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Research Article

A New Approach to Approximate Solutions for Nonlinear Differential Equation

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The question discussed in this study concerns one of the most helpful approximation methods, namely, the expansion of a solution of a differential equation in a series in powers of a small parameter. We used the Lindstedt-Poincaré perturbation method to construct a solution closer to uniformly valid asymptotic expansions for periodic solutions of second-order nonlinear differential equations.

1. Introduction

In the last few years, the study of approximations methods for systems of differential equations has been extensively developed; see, for example, [1]. This technique, known as the perturbation method (see [2]), has many applications in the theory of fractional differentiation operators (see [3]), in reaction-diffusion equations, stochastic stability, and asymptotic stability (see [4–9]), and for some numerical considerations (see, for example, [10–12]).

In current applications, some considerations require only the use of a small number of terms in the perturbation expansion, but the simple application of the perturbation is problematic if we want to calculate a uniformly valid solution.

Therefore, to structure a uniformly valid solution, one must look for an approximation that eliminates the terms causing the problem (secular terms). A technique to avoid the presence of these terms has been developed by Lindstedt. The principle of the Lindstedt method is to find approximations for periodic solutions, by convergent series using the expansion theorem and the periodicity of the solution [13, 14]. This method has various applications and properties; see, for example, [15]. Later, Poincaré proved that the expansion obtained by the Lindstedt technique is both asymptotic and uniformly valid.

The aim of this work is to present an analytical approximation study of periodic solutions for systems of second-order nonlinear differential equations. Although our analysis is based on the Lindstedt method, nevertheless the chosen development is according to a different approach from the one usually used. Thus, we recover an improvement in the process of the approximation.

Our paper consists of three sections. In the first section we present the general framework of our study. In the second, we recall most of the preliminary notions and the necessary definitions, and we prove the third approximation in the general case. Finally, in Section 3 we define and study the approximations of a new nonclassical equation.

2. Preliminaries and Definitions

In this section, we present an approximation method, based on the expansion of a solution of a differential equation in a series in a small parameter. It is used to construct uniformly valid periodic solutions to second-order nonlinear differential equations in the form

$$\frac{d^2 y}{dt^2}(t,\epsilon) + y(t,\epsilon) = \epsilon F\left(y(t,\epsilon), \frac{dy}{dt}(t,\epsilon)\right), \quad 0 < \epsilon << 1, \tag{1}$$

with $y(0,\epsilon) = A, (dy/dt)(0,\epsilon)(0) = 0$, where $0 < \epsilon << 1$ means that the positive parameter ϵ is small enough to be close to zero and F is supposed to be an analytical function of $y(t,\epsilon)$ and $(dy/dt)(t,\epsilon)$.

If $\epsilon = 0$ we obtain the following nonperturbed problem:

$$\frac{d^2 y}{dt^2}(t,0) + y(t,0) = 0. \tag{2}$$

Before we discuss our subject, we present some basic concepts concerning the perturbation theory. Then we introduce the Lindstedt method, which we use to determine uniformly valid solutions, in order to find a closer approximate solution for (2) (y^{**} is closer to y than y^* means that $|y - y^{**}| < |y - y^*|$). For further developments concerning the Lindstedt method see [16, 17].

2.1. Approximation Technique. We assume that the $(n+1)$th approximate solution of (1) can be written as

$$y(t,\epsilon) = \sum_{m=0}^{n} \epsilon^m y_m(t,0) + O\left(\epsilon^{n+1}\right). \tag{3}$$

The general procedure of the simple approximation is to substitute (3) into (1), develop in powers of ϵ, and put all coefficients of the powers of ϵ equal to zero. This gives a system of linear nonhomogeneous differential equations that we can solve recursively.

But the simple approximation takes us on a problem, if we need to calculate an analytical approximations of periodic solutions of nonlinear differential equations in the form given by (1). We illustrate this type of difficulty in the following example.

2.1.1. Example. We apply the simple approximation to the following equation:

$$\frac{d^2 y}{dt^2}(t,\epsilon) + \epsilon\left(\frac{dy}{dt}(t,\epsilon)\right)^2 + y(t,\epsilon) = 0, \quad 0 < \epsilon << 1, \tag{4}$$

with the initial values $y(0,\epsilon) = A, (dy/dt)(0,\epsilon) = 0$.

The fourth approximate solution of (4) is $y(t,\epsilon) = y_0(t,0) + \epsilon y_1(t,0) + \epsilon^2 y_2(t,0) + \epsilon^3 y_3(t,0) + O(\epsilon^4)$. After substituting and calculating, we find

$$\begin{aligned} y(t,\epsilon) &= A\cos t + \epsilon\frac{A^2}{6}(-3 + 4\cos t - \cos 2t) + \epsilon^2 \\ &\cdot \frac{A^3}{72}(-48 + 61\cos t - 16\cos 2t + 12t\sin t \\ &- 3\cos 3t) + \epsilon^3 A^4\left(-\frac{23}{24} + \frac{659}{540}\cos t - \frac{1}{3}\cos 2t\right. \\ &\left. + \frac{1}{12}\cos 3t - \frac{13}{1080}\cos 4t + \frac{1}{3}t\sin t - \frac{1}{18}t\sin 2t\right) \\ &+ O\left(\epsilon^4\right). \end{aligned} \tag{5}$$

We remark that the terms $y_2(t,0)$ and $y_3(t,0)$ are nonperiodic and unbounded as $t \longrightarrow +\infty$. This leads to the notion of secular terms.

2.2. Secular Terms. The conservation of a finite numbers of terms on the right-side of expansion (5) determines a function that is not only nonperiodic, but also unbounded as $t \longrightarrow +\infty$.

Definition 1. Terms such as $t^m \cos(pt)$ or $t^m \sin(nt)$ where $m, n \in \mathbb{N}^*$, $p \in \mathbb{N}$ are called secular terms.

These expressions appear because expansion (5) is not uniformly valid. The existence of such expressions destroys the periodicity of expansion (5) when only a finite number of terms is conserved. Therefore, to obtain a uniformly valid solution, we must look for an approximation that eliminates secular terms. A technique to avoid the presence of secular terms and allows for an approximation that is valid for all time has been developed by Lindstedt-Poincaré as described above in what follows.

2.3. Lindstedt-Poincaré Method. The substance of this method is to introduce a new independent variable linearly linked to the old independent variable. This transformation completely eliminates the secular terms. The basic idea came from the astronomer Lindstedt, based on the change of variable $\theta = \omega(\epsilon)t$ with $\omega_0 = w(0) = 1, \omega(\epsilon) \neq 1$, and both $y(\theta,\epsilon)$ and $\omega(\epsilon)$ are expanded in powers of ϵ as follows:

$$\begin{aligned} y(\theta,\epsilon) &= y_0(\theta,0) + \epsilon y_1(\theta,0) + \ldots + \epsilon^n y_n(\theta,0) + \ldots, \\ \omega(\epsilon) &= 1 + \epsilon\omega_1 + \ldots + \epsilon^n\omega_n + \ldots, \end{aligned} \tag{6}$$

and we note that, in this step, ω_j are unknowns; we obtain them by elimination of the secular terms.

First, we introduce the following notations:

$$\begin{aligned} \dot{y} &\equiv \frac{dy}{d\theta}(\theta,\epsilon), \\ \ddot{y} &\equiv \frac{d^2 y}{d\theta^2}(\theta,\epsilon), \\ F_y(y,\omega\dot{y}) &\equiv \frac{\partial F(y(\theta,\epsilon),\dot{y})}{\partial y(\theta,\epsilon)}, \\ F_{\dot{y}}(y,\dot{y}) &\equiv \frac{\partial F(y(\theta,\epsilon),\dot{y})}{\partial \dot{y}}, \end{aligned} \tag{7}$$

and (1) becomes

$$\omega^2\ddot{y} + y = \epsilon F(y,\omega\dot{y}), \quad 0 < \epsilon << 1, \tag{8}$$

with $y(0,\epsilon) = A$, $\dot{y}(0,\epsilon) = 0$. When we substitute expansion (6) into (8) we have

$$\begin{aligned} &\left(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \ldots\right)^2 \\ &\cdot\left(\ddot{y}_0 + \epsilon\ddot{y}_1 + \epsilon^2\ddot{y}_2 + \epsilon^3\ddot{y}_3 + \ldots\right) + y_0 + \epsilon y_1 + \epsilon^2 y_2 \\ &+ \epsilon^3 y_3 + \ldots = \epsilon F(y_0,\dot{y}_0) + \epsilon^2\frac{\partial F(y_0,\dot{y}_0)}{\partial\epsilon} + \frac{\epsilon^3}{2} \\ &\cdot\frac{\partial^2 F(y_0,\dot{y}_0)}{\partial\epsilon^2} + \ldots. \end{aligned} \tag{9}$$

Then we put all different powers of ϵ to zero, and we obtain (10), (11), (12), (13), and (15), such that

$$\ddot{y}_0 + y_0 = 0, \quad (10)$$

$$\ddot{y}_1 + y_1 = -2\omega_1 \ddot{y}_0 + F(y_0, \dot{y}_0) =: G_1(y_0(\theta, 0), \dot{y}_0(\theta, 0)) = G_1(\theta), \quad (11)$$

$$\ddot{y}_2 + y_2 = -2\omega_1 \ddot{y}_1 - (\omega_1^2 + 2\omega_2)\ddot{y}_0 + F_y(y_0, \dot{y}_0) y_1 + F_{\dot{y}}(y_0, \dot{y}_0)(\omega_1 \dot{y}_0 + \dot{y}_1) =: G_2(y_0(\theta, 0), y_1(\theta, 0), \dot{y}_0(\theta, 0), \dot{y}_1(\theta, 0)) = G_2(\theta), \quad (12)$$

$$\ddot{y}_3 + y_3 =: G_3(y_0(\theta, 0), y_1(\theta, 0), y_2(\theta, 0); \dot{y}_0(\theta, 0), \dot{y}_1(\theta, 0), \dot{y}_2(\theta, 0)) = G_3(\theta), \quad (13)$$

$$\cdots\cdots\cdots \quad (14)$$

$$\ddot{y}_n + y_n = G_n(y_0(\theta, 0), y_1(\theta, 0), \dots, y_{n-1}(\theta, 0); \dot{y}_0(\theta, 0), \dot{y}_1(\theta, 0), \dots, \dot{y}_{n-1}(\theta, 0)) = G_n(\theta); \quad (15)$$

note here that $G_i, i = 1, .., n$, is also an analytical function of $y_0, y_1, \dots, y_{i-1}; \dot{y}_0, \dot{y}_1, \dots, \dot{y}_{i-1}$.

To calculate an approximate periodic solutions of (8), we must solve (11), (12), (13), and (15). The following proposition gives the general formula of periodic solutions. Although these results are in [17], they are not detailed.

Proposition 2. *We consider the following equation:*

$$\ddot{y} + y = G(\theta), \quad y(0) = 0, \ \dot{y}(0) = 0, \ \text{with } G(\theta) \neq 0, \quad (16)$$

and the solution of problem (16) is

$$y(\theta) = \int_0^\theta \sin(\theta - \tau) G(\tau) \, d\tau. \quad (17)$$

Moreover, problem (16) has a periodic solution $y_1(\theta, 0)$ if and only if

$$\int_0^{2\pi} F(A\cos\theta, -A\sin\theta) \sin\theta \, d\theta = 0,$$
$$2\pi\omega_1 A + \int_0^{2\pi} F(A\cos\theta, -A\sin\theta) \cos\theta \, d\theta = 0. \quad (18)$$

Proof. We know that the solution of (16) is $y(\theta) = C_1\cos\theta + C_2\sin\theta + y_p(\theta)$ such that $y_p(\theta) = C_1(\theta)\cos\theta + C_2(\theta)\sin\theta$. By variation of constants we find

$$\begin{gathered} C_1'(\theta)\cos\theta + C_2'(\theta)\sin\theta = 0, \\ -C_1'(\theta)\sin\theta + C_2'(\theta)\cos\theta = G(\theta) \\ \Downarrow \\ -C_1'(\theta) = -\sin\theta G(\theta) \Longrightarrow \\ C_1(\theta) = -\int_0^\theta \sin\tau G(\tau)\, d\tau, \\ C_1(0) = 0, \\ -C_2'(\theta) = \cos\theta G(\theta) \Longrightarrow \\ C_1(\theta) = \int_0^\theta \cos\tau G(\tau)\, d\tau, \\ C_2(0) = 0 \end{gathered} \quad (19)$$

$\Longrightarrow y_p(\theta) = (-\int_0^\theta \sin\tau G(\tau) d\tau)\cos\theta + (\int_0^\theta \cos\tau G(\tau) d\tau)\sin\theta = \int_0^\theta (-\sin\tau\cos\theta + \cos\tau\sin\theta) G(\tau) d\tau$. $\Longrightarrow y_p(\theta) = \int_0^\theta \sin(\theta - \tau) G(\tau) d\tau \Longrightarrow y(\theta) = C_1\cos\theta + C_2\sin\theta + \int_0^\theta \sin(\theta - \tau) G(\tau) d\tau$ with the initial values $y(0) = 0$, $\dot{y}(0) = 0$; we have $C_1 = C_2 = 0$, so we deduce that problem (16) admits (17) as a solution.

Moreover, (16) gives

$$\begin{gathered} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 + G(\tau). \end{gathered} \quad (20)$$

On the other hand, the condition of periodicity for the new variable θ can be expressed as $y(\theta) = y(\theta + 2\pi)$, so the corresponding conditions for $y_n(\theta)$ are $y_n(\theta) = y_n(\theta + 2\pi), n = 1, 2, \dots$

$$\Longrightarrow \begin{cases} y_1(2\pi) = y_1(0) = 0 \\ y_2(2\pi) = y_2(0) = 0 \end{cases} \quad (21)$$

which yields to the periodicity condition $\int_\theta^{\theta+2\pi} \sin(\theta - \tau) G(\tau) d\tau = 0$,

$$\Longrightarrow \begin{cases} \int_0^{2\pi} \cos\theta G(\theta)\, d\theta = 0, \\ \int_0^{2\pi} \sin\theta G(\theta)\, d\theta = 0. \end{cases} \quad (22)$$

According to (11) we have $G(\theta) = -2\omega_1 \ddot{y}_0 + F(y_0, \dot{y}_0)$, $y_0 = A\cos\theta$; we rewrite (22) as

$$\Longrightarrow \begin{cases} \int_0^{2\pi} \cos\theta\left[2\omega_1 A\cos\theta + F\left(A\cos\theta, -A\sin\theta\right)\right] d\theta = 0, \\ \int_0^{2\pi} \sin\theta\left[2\omega_1 A\cos\theta + F\left(A\cos\theta, -A\sin\theta\right)\right] d\theta = 0, \end{cases}$$
$$\Longrightarrow \begin{cases} 2\omega_1 \pi A + \int_0^{2\pi} \cos\theta F\left(A\cos\theta, -A\sin\theta\right) d\theta = 0, \\ \int_0^{2\pi} \sin\theta F\left(A\cos\theta, -A\sin\theta\right) d\theta = 0, \end{cases} \tag{23}$$

which is required. □

2.3.1. Example. We apply the method of Lindstedt to (4) with the initial values $y(0,\epsilon) = A$, $\dot{y}(0,\epsilon) = 0$, and we calculate $y_i(\theta, 0)$, $i = 1, 2, 3$ according to Proposition 2. Thus, we find that the fourth approximation of the periodic solution of (4) is

$$\begin{aligned} y(\theta,\epsilon) &= A\cos\theta + \epsilon\left(\frac{A^2}{6}\right)(-3 + 4\cos\theta - \cos 2\theta) \\ &\quad + \epsilon^2\left(\frac{A^3}{3}\right)\left[-2 + \left(\frac{61}{24}\right)\cos\theta - \left(\frac{2}{3}\right)\cos 2\theta \right. \\ &\quad \left. - \left(\frac{1}{8}\right)\cos 3\theta\right] + \epsilon^3\left[-\frac{23}{24}A^4 + \frac{659}{540}A^4\cos\theta \right. \\ &\quad \left. - \frac{1}{3}A^4\cos 2\theta + \frac{A^4}{12}\cos 3\theta - \frac{13}{1080}A^4\cos 4\theta\right] \\ &\quad + O\left(\epsilon^4\right), \end{aligned} \tag{24}$$

with $\theta = \omega(\epsilon)t$ such that $\omega(\epsilon) = 1 - \epsilon^2(A^2/6) - \epsilon^3((2/9)A^3) + O(\epsilon^4)$.

Remark 3. Although the calculation of $y_3(\theta)$ is very long, usually in applications, the fourth approximation is among the high orders that are often useful. For this reason, we give its equation in the next proposition.

Remark 4. The Lindstedt method gives only periodic solutions.

3. Our Results

3.1. General Formula. Practically, for many considerations we are forced to use a small number of terms in the perturbation expansion. We note here that the second and third terms are determined by (11) and (12) in [17]. In the following proposition, (13) which determines the fourth term is explicitly stated.

Proposition 5. *The general formula of (13) is*

$$\begin{aligned} \ddot{y}_3 + y_3 &= G_3(\theta) \\ &= -2\omega_1\ddot{y}_2 - \left(\omega_1^2 + 2\omega_2\right)\ddot{y}_1 \\ &\quad - \left(2\omega_3 + 2\omega_1\omega_2\right)\ddot{y}_0 + y_2 F_y\left(y_0, \dot{y}_0\right) \\ &\quad + \frac{{y_1}^2}{2} F_{yy}\left(y_0, \dot{y}_0\right) \\ &\quad + + y_1\left(\omega_1\dot{y}_0 + \dot{y}_1\right) F_{y\dot{y}}\left(y_0, \dot{y}_0\right) \\ &\quad + \left(\omega_2\dot{y}_0 + \omega_1\dot{y}_1 + \dot{y}_2\right) F_{\dot{y}}\left(y_0, \dot{y}_0\right) \\ &\quad + \frac{1}{2}\left(\omega_1\dot{y}_0 + \dot{y}_1\right)^2 F_{\dot{y}\dot{y}}\left(y_0, \dot{y}_0\right). \end{aligned} \tag{25}$$

Proof. First, (9) gives

$$\begin{aligned} &\epsilon^3\left(\ddot{y}_3 + y_3 + 2\omega_1\ddot{y}_2 + \left({\omega_1}^2 + 2\omega_2\right)\ddot{y}_1 \right. \\ &\quad \left. + \left(2\omega_3 + 2\omega_1\omega_2\right)\ddot{y}_0\right) = \frac{\epsilon^3}{2}\frac{\partial^2 F\left(y_0, \dot{y}_0\right)}{\partial\epsilon^2} \Longrightarrow \\ &\ddot{y}_3 + y_3 = -2\omega_1\ddot{y}_2 - \left({\omega_1}^2 + 2\omega_2\right)\ddot{y}_1 - \left(2\omega_3 + 2\omega_1\omega_2\right) \\ &\quad \cdot \ddot{y}_0 + \frac{1}{2}\frac{\partial^2 F\left(y_0, \dot{y}_0\right)}{\partial\epsilon^2}, \end{aligned} \tag{26}$$

such that

$$\begin{aligned} \frac{\partial^2 F(y, \omega\dot{y})}{\partial\epsilon^2} &= \frac{\partial}{\partial\epsilon}\left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial\epsilon} + \frac{\partial F}{\partial\dot{y}}\frac{\partial\omega\dot{y}}{\partial\epsilon}\right) \\ &= \frac{\partial y}{\partial\epsilon}\left(\frac{\partial}{\partial\epsilon}\frac{\partial F}{\partial y}\right) + \frac{\partial^2 y}{\partial\epsilon^2}\frac{\partial F}{\partial y} \\ &\quad + \frac{\partial\omega\dot{y}}{\partial\epsilon}\left(\frac{\partial}{\partial\epsilon}\frac{\partial F}{\partial\dot{y}}\right) + \frac{\partial^2\omega\dot{y}}{\partial\epsilon^2}\frac{\partial F}{\partial\dot{y}} \\ &= \frac{\partial y}{\partial\epsilon}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial y}{\partial\epsilon} + \frac{\partial^2 F}{\partial y\partial\dot{y}}\frac{\partial\omega\dot{y}}{\partial\epsilon}\right) + \frac{\partial^2 y}{\partial\epsilon^2}F_y \\ &\quad + \frac{\partial^2\omega\dot{y}}{\partial\epsilon^2}F_{\dot{y}} \\ &\quad + \frac{\partial^2\omega\dot{y}}{\partial\epsilon}\left(\frac{\partial^2 F}{\partial y\partial\dot{y}}\frac{\partial y}{\partial\epsilon} + \frac{\partial^2 F}{\partial y^2}\frac{\partial\omega\dot{y}}{\partial\epsilon}\right) \\ &= \left(\frac{\partial y}{\partial\epsilon}\right)^2 F_{yy} + 2\frac{\partial y}{\partial\epsilon}\frac{\partial\omega\dot{y}}{\partial\epsilon}F_{y\dot{y}} + \frac{\partial^2 y}{\partial\epsilon}F_y \\ &\quad + \frac{\partial^2\omega\dot{y}}{\partial\epsilon^2}F_{\dot{y}} + \left(\frac{\partial\omega\dot{y}}{\partial\epsilon}\right)^2 F_{\dot{y}\dot{y}}, \end{aligned} \tag{27}$$

with

$$F_{yy} \equiv \frac{\partial^2 F}{\partial y^2}, \quad F_{y\dot{y}} \equiv \frac{\partial^2 F}{\partial y \partial \dot{y}}, \quad F_{\dot{y}\dot{y}} \equiv \frac{\partial^2 F}{\partial \dot{y} \partial \dot{y}}. \tag{28}$$

On the other hand, in the third order we have

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 \Longrightarrow$$
$$\frac{\partial y}{\partial \epsilon} = y_1 + 2\epsilon y_2 + 3\epsilon^2 y_3 \Longrightarrow$$
$$\frac{\partial^2 y}{\partial \epsilon^2} = 2y_2 + 6\epsilon y_3, \tag{29}$$

and

$$\begin{aligned}\frac{\partial \omega \dot{y}}{\partial \epsilon} &= \frac{\partial \omega}{\partial \epsilon}\dot{y} + \frac{\partial \dot{y}}{\partial \epsilon}\omega \\ &= \left(\omega_1 + 2\epsilon\omega_2 + 3\epsilon^2\omega_3\right)\left(\dot{y}_0 + \epsilon\dot{y}_1 + \epsilon^2\dot{y}_2 + \epsilon^3\dot{y}_3\right) \\ &\quad + \left(\dot{y}_1 + 2\epsilon\dot{y}_2 + 3\epsilon^2\dot{y}_3\right)\left(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3\right),\end{aligned} \tag{30}$$

and also

$$\begin{aligned}\frac{\partial^2 \omega \dot{y}}{\partial \epsilon^2} &= (2\omega_2 + 6\epsilon\omega_3)\left(\dot{y}_0 + \epsilon\dot{y}_1 + \epsilon^2\dot{y}_2 + \epsilon^3\dot{y}_3\right) + 2\left(\dot{y}_1 + 2\epsilon\dot{y}_2 + 3\epsilon^2\dot{y}_3\right)\left(\omega_1 + 2\epsilon\omega_2 + 3\epsilon^2\omega_3\right) \\ &\quad + (2\dot{y}_2 + 6\epsilon\dot{y}_3)\left(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3\right) \Longrightarrow \\ \frac{1}{2}\frac{\partial^2 F(y_0, \dot{y}_0)}{\partial \epsilon^2} &= \frac{{y_1}^2}{2} F_{yy}(y_0, \dot{y}_0) + (\omega_1\dot{y}_0 + \dot{y}_1)\, y_1 F_{y\dot{y}}(y_0, \dot{y}_0) \\ &\quad + y_2 F_y(y_0, \dot{y}_0) + (\omega_2\dot{y}_0 + \omega_1\dot{y}_1 + \dot{y}_2) F_{\dot{y}}(y_0, \dot{y}_0) + (\omega_1\dot{y}_0 + \dot{y}_1)^2 F_{\dot{y}\dot{y}}(y_0, \dot{y}_0),\end{aligned} \tag{31}$$

and when we substitute (31) into (26), we get (25). □

3.2. Main Result. In this essential part of our work, we deal with some nonclassical equations, more general than (1), and also different from the equation studied in [2]. We consider equations in the following form:

$$\frac{d^2\tilde{y}}{dt^2}(t,\epsilon) + \tilde{y}(t,\epsilon) = g(\epsilon) F\left(\tilde{y}(t,\epsilon), \frac{d\tilde{y}}{dt}(t,\epsilon)\right), \quad 0 < \epsilon << 1, \tag{32}$$

with $\tilde{y}(0,\epsilon) = A$, $(d\tilde{y}/dt)(0,\epsilon) = 0$, where ϵ is a small positive parameter and F is supposed to be an analytical function of $\tilde{y}(t,\epsilon)$ and $d\tilde{y}/dt(t,\epsilon)$.

To compute an uniformly approximate periodic solution, a new variable $\tilde{\theta} = \tilde{\omega}t$ is introduced, and both $\tilde{y}$ and $\tilde{\omega}$ are expanded in powers of ϵ as follows:

$$\tilde{y}\left(\tilde{\theta},\epsilon\right) = \tilde{y}_0\left(\tilde{\theta},0\right) + \epsilon\tilde{y}_1\left(\tilde{\theta},0\right) + \ldots + \epsilon^n\tilde{y}_n\left(\tilde{\theta},0\right) + \ldots, \tag{33}$$

with

$$\tilde{\omega}(\epsilon) = 1 + \epsilon\tilde{\omega}_1 + \ldots + \epsilon^n\tilde{\omega}_n + \ldots. \tag{34}$$

We note that, in this step, $\tilde{\omega}_j$ are unknowns, and we obtain them by elimination of the secular terms.

To use the uniformly approximate periodic solution (33), we give firstly the general formula of $\tilde{y}_0(\tilde{\theta},0)$, $\tilde{y}_1(\tilde{\theta},0)$, and $\tilde{y}_2(\tilde{\theta},0)$ in the following proposition.

Proposition 6. *The terms $\tilde{y}_0(\tilde{\theta},0)$, $\tilde{y}_1(\tilde{\theta},0)$ and $\tilde{y}_2(\tilde{\theta},0)$ are, respectively, solutions of (35), (36), and (37) such that*

$$\ddot{\tilde{y}}_0 + \tilde{y}_0 = 0, \tag{35}$$

$$\ddot{\tilde{y}}_1 + \tilde{y}_1 = \widetilde{G}_1\left(\tilde{\theta}\right), \tag{36}$$

$$\ddot{\tilde{y}}_2 + \tilde{y}_2 = \widetilde{G}_2\left(\tilde{\theta}\right), \tag{37}$$

with $\widetilde{G}_1(\tilde{\theta}) = -2\tilde{\omega}_1\ddot{\tilde{y}}_0 + c_1 F(\tilde{y}_0(\tilde{\theta},0), \dot{\tilde{y}}_0(\tilde{\theta},0))$,

$$\begin{aligned}\widetilde{G}_2\left(\tilde{\theta}\right) &= -2\tilde{\omega}_1\ddot{\tilde{y}}_0 + c_1 F\left(\tilde{y}_0\left(\tilde{\theta},0\right), \dot{\tilde{y}}_0\left(\tilde{\theta},0\right)\right) \\ &\quad - 2\tilde{\omega}_1\ddot{\tilde{y}}_1 - \left(\tilde{\omega}_1^2 + 2\tilde{\omega}_2\right)\ddot{\tilde{y}}_0 \\ &\quad + + c_1\left(F_{\tilde{y}}\left(\tilde{y}_0\left(\tilde{\theta},0\right), \dot{\tilde{y}}_0\left(\tilde{\theta},0\right)\right)\tilde{y}_1\right. \\ &\quad \left. + F_{\dot{\tilde{y}}}\left(\tilde{y}_0\left(\tilde{\theta},0\right), \dot{\tilde{y}}_0\left(\tilde{\theta},0\right)\right)\left(\tilde{\omega}_1\dot{\tilde{y}}_0 + \dot{\tilde{y}}_1\right)\right) \\ &\quad + c_2 F\left(\tilde{y}_0\left(\tilde{\theta},0\right), \dot{\tilde{y}}_0\left(\tilde{\theta},0\right)\right).\end{aligned} \tag{38}$$

Proof. Equation (32) will be written as

$$\begin{aligned}\frac{d^2\tilde{y}(\tilde{\theta},\epsilon)}{dt^2}+\tilde{y}(\tilde{\theta}) &= \sum_{k\geq 1}\epsilon^k c_k F\left(\tilde{y}(\tilde{\theta},\epsilon),\frac{d\tilde{y}(\tilde{\theta},\epsilon)}{dt}\right)\\ &= \epsilon\left(\sum_{k\geq 0}\epsilon^k c_{k+1}F\left(\tilde{y}(\tilde{\theta},\epsilon),\frac{d\tilde{y}(\tilde{\theta},\epsilon)}{dt}\right)\right)\\ &= \epsilon K\left(\tilde{y}(\tilde{\theta},\epsilon),\frac{d\tilde{y}(\tilde{\theta},\epsilon)}{dt}\right),\end{aligned}\tag{39}$$

where K is an analytical function of y and dy/dt.

Substituting (33) into (32), we have

$$\begin{aligned}&\left(1+\epsilon\tilde{\omega}_1+\epsilon^2\tilde{\omega}_2+\epsilon^3\tilde{\omega}_3+\ldots\right)^2\left(\ddot{\tilde{y}}_0+\epsilon\ddot{\tilde{y}}_1+\epsilon^2\ddot{\tilde{y}}_2\right.\\ &\quad\left.+\epsilon^3\ddot{\tilde{y}}_3+\ldots\right)+\tilde{y}_0+\epsilon\tilde{y}_1+\epsilon^2\tilde{y}_2+\epsilon^3\tilde{y}_3+\ldots\\ &\quad=\epsilon K\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)+\epsilon^2\frac{\partial K\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)}{\partial\epsilon}+\frac{\epsilon^3}{2}\frac{\partial^2 K\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)}{\partial\epsilon^2}\\ &\quad+\ldots,=\epsilon c_1F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)+\epsilon^2\left[c_1\frac{\partial F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)}{\partial\epsilon}\right.\\ &\quad\left.+c_2F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\right]+\ldots,=\epsilon c_1F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\\ &\quad+\epsilon^2\left[c_1\left(F_{\tilde{y}}\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\tilde{y}_1\right.\right.\\ &\quad\left.\left.+F_{\dot{\tilde{y}}}\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\left(\widetilde{\omega_1}\dot{\tilde{y}}_0+\dot{\tilde{y}}_1\right)\right)+c_2F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\right]+\cdots,\end{aligned}\tag{40}$$

then we put all different powers of ϵ to zero, and we obtain (35), (36), (37), and so on. □

The aim of this study is to construct a new approach to (2), which gives a closer approximate solution of (2) more than an approximate solution of (1). The relations between an approximate solution of (32) and that of (1) are determined by the following lemma.

Lemma 7. *If the function $g(\epsilon)$ is expanded in powers of ϵ with $g(0)=0$ i.e $g(\epsilon)=\sum_{k\geq 1}\epsilon^k c_k$, where c_k are real constants, we have*

(1) $\tilde{y}_0(\tilde{\theta},0)=y_0(\tilde{\theta},0)$.

(2) $\tilde{y}_1(\tilde{\theta},0)=c_1y_1(\tilde{\theta},0)$.

(3) $\tilde{y}_2(\tilde{\theta},0)=c_1^2y_2(\tilde{\theta},0)+c_2y_1(\tilde{\theta},0)$.

Proof. (1) Equation (35) gives $\tilde{y}_0(\tilde{\theta},0)=A\cos\tilde{\theta}=A\cos(\tilde{\omega}(0)t)=A\cos t=A\cos(\omega(0)t)=A\cos\theta=y_0(\theta,0)$.

(2) When we apply the periodicity condition (22) to (36), we have

$$\begin{aligned}&\int_0^{2\pi}\cos\tilde{\theta}\widetilde{G}_1\left(\tilde{\theta}\right)d\tilde{\theta}=0,\\ &\int_0^{2\pi}\sin\tilde{\theta}\widetilde{G}_1(\theta)\,d\tilde{\theta}=0.\\ &\Downarrow\\ &\int_0^{2\pi}\cos\tilde{\theta}\left[-2\tilde{\omega}_1\ddot{\tilde{y}}_0+c_1F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\right]d\tilde{\theta}=0,\\ &\int_0^{2\pi}\sin\tilde{\theta}\left[-2\tilde{\omega}_1\ddot{\tilde{y}}_0+c_1F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\right]d\tilde{\theta}=0.\\ &\Downarrow\\ &\tilde{\omega}_1=\frac{-c_1}{2\pi A}\int_0^{2\pi}\cos\tilde{\theta}F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)d\tilde{\theta}=c_1\omega_1,\\ &\int_0^{2\pi}\sin\tilde{\theta}F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)d\tilde{\theta}=0.\end{aligned}\tag{41}$$

On the other hand, according to (17) the solution $\tilde{y}_1(\tilde{\theta},0)$ of (36) is given by

$$\begin{aligned}\tilde{y}_1\left(\tilde{\theta},0\right)&=\int_0^{\tilde{\theta}}\sin\left(\tilde{\theta}-\tilde{\tau}\right)\widetilde{G}_1(\tilde{\tau})\,d\tilde{\tau}\\ &=c_1\int_0^{\tilde{\theta}}\sin\left(\tilde{\theta}-\tilde{\tau}\right)G_1(\tilde{\tau})\,d\tilde{\tau}=c_1y_1\left(\tilde{\theta},0\right).\end{aligned}\tag{42}$$

(3) When we apply the periodicity condition (22) to (12), we have

$$\begin{aligned}&\int_0^{2\pi}\cos\tilde{\theta}\widetilde{G}_2\left(\tilde{\theta}\right)d\tilde{\theta}=0,\\ &\int_0^{2\pi}\sin\tilde{\theta}\widetilde{G}_2(\theta)\,d\tilde{\theta}=0.\\ &\Downarrow\\ &\int_0^{2\pi}\cos\tilde{\theta}\left[c_1^2G_2\left(\tilde{\theta}\right)+c_2F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\right]d\tilde{\theta}=0,\\ &\int_0^{2\pi}\sin\tilde{\theta}\left[c_1^2G_2\left(\tilde{\theta}\right)+c_2F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)\right]d\tilde{\theta}=0.\\ &\Downarrow\\ &\tilde{\omega}_2=c_1^2\omega_2+c_2\omega_1,\\ &\int_0^{2\pi}\sin\tilde{\theta}F\left(\tilde{y}_0,\dot{\tilde{y}}_0\right)d\tilde{\theta}=0.\end{aligned}\tag{43}$$

On the other hand, according to (17) the solution $\tilde{y}_2(\tilde{\theta}, 0)$ of (11) is given by

$$\begin{aligned}\tilde{y}_2(\tilde{\theta}, 0) &= \int_0^{\tilde{\theta}} \sin(\tilde{\theta} - \tilde{\tau}) \tilde{G}_2(\tilde{\tau})\, d\tilde{\tau} \\ &= c_1^2 \int_0^{\tilde{\theta}} \sin(\tilde{\theta} - \tilde{\tau}) G_2(\tilde{\tau})\, d\tilde{\tau} \\ &\quad + c_2 \int_0^{\tilde{\theta}} \sin(\tilde{\theta} - \tilde{\tau}) \left[-2\tilde{\omega}_1 \ddot{\tilde{y}}_0 + c_1 F(\tilde{y}_0, \dot{\tilde{y}}_0)\right] d\tilde{\tau} \Longrightarrow \\ \tilde{y}_2(\tilde{\theta}, 0) &= c_1^2 y_2(\tilde{\theta}, 0) + c_2 y_1(\tilde{\theta}, 0).\end{aligned} \tag{44}$$

□

Theorem 8. *If the function $g(\epsilon)$ is expanded in powers of ϵ with $g(0) = 0$ i.e $g(\epsilon) = \sum_{k\geq 1} \epsilon^k c_k$, where c_k are real constants, so one has the following:*

(1) If $c_1 \neq 0$, with $|c_1| < 1$, the approximate solutions of (32) are closer to the solutions of (2) more than the approximate solutions of (1).

(2) If $c_1 = 0$ and $|c_2| < 1$, with $c_2 \neq 0$, the approximate solutions of (32) are closer to the solutions of (2) more than the approximate solutions of (1).

Moreover, the approximate solutions of (32) are closer to the solutions of (2) more than the approximate solutions of (1) where $c_1 = 0, c_2 \neq 0$.

Proof. According to the results given by Lemma 7, we have the following:

(1) If $|c_1| < 1$ and $c_1 \neq 0$ we have

$$\begin{aligned}&\left|y_0(\theta, 0) - \tilde{y}(\tilde{\theta}, \epsilon)\right| \\ &\quad = \left|y_0(\theta, 0) - y_0(\tilde{\theta}, 0) - \epsilon \tilde{y}_1(\tilde{\theta}, 0) - o(\epsilon)\right| \\ &\quad = \left|\epsilon \tilde{y}_1(\tilde{\theta}, 0) + o(\epsilon)\right| = \left|c_1 \epsilon y_1(\tilde{\theta}, 0) + o(\epsilon)\right| \\ &\quad = |c_1| \left|\epsilon y_1(\tilde{\theta}, 0) + o(\epsilon)\right| < \left|\epsilon y_1(\tilde{\theta}, 0) + o(\epsilon)\right| \\ &\quad = \left|y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)\right|.\end{aligned} \tag{45}$$

So $\tilde{y}(\tilde{\theta}, \epsilon)$ is an approximation to $y_0(\theta, 0)$ closer than $y(\tilde{\theta}, \epsilon)$.

(2) If $c_1 = 0$ and $|c_2| < 1$ with $c_2 \neq 0$, we have

$$\begin{aligned}&\left|y_0(\theta, 0) - \tilde{y}(\tilde{\theta}, \epsilon)\right| \\ &\quad = \left|y_0(\theta, 0) - y_0(\tilde{\theta}, 0) - \epsilon^2 \tilde{y}_2(\tilde{\theta}, 0) - o(\epsilon^2)\right| \\ &\quad = \left|\epsilon^2 \tilde{y}_2(\tilde{\theta}, 0) + o(\epsilon^2)\right| = \left|\epsilon^2 c_2 y_1(\tilde{\theta}, 0) + o(\epsilon^2)\right| \\ &\quad = |c_2| \left|\epsilon^2 y_1(\tilde{\theta}, 0) + o(\epsilon^2)\right| \\ &\quad = |c_2| \left|\epsilon^2 \left(y_0(\theta, 0) - y(\tilde{\theta}, \epsilon) + \epsilon^2 y_2(\tilde{\theta}, 0)\right)\right| \\ &\quad < \left|\epsilon^2 \left(y_0(\theta, 0) - y(\tilde{\theta}, \epsilon) + \epsilon^2 y_2(\tilde{\theta}, 0)\right)\right| \\ &\quad < \epsilon^2 \left|y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)\right| + \epsilon^4 \left|y_2(\tilde{\theta}, 0)\right|.\end{aligned} \tag{46}$$

Since $|y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)| = o(\epsilon)$, for ϵ small enough, there exists a positive real constant C such that $|y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)| \leq C\epsilon$.

Let h be a function defined by $h(\epsilon) := (\epsilon^2 - 1)C + \epsilon^3 |y_2(\tilde{\theta}, 0)|$. So, h is continuous with $h(0) = -C < 0$, then $\exists \epsilon_0 > 0$ such that $\forall \epsilon \in]0, \epsilon_0[$, we have $h(\epsilon) < 0$.

Therefore, for all $\epsilon \in]0, \epsilon_0[$ we get

$$\begin{aligned}&(\epsilon^2 - 1)\left|y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)\right| + \epsilon^4 \left|y_2(\tilde{\theta}, 0)\right| \\ &\quad \leq \epsilon\left((\epsilon^2 - 1) C + \epsilon^3 \left|y_2(\tilde{\theta}, 0)\right|\right) = \epsilon h(\epsilon) < 0\end{aligned} \tag{47}$$

which implies from what precedes that $|y_0(\theta, 0) - \tilde{y}(\tilde{\theta}, \epsilon)| < |y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)|$. So $\tilde{y}(\tilde{\theta}, \epsilon)$ is an approximation to $y_0(\theta, 0)$ closer than $y(\tilde{\theta}, \epsilon)$.

Moreover, if $c_1 = 0$ and $c_2 \neq 0$, then $\exists \epsilon_0 (< 1/|c_2|) > 0$ such that, $\forall \epsilon \in]0, \epsilon_0[$,

$$\begin{aligned}&\left|y_0(\theta, 0) - \tilde{y}(\tilde{\theta}, \epsilon)\right| \\ &\quad = \left|y_0(\theta, 0) - y_0(\tilde{\theta}, 0) - \epsilon^2 \tilde{y}_2(\tilde{\theta}, 0) - o(\epsilon^2)\right| \\ &\quad = \left|\epsilon^2 \tilde{y}_2(\tilde{\theta}, 0) + o(\epsilon^2)\right| = \left|\epsilon^2 c_2 y_1(\tilde{\theta}, 0) + o(\epsilon^2)\right| \\ &\quad = \epsilon |c_2| \left|\epsilon y_1(\tilde{\theta}, 0) + o(\epsilon)\right| \\ &\quad = \epsilon |c_2| \left|y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)\right| \\ &\quad < \left|y_0(\theta, 0) - y(\tilde{\theta}, \epsilon)\right|.\end{aligned} \tag{48}$$

So $\tilde{y}(\tilde{\theta}, \epsilon)$ is an approximation to $y_0(\theta, 0)$ closer than $y(\tilde{\theta}, \epsilon)$.

□

Remark 9. Let g be a real function such that $g(\epsilon) = \epsilon^m h(\epsilon), m \in \mathbb{N}^*$, where $h(\epsilon)$ can be expanded in powers of ϵ as $h(\epsilon) = \sum_{k\geq 1} \epsilon^k a_k$, with a_k real constants.

(1) If $m = 1$ or 2, according to the conditions of Theorem 8, we conclude that $\tilde{y}(\tilde{\theta}, \epsilon)$ is an approximation to $y_0(\theta, 0)$ closer than $y(\tilde{\theta}, \epsilon)$.

(2) If $m > 2$ (in the case where $m = 3$, the fourth term is given by Proposition 5), we can expect that $\tilde{y}(\tilde{\theta}, \epsilon)$ is an approximation to $y_0(\theta, 0)$ closer than $y(\tilde{\theta}, \epsilon)$.

Remark 10. We note here that, in the fractional case, the existence of a positive solution of (32) is studied in [18].

Remark 11. Although the Lindstedt-Poincaré method gives uniformly valid asymptotic expansions for periodic solutions of weakly nonlinear oscillations, i.e., $0 < \epsilon < C_1$, the technique does not work if the amplitude of the oscillation is a function of time (see [16, 17]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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