New Branch of Intuitionistic Fuzzification in Algebras with Their Applications

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Received 28 February 2018; Revised 22 May 2018; Accepted 11 June 2018; Published 10 July 2018

Academic Editor: Susana Montes

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The intuitionistic fuzzification in $\rho-$algebras about the concepts of ideals and subalgebras given with several related characterizations is considered. Some new concepts like intuitionistic fuzzy $\rho-$ideal ($IF_{\rho}i$), intuitionistic fuzzy $\rho-$subalgebra ($IF_{\rho}s$), $\rho-$homomorphism, and intuitionistic fuzzy $\bar{\rho}-$ideal ($IF_{\bar{\rho}}i$) are introduced and some of their descriptions are given in this work. Further, we show some applications on the family of all intuitionistic fuzzy $\rho-$subalgebras $IF_{\rho}(\mathcal{R})$ in $\rho-$algebra like the binary relations $\approx_{\mu_{\rho}}$ and $\Gamma_{\rho}$ on $IF_{\rho}(\mathcal{R})$. Also, their equivalence classes are given and studied.

1. Introduction

The fuzzy set (FS) as suggested by Zadeh [1] in 1965 is a regulation to vagueness and encounter uncertainty. A FS maps each element of the universe of discourse to the interval $[0, 1]$. After the introduction of fuzzy sets theory by him, many mathematicians were conducted on the generalizations of the this concept and studied in the groups, algebras, and soft spaces (see [2–5]). By including a fuzzy set the degree of nonmembership, Atanassov [6] in 1986 suggested the intuitionistic fuzzy set (IFS), which seems more precise for provides opportunities and uncertainty quantification to accurately model a problem based on existing knowledge and monitoring. Also, this notion is discussed in different fields (see [7–11]).

$BCK-$algebra, class of algebra of logic, was investigated by Imai and Iseki [12]. After that, the notion of $d-$algebras was investigated by Neggers and Kim [13]. In 2017, the concepts of $\rho-$algebra, $\bar{\rho}-$ideal, $\rho-$ideal, $\rho-$subalgebra, and permutation topological $\rho-$algebra were first proposed by Mahmood and Abud Alradha [14]. Next, they showed the notion of the soft $\rho-$algebra and soft edge $\rho-$algebra [15].

In the present work, the notions of intuitionistic fuzzy $\rho-$ideal ($IF_{\rho}i$), intuitionistic fuzzy $\rho-$subalgebra ($IF_{\rho}s$), $\rho-$homomorphism, and intuitionistic fuzzy $\bar{\rho}-$ideal ($IF_{\bar{\rho}}i$) are introduced. Further, we show some applications on the family of all intuitionistic fuzzy $\rho-$subalgebras $IF_{\rho}(\mathcal{R})$ in $\rho-$algebra like the binary relations $\approx_{\mu_{\rho}}$ and $\Gamma_{\rho}$ on $IF_{\rho}(\mathcal{R})$. Also, their equivalence classes are given and studied.

2. Preliminaries and Notations

We will recall basic definitions and results to obtain properties developed in this work.

Definition 1 (see [16]). An intuitionistic fuzzy set $\alpha$ (IFS, in short) over the universe $\mathcal{R}$ is defined by $\alpha = \{< a, \mu_{\alpha}(a), \nu_{\alpha}(a) > | a \in \mathcal{R} \}$, where $\mu_{\alpha}(a): \mathcal{R} \rightarrow [0, 1]$, $\nu_{\alpha}(a): \mathcal{R} \rightarrow [0, 1]$ with $0 \leq \mu_{\alpha}(a) + \nu_{\alpha}(a) \leq 1, \forall a \in \mathcal{R}$. $\mu_{\alpha}(a)$ and $\nu_{\alpha}(a)$ are real numbers and their values represent the degree of membership and nonmembership of $a$ to $\alpha$, respectively.

Definition 2 (see [6]). The IF whole and empty sets of $\mathcal{R}$ are defined by $\overline{\mathcal{R}} = \{< a, (1, 0) > | a \in \mathcal{R} \}$ and $\overline{\emptyset} = \{< a, (0, 1) > | a \in \mathcal{R} \}$, respectively.
2.1. Basic Relations and Operations on Intuitionistic Fuzzy Sets

Assume \( \alpha = \{ < a, (\mu_\alpha(a), \nu_\alpha(a)) > | a \in \mathbb{R} \} \) and \( \beta = \{ < a, (\mu_\beta(a), \nu_\beta(a)) > | a \in \mathbb{R} \} \) are two IFSs of \( \mathbb{R} \). We deduced the following relations:

1. [inclusion] \( \alpha \subseteq \beta \) iff \( \mu_\alpha(a) \leq \mu_\beta(a) \) and \( \nu_\alpha(a) \geq \nu_\beta(a) \), \( \forall a \in \mathbb{R} \).
2. [equality] \( \alpha = \beta \) iff \( \alpha \subseteq \beta \) and \( \beta \subseteq \alpha \).
3. [intersection] \( \alpha \cap \beta = \{ (a, \min\{\mu_\alpha(a), \mu_\beta(a)\}, \min\{\nu_\alpha(a), \nu_\beta(a)\}) : a \in \mathbb{R} \} \).
4. [union] \( \alpha \cup \beta = \{ (a, \max\{\mu_\alpha(a), \mu_\beta(a)\}, \max\{\nu_\alpha(a), \nu_\beta(a)\}) : a \in \mathbb{R} \} \).
5. [complement] \( \alpha^c = \{ (a, \nu_\alpha(a), \mu_\alpha(a)) : a \in \mathbb{R} \} \).

Definition 3 (see [14]). We say \((\mathbb{R}, \cdot, 0)\) is \(\rho\)-algebra if \(\cdot\) is a binary operation on \(\mathbb{R}\) with a constant \(0 \in \mathbb{R}\) and such that

1. \(a \cdot a = 0\),
2. \(0 \cdot a = 0\),
3. \(a \cdot b = 0 \Rightarrow b \cdot a = 0\) imply that \(a = b\),
4. For all \(a \neq b \in \mathbb{R} - \{0\}\) imply that \(a \cdot b = b \cdot a \neq 0\).

Definition 4 (see [14]). Assume \((\mathbb{R}, \cdot, 0)\) is a \(\rho\)-algebra and \(\phi \neq K \subseteq \mathbb{R}\). We say \(K\) is a \(\rho\)-subalgebra of \(\mathbb{R}\) if \(a \cdot b \in K, \forall a, b \in K\).

Definition 5 (see [14]). Assume \((\mathbb{R}, \cdot, 0)\) is a \(\rho\)-algebra and \(\phi \neq K \subseteq \mathbb{R}\). We say \(K\) is a \(\rho\)-ideal of \(\mathbb{R}\) if

1. \(a \cdot b \in K\) imply \(a \cdot b \in K\),
2. \(a \cdot b \in K\) and \(b \cdot K\) imply \(a \in K, \forall a, b \in \mathbb{R}\).

Definition 6 (see [14]). Assume \((\mathbb{R}, \cdot, 0)\) is a \(\rho\)-algebra and \(K\) subset of \(\mathbb{R}\). We say \(K\) is a \(\rho\)-ideal of \(\mathbb{R}\) if

1. \(0 \in K\),
2. \(a \in K\) and \(a \cdot b \in \mathbb{R}\), \(a \cdot b \in K, \forall a, b \in \mathbb{R}\).

Definition 7 (see [11]). Assume that \(\alpha = \{ < a, (\mu_\alpha(a), \nu_\alpha(a)) > | a \in \mathbb{R} \}\) is an IFS in \(\mathbb{R}\) and \(r \in [0,1]\). The set \(W(\mu_\alpha, r) = \{ a \in \mathbb{R} | \mu_\alpha(a) \geq r \}\) (resp., \(L(\nu_\alpha, r) = \{ a \in \mathbb{R} | \nu_\alpha(a) \leq r \}\)) is said to be \(\mu\)-level \(r\)-cut (resp., \(\nu\)-level \(r\)-cut) of \(\alpha\).

3. Intuitionistic Fuzzy

\(\rho\)-Subalgebras in \(\rho\)-Algebras

In this section, we introduce some new concepts, such as (IFPs), (IFpi), (IFp\(\bar{\rho}\)), and \(\rho\)-homomorphism which are introduced and discussed. Further, some binary relations \(\preceq\), \(\succeq\), and \(\Gamma\), on IFSs of \(\mathbb{R}\) are given, and some basic properties are shown.

Definition 8. Assume \((\mathbb{R}, \cdot, 0)\) is a \(\rho\)-algebra and \(\alpha = \{ < a, (\mu_\alpha(a), \nu_\alpha(a)) > | a \in \mathbb{R} \}\) is IFS of \(\mathbb{R}\). We say \(\alpha\) is an (IFPs) of \(\mathbb{R}\) if \(\mu_\alpha(a \cdot b) \geq \min\{\mu_\alpha(a), \mu_\beta(b)\}\) and \(\nu_\alpha(a \cdot b) \leq \max\{\nu_\alpha(a), \nu_\beta(b)\}, \forall a, b \in \mathbb{R}\).
Let \( \mu \) be a function from a set \( A \) to \([0,1]\). We define \( \mu(A) = \sum_{a \in A} \mu(a) \).

\[ \int \mu \leq \sum_{a \in A} \mu(a) \leq 1 \]

Proof. Let \( a, b \in \mathcal{R} \). Thus we consider that
\[
\min\{\mu_a(a \cdot b), \mu_b(a \cdot b)\} \leq \max\{\mu_a(a), \mu_b(a)\} \leq \min\{\mu_a(a \cdot b), \mu_b(a \cdot b)\}.
\]

Thus \( \int \mu \leq \mu \). Hence we consider that
\[
\min\{\mu_a(a \cdot b), \mu_b(a \cdot b)\} \leq \max\{\mu_a(a), \mu_b(a)\} \leq \min\{\mu_a(a \cdot b), \mu_b(a \cdot b)\}.
\]

Therefore \( \mu \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \).

Theorem 17. If \( \alpha = \{< a, (\mu_a(a), \nu_a(a)) | a \in \mathcal{A} \} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \), then \( \mathcal{K} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \).

Proof. We need only to show that \( 1 - \mu(a) \) satisfies the first condition in Definition 10. Assume \( a, b \in \mathcal{R} \).

Then \( 1 - \mu(a \cdot b) \leq 1 - \min\{\mu_a(a), \mu_b(b)\} = \max\{1 - \mu_a(a), 1 - \mu_b(b)\} \).

Furthermore, \( 1 - \mu(a \cdot b) \leq 1 - \min\{\mu_a(a), \mu_b(b)\} = \max\{1 - \mu_a(a), 1 - \mu_b(b)\} \).

Therefore \( \mu \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \).

Theorem 22. If \( \alpha = \{< a, (\mu_a(a), \nu_a(a)) | a \in \mathcal{A} \} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \), then \( \mathcal{K} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \).

Proof. Suppose there are two members \( t_1 \) and \( t_2 \) in \( \mathcal{R} \) with \( \mu_a(t_1 \cdot t_2) < \min\{\mu_a(t_1), \mu_a(t_2)\}/2 \). Hence \( \mu_a(t_1 \cdot t_2) < \min\{\mu_a(t_1), \mu_a(t_2)\} \) and so \( t_1 \cdot t_2 \neq W(\mu_a, t) \) but \( t_1, t_2 \in W(\mu_a, t) \). This is a contradiction, and therefore \( \mu_a(a \cdot b) \geq \min\{\mu_a(a), \mu_b(b)\} \).

Now assume that \( \nu_b(t_1 \cdot t_2) > k \max\{\nu_a(t_1), \nu_a(t_2)\} \) for some \( t_1, t_2 \in \mathcal{R} \). Taking \( k = \nu_a(t_1 \cdot t_2) + \min\{\nu_a(t_1), \nu_a(t_2)\}/2 \), then we consider that \( \nu_b(t_1 \cdot t_2) > k \max\{\nu_a(t_1), \nu_a(t_2)\} \).

It follows that \( t_1, t_2 \in L(\nu_a, k) \) and \( t_1 \cdot t_2 \notin L(\nu_a, k) \). This is a contradiction. Therefore, we consider that \( \nu_b(a \cdot b) \leq \max\{\nu_a(a), \nu_b(b)\} \).

Theorem 23. If \( \mathcal{H} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \), then there exists \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \).

Proof. Assume \( \mathcal{H} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \) and let \( \mu_a \) and \( \nu_a \) be fuzzy sets in \( \mathcal{R} \) defined by

\[ \mu_a(a) = \begin{cases} k, & \text{if } a \in H \\ 1, & \text{Otherwise} \end{cases} \]

and

\[ \nu_a(a) = \begin{cases} m, & \text{if } a \in H \\ 1, & \text{Otherwise} \end{cases} \]

for all \( a \in \mathcal{A} \), where \( k, m \in (0, 1) \) are fixed real numbers with \( k + m < 1 \). Assume \( a, b \in \mathcal{A} \). Then \( a \cdot b \in H \) whenever \( a, b \in H \). This implies that \( \mu_a(a) = \mu_b(b) \).

If at least one of \( a \) or \( b \) does not belong to \( H \), then either \( \nu_a(a) = 0 \) or \( \nu_b(b) = 0 \) and hence \( \nu_a(a) = 1 \) or \( \nu_b(b) = 1 \). It follows that \( \mu_a(b) = \mu_b(a) = \mu_a(a) = \mu_b(b) \).

Hence \( \alpha = \{< a, (\mu_a(a), \nu_a(a)) | a \in \mathcal{A} \} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \).

Obviously, \( W(\mu_a, k) = H = L(\nu_a, k) \).

Definition 24. Assume \( \Theta : \mathcal{R} \rightarrow Y \) is a mapping of \( \mathcal{R} \)-algebras. We say \( \Theta \) is a \( \mathcal{R} \)-homomorphism if \( \Theta(a \cdot b) = \Theta(a) \cdot \Theta(b) \), \( \forall a, b \in \mathcal{R} \). And \( \Theta^{-1}(\beta) = \{< a, (\Theta^{-1}(\mu_a), \Theta^{-1}(\nu_a)) | a \in \mathcal{A} \} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \) for any \( \mathcal{R} \)-algebra \( \mathcal{B} \). Also, if \( \alpha = \{< a, (\mu_a(a), \nu_a(a)) | a \in \mathcal{A} \} \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \), then \( \Theta(\alpha) \) is \((\mathcal{R},\mathcal{R})\)-super-algebra of \( \mathcal{A} \) and defined by

\[ \Theta(\alpha) = \{< c, (\Theta^{-1}(\mu_c), \Theta^{-1}(\nu_c)) | c \in Y \} \]
Theorem 25. Let \( \Theta \) be \( p \)-homomorphism of \( p \)-algebra \( R \) into \( p \)-algebra \( Y \) and \( K \) be \((IPf)\) of \( Y \). Then \( \Theta^{-1}(K) \) is \((IPf)\) of \( R \).

Proof. Assuming \( a, b \in R \), we have \( \mu_{\Theta^{-1}(K)}(a \cdot b) = \mu_{K}(\Theta(a) \cdot \Theta(b)) \geq \min\{\mu_{K}(\Theta(a)), \mu_{K}(\Theta(b))\} = \min\{\mu_{\Theta^{-1}(K)}(a), \mu_{\Theta^{-1}(K)}(b)\} \) and \( \nu_{\Theta^{-1}(K)}(a \cdot b) = \nu_{K}(\Theta(a) \cdot \Theta(b)) \leq \max\{\nu_{K}(\Theta(a)), \nu_{K}(\Theta(b))\} = \max\{\nu_{\Theta^{-1}(K)}(a), \nu_{\Theta^{-1}(K)}(b)\} \). Thus \( \Theta^{-1}(K) \) is \((IPf)\) of \( R \).

Theorem 26. Assume \( \Theta : R \rightarrow Y \) is \( p \)-homomorphism of \( p \)-algebra \( R \) into \( p \)-algebra \( Y \) and \( a = \{< a, (\mu_{\alpha}(a), \nu_{\alpha}(a)) | a \in R \} \) is \((IPf)\) of \( R \). Then \( \Theta(a) = \{< b, (\Theta_{sup}(\mu_{\alpha}), \Theta_{inf}(\nu_{\alpha})) | a \in R \} \) is \((IPf)\) of \( Y \).

Proof. Let \( \alpha = \{< a, (\mu_{\alpha}(a), \nu_{\alpha}(a)) | a \in R \} \) be \((IPf)\) of \( R \) and \( \alpha \), \( \beta \) be elements of \( \Theta^{-1}(\alpha) \). Noticing that \( \{a, a_1 | a_1 \in \Theta^{-1}(t_1) \} \subseteq \{a | a \in \Theta^{-1}(t_1) \} \) and \( \{a, a_2 | a_2 \in \Theta^{-1}(t_2) \} \subseteq \{a | a \in \Theta^{-1}(t_2) \} \), we have \( \Theta_{sup}(\mu_{\alpha}(a_1 \cdot t_1)) = \max\{\Theta_{sup}(\mu_{\alpha}(a_1)), \Theta_{sup}(\mu_{\alpha}(t_1))\} \) and \( \Theta_{inf}(\nu_{\alpha}(a_1 \cdot t_1)) = \min\{\Theta_{inf}(\nu_{\alpha}(a_1)), \Theta_{inf}(\nu_{\alpha}(t_1))\} \). Also, we consider that \( \Theta_{sup}(\mu_{\alpha}(a_1 \cdot t_1)) = \max\{\Theta_{sup}(\mu_{\alpha}(a_1)), \Theta_{sup}(\mu_{\alpha}(t_1))\} \) and \( \Theta_{inf}(\nu_{\alpha}(a_1 \cdot t_1)) = \min\{\Theta_{inf}(\nu_{\alpha}(a_1)), \Theta_{inf}(\nu_{\alpha}(t_1))\} \). Hence \( \Theta(\alpha) = \{< b, (\Theta_{sup}(\mu_{\alpha}), \Theta_{inf}(\nu_{\alpha})) | a \in R \} \) is \((IPf)\) of \( Y \).

Theorem 27. Assume \( \Theta : R \rightarrow Y \) is \( p \)-homomorphism of \( p \)-algebra \( R \) into \( p \)-algebra \( Y \) and \( a = \{< a, (\mu_{\alpha}(a), \nu_{\alpha}(a)) | a \in R \} \) is \((IPf)\) of \( R \). Then \( \Theta(a) = \{< b, (\Theta_{sup}(\mu_{\alpha}), \Theta_{inf}(\nu_{\alpha})) | a \in R \} \) is \((IPf)\) of \( Y \).

Proof. Since \( \alpha = \{< a, (\mu_{\alpha}(a), \nu_{\alpha}(a)) | a \in R \} \) is \((IPf)\) of \( R \), then by Theorem 26 and Remark 14 we have \( \Theta(a) = \{< b, (\Theta_{sup}(\mu_{\alpha}), \Theta_{inf}(\nu_{\alpha})) | a \in R \} \) is \((IPf)\) of \( Y \). Hence condition (1) in Definition 10 is satisfied. Since \( \Theta \) is surjective, then for any \( t_1, t_2 \in Y \), \( \exists a, a_1, a_2 \in R \) such that \( a_1 \in \Theta^{-1}(\Theta(a)) = \Theta^{-1}(t_1) \) and \( a_2 \in \Theta^{-1}(\Theta(a)) = \Theta^{-1}(t_2) \). Also, \( a, a_1 \in \Theta^{-1}(\Theta(a) \cdot t_1) \) and \( a, a_2 \in \Theta^{-1}(\Theta(a) \cdot t_2) \). Further, noticing that \( \mu_{\Theta(a)}(a_1) \geq \min\{\mu_{\Theta(a)}(a_1), \mu_{\Theta(a)}(a_2)\} \) and \( \nu_{\Theta(a)}(a_1) \leq \max\{\nu_{\Theta(a)}(a_1), \nu_{\Theta(a)}(a_2)\} \), for any \( t_1, t_2 \in Y \), we have \( \Theta_{sup}(\mu_{\Theta(a)}(a_1 \cdot t_1)) = \sup\{\mu_{\Theta(a)}(a_1 \cdot t_1) \} \) and \( \Theta_{inf}(\nu_{\Theta(a)}(a_1 \cdot t_1)) = \inf\{\nu_{\Theta(a)}(a_1 \cdot t_1) \} \). Also, \( \Theta_{sup}(\mu_{\Theta(a)}(a_1 \cdot t_1)) = \sup\{\mu_{\Theta(a)}(a_1 \cdot t_1) \} \) and \( \Theta_{inf}(\nu_{\Theta(a)}(a_1 \cdot t_1)) = \inf\{\nu_{\Theta(a)}(a_1 \cdot t_1) \} \). Hence \( \Theta(a) = \{< b, (\Theta_{sup}(\mu_{\alpha}), \Theta_{inf}(\nu_{\alpha})) | a \in R \} \) is \((IPf)\) of \( Y \).

4. Some Applications on \( IPf_{\mathcal{P}}(R) \)

In this section, some applications on \( IPf_{\mathcal{P}}(R) \) are shown like the binary relations \( = \) and \( \supseteq \) on \( IPf_{\mathcal{P}}(R) \). Also, in this section the equivalence classes for these binary relations are given, and some of their basic properties are studied.

4.1. Equivalence Classes Modulo \( (=, \supseteq) \).

Denote the collection of all \( (IPf_{\mathcal{P}}(R)) \) by \( IPf_{\mathcal{P}}(R) \) and let \( r \in [0, 1] \). Define binary relations \( = \) and \( \supseteq \) on \( IPf_{\mathcal{P}}(R) \) as follows.

\[ \alpha = \beta \iff W(\mu_{\beta}, r) = W(\mu_{\beta}, r) \]
\[ \alpha \supseteq \beta \iff L(\nu_{\beta}, r) = L(\nu_{\beta}, r) \]

for \( \alpha, \beta \in IPf_{\mathcal{P}}(R) \). Moreover, it is clear that \( = \) and \( \supseteq \) are equivalence relations on \( IPf_{\mathcal{P}}(R) \). If \( \alpha = \beta \), then we refer to the equivalence class of \( \alpha \) as \( \alpha \) and \( \beta \), respectively. For \( \alpha = {\alpha}_{\mu}, \nu_{\alpha} \in IPf_{\mathcal{P}}(R) \), then we refer to the family of all \( p \)-ideals of \( R \) by \( \rho_{\mathcal{P}}(R) \) and let \( r \in [0, 1] \).

Let \( \sigma, \eta, \) be maps from \( IPf_{\mathcal{P}}(R) \) to \( \rho_{\mathcal{P}}(R) \) and let \( \delta, \eta, \) be maps from \( IPf_{\mathcal{P}}(R) \) to \( \rho_{\mathcal{P}}(R) \). Then \( \sigma, \eta, \) are surjective, for each \( r \in (0, 1) \).
Proof. Let $r \in (0, 1)$. Then $\tilde{\alpha} = a, \tilde{\beta} >$ is in $IF_{\mathcal{P}}(\mathcal{R})$, where each one of $\tilde{\alpha}$ and $\tilde{\beta}$ is (FS) in $\mathcal{R}$ defined by $\tilde{\alpha}(a) = 0$ and $\tilde{\beta}(a) = 1$, $\forall a \in \mathcal{R}$. Furthermore, $\tilde{\alpha}(a) = W(\tilde{\alpha}, r) = \phi = L(\tilde{\beta}, r) = \eta(\tilde{\beta})$. Also, $\eta(\hat{H}) = W(\mu(\hat{H}), r) = H = L(v(\hat{H}), r) = \eta(\hat{H})$. Therefore, we want to prove that $\hat{H} = x_\alpha, \mu(\hat{H}), v(\hat{H}) \in IF_{\mathcal{P}}(\mathcal{R})$. Since $H \in \rho_1(\mathcal{R})$, then by condition (i) in Definition 5 we have $H$ as $p$-subalgebra of $\mathcal{R}$ and this implies that $W(\mu(\hat{H}), r)$ and $L(v(\hat{H}), r)$ are $p$-subalgebras of $\mathcal{R}$. By Theorem 22 we consider $\tilde{\alpha} = a, \tilde{\beta} > \in IF_{\mathcal{P}}(\mathcal{R})$. Therefore, $\forall H \in \rho_1(\mathcal{R})$ we consider $\sigma(\hat{H}) = H$ and $\eta(\hat{H}) = H$ for some $H \in IF_{\mathcal{P}}(\mathcal{R})$. This completes the proof.

**Theorem 30.** Let $IF_{\mathcal{P}}(\mathcal{R})/\rho_1$ and $IF_{\mathcal{P}}(\mathcal{R})/\rho_2$ be quotient sets. Then they are equivalent to $\rho_1(\mathcal{R}) \cup \{\phi\}$, $\forall r \in (0, 1)$.

**Proof.** Assume $r \in (0, 1)$ and let $\alpha'(\rho_1(\mathcal{R}))$ be a map from $IF_{\mathcal{P}}(\mathcal{R})/\rho_1$ to $\rho_1(\mathcal{R}) \cup \{\phi\}$ and they are defined by $\alpha'(\tilde{\alpha}(\rho_1(\mathcal{R}))) = \sigma(\alpha)$, $\forall \alpha = \langle a, \mu(\alpha), v(\alpha) \rangle \in IF_{\mathcal{P}}(\mathcal{R})$. Hence, $\alpha = \beta$ and $\alpha = \beta$, $\forall \alpha = \langle a, \mu(\alpha), v(\alpha) \rangle$ and $\beta = \langle a, \mu(\beta), v(\beta) \rangle \in IF_{\mathcal{P}}(\mathcal{R})$, if $W(\mu(\alpha), r) = W(\mu(\beta), r)$ and $L(v(\alpha), r) = L(v(\beta), r)$. Then $\alpha(\tilde{\alpha}(\rho_1(\mathcal{R}))) = \beta(\tilde{\beta}(\rho_1(\mathcal{R})))$ and $(\alpha) = (\beta)$. This implies the maps $\alpha'$ and $\beta'$ are injective. Moreover, let $\phi = H \in \rho_1(\mathcal{R})$ and $\forall r \in (0, 1)$, let

$$
\begin{align*}
\mu_H(a) = \begin{cases} 
1, & \text{if } a \in H \\
0, & \text{if } a \notin H,
\end{cases}
\end{align*}
$$

and $\eta_H(a) = 1 - \mu_H(a)$ such that $\eta(H) = \sigma(\hat{H}) \cap \eta(\hat{H}) = W(\mu(\hat{H}), r) \cap L(v(\hat{H}), r) = H$. This completes the proof.

**Theorem 32.** For any $r \in (0, 1)$, the quotient set $IF_{\mathcal{P}}(\mathcal{R})/\Gamma_r$ is equivalent to $\rho_1(\mathcal{R}) \cup \{\phi\}$.

**Proof.** Assume $r \in (0, 1)$ and $\psi_1 : IF_{\mathcal{P}}(\mathcal{R})/\Gamma_r \to \rho_1(\mathcal{R}) \cup \{\phi\}$ is a map defined by $\psi_1((\alpha)) = \sigma(\alpha)$, $\forall (\alpha) \in IF_{\mathcal{P}}(\mathcal{R})/\Gamma_r$. Suppose that $\psi((\alpha)) = \psi((\beta))$ for any $(\alpha), (\beta) \in IF_{\mathcal{P}}(\mathcal{R})/\Gamma_r$. We consider that $\sigma(\alpha) \cap \eta(\tilde{\beta}) = \sigma(\beta) \cap \eta(\beta)$, $\forall (\alpha) \in IF_{\mathcal{P}}(\mathcal{R})/\Gamma_r$. Hence $\alpha = \beta$, $\forall (\alpha) \in IF_{\mathcal{P}}(\mathcal{R})/\Gamma_r$. Therefore $\psi$ is injective. Furthermore, for $\phi = a, \tilde{\alpha} \in \rho_1(\mathcal{R})$ we get $\psi((\tilde{\alpha})) = \psi((\tilde{\beta})) = \phi$. Let $H = a, \tilde{\alpha} \in \rho_1(\mathcal{R})$, $\forall H \in IF_{\mathcal{P}}(\mathcal{R})$, be the same $(IF_{\mathcal{P}}(\mathcal{R}))$ of $X$ that is defined in the proof of Theorem 22. Then we have $\psi((\tilde{\alpha})) = \psi((\tilde{\beta})) = \sigma(H) \cap \eta(\tilde{H}) = W(\mu(\tilde{H}), r) \cap L(v(\tilde{H}), r) = H$. Hence $\psi$ is surjective. This completes the proof.

**5. Conclusion**

In this work, we introduce the notions of $(IF_{\mathcal{P}}(\mathcal{R}))$, $(IF_{\mathcal{P}}(\mathcal{R}))$, and others; then we proved that for any $p$-subalgebra of $X$ can be considered as both $p$-level $p$-subalgebra and $v$-level $p$-subalgebra of some $(IF_{\mathcal{P}}(\mathcal{R}))$. At the same time, we proved that intersection of any family of $(IF_{\mathcal{P}}(\mathcal{R}))$ of $(IF_{\mathcal{P}}(\mathcal{R}))$ of $X$. Also, we show that if $\psi \in \rho_1(\mathcal{R})$, then $\psi \in \rho_1(\mathcal{R})$ and $\psi$ and $\psi$ are $\rho$-homomorphism are given. Finally, some binary relations $\equiv$, $\approx$ and $\Gamma_r$ on $IF_{\mathcal{P}}(\mathcal{R})$ are obtained, and some of their basic properties are discussed. In future work, we will investigate IF in new types of algebras like $BCL^*$-algebras, $BCL^*$-subalgebras, $BCL^*$-ideals and others. Next, we will study their characteristics.

**Data Availability**

Data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare no conflicts of interest.

**References**


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