authors [7, 8] assumed these three conditions in their corresponding results. Hence the results of this article generalize and improve the corresponding results of these papers.

Combining all the above results, i.e., Theorems 1, 2, 4, 6, 12, and 13, we obtain the following theorem.

Theorem 15. *Suppose that (H1)-(H3) hold. Assume p_n^j in (1) to satisfy one of the four conditions (12), (22), (26), and (30). Then (H4) is both necessary and sufficient condition for every solution of (1) to be oscillatory or tending to zero as $n \longrightarrow \infty$.*

Remark 16. The results of this work hold for $G(x) = x$ and $f(x) = 0$, i.e., for the linear homogeneous equation associated with (1).

Data Availability

Previously reported data were used to support this study and are available at [DOI or OTHER PERSISTENT IDENTIFIER].

These prior studies (and datasets) are cited at relevant places within the text as references [#-#].

Disclosure

This work is done for the Ph.D. thesis work of the second author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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