Research Article

Structure of n-Lie Algebras with Involutive Derivations

Ruipu Bai,1 Shuai Hou,2 and Yansha Gao2

1College of Mathematics and Information Science, Hebei University, Key Laboratory of Machine Learning and Computational Intelligence of Hebei Province, Baoding 071002, China
2College of Mathematics and Information Science, Hebei University, Baoding 071002, China

Correspondence should be addressed to Ruiru Bai; bairuipu@hbu.edu.cn

Received 30 January 2018; Revised 28 June 2018; Accepted 11 July 2018; Published 2 September 2018

1. Introduction

Derivation is an important tool in studying the structure of n-Lie algebras [1]. The derivation algebra $\text{Der}(A)$ of an n-Lie algebra $A$ over the field of real numbers is the Lie algebra of the automorphism group $\text{Aut}(A)$, which is a Lie group if $\dim A < \infty$ [2]. Any n-Lie algebra module $(V, \rho)$ is a module of the inner derivation algebra $\text{ad}(A)$, which is a linear Lie algebra [3]. Also, derivations have close relationship with extensions of n-Lie algebras.

The concept of 3-Lie classical Yang-Baxter equations is introduced in [4]. It is known that if there is an involutive derivation $D$ on $A$, then $(A, \{, , \})$ is a 3-pre-Lie algebra, where $\{x, y, z\}_D = D(\text{ad}(x, y)D(z)), \forall x, y, z \in A$, and the 3-Lie algebra $A$ is a subadjacent 3-Lie algebra of $(A, \{, , \}_{D})$, and $r = \sum e_i^* \otimes D(e_i) - D(e_i) \otimes e_i^*$ is a skew-symmetric solution of the 3-Lie classical Yang-Baxter equation in the 3-Lie algebra $\text{ad}^* A\ast$, where $\{e_1, \ldots, e_m\}$ is a basis of $A$ and $\{e_1^*, \ldots, e_m^*\}$ is the dual basis of $A^*$. Due to this importance of involutive derivations on 3-Lie algebras, we investigate in this paper the existence of involutive derivations on finite dimensional n-Lie algebras. More specifically, in Section 2, we discuss the properties of involutive derivations on n-Lie algebras. In Section 3, we study the existence of involutive derivations on $(2s + 2)$-dimensional $(2s + 1)$-Lie algebras. In Section 4, we consider the existence of involutive derivations on $(2s + 3)$-dimensional $(2s + 1)$-Lie algebras. In Section 5, we investigate a class of 3-Lie algebras with involutive derivations which are two-dimensional extension of Lie algebras.

In the following, we assume that all algebras are over an algebraically closed field $\mathbb{F}$ with characteristic zero, $\text{Id}$ is the identity mapping, and $\mathbb{Z}$ is the set of integers. For $\lambda \in \mathbb{F}$ and an $\mathbb{F}$-linear mapping $D$ on a vector space $A$, $A_\lambda$ denotes the subspace $\{x \in A \mid D(x) = \lambda x\}$.

2. n-Lie Algebras with Involutive Derivations

An n-Lie algebra [1] is a vector space $A$ over a field $\mathbb{F}$ equipped with a linear multiplication $[\cdot, \cdot, \cdot] : \wedge^n A \rightarrow A$ satisfying, for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$,

$$
\left[ [x_1, \ldots, x_n], y_2, \ldots, y_n \right] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].
$$

Equation (1) is usually called the generalized Jacobi identity, or Filippov identity.
The derived algebra of an \( n \)-Lie algebra \( A \) is a subalgebra of \( A \) generated by \( [x_1, \cdots, x_n] \) for all \( x_1, \cdots, x_n \in A \), and is denoted by \( A^1 \). We use \( Z(A) \) to denote the center of \( A \); that is, \( Z(A) = \{ x \mid x \in A, [x, A, \cdots, A] = 0 \} \).

A derivation of \( A \) is an endomorphism of \( A \) satisfying

\[
D([x_1, \cdots, x_n]) = \sum_{i=1}^{n} D(x_i) [x_1, \cdots, \hat{x_i}, \cdots, x_n],
\]

\( \forall x_1, \cdots, x_n \in A. \) (2)

If a derivation \( D \) satisfies that \( D^2 = Id \), then \( D \) is called an involutive derivation on \( A \). Der(\( A \)) denotes the derivation algebra of \( A \).

For \( x_1, \cdots, x_{n-1} \in A \), map \( \text{ad}(x_1, \cdots, x_{n-1}) : A \rightarrow A, \)

\( \text{ad}(x_1, \cdots, x_{n-1})(x) = [x_1, \cdots, x_{n-1}, x], \forall x \in A \) (3)

is called a left multiplication defined by elements \( x_1, \cdots, x_{n-1} \). From (1), left multiplications are derivations.

The following lemma can be easily verified.

**Lemma 1.** Let \( V \) be a finite dimensional vector space over \( \mathbb{F} \) and \( D \) be an endomorphism of \( V \) with \( D^2 = Id \). Then \( V \) can be decomposed into the direct sum of subspaces \( V = V_1 + V_{-1} \), where \( V_1 = \{ v \in V \mid Dv = v \} \) and \( V_{-1} = \{ v \in V \mid Dv = -v \} \).

If \( A \) is a finite dimensional \( n \)-Lie algebra with an involutive derivation \( D \), then we have

\[
A = A_1 + A_{-1}.
\]

(4)

**Lemma 2.** Let \( A \) be an \( n \)-Lie algebra over \( \mathbb{F} \). If \( D \in \text{Der}(A) \) is an involutive derivation, then, for all \( x_1, \cdots, x_n \in A \),

\[
[x_1, \cdots, x_n] = \frac{-2}{n} \sum_{1 \leq i < j \leq n} [x_1, \cdots, x_j, x_{j+1}, \cdots, x_n],
\]

\[
\frac{1}{n} [D(x_1), \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_n],
\]

\[
[D(x_1), \cdots, D(x_n)] = \frac{-2}{n} \sum_{1 \leq i < j \leq n} D(x_1, \cdots, D(x_{j-1}), x_j),
\]

\[
D([x_1, \cdots, x_n]) = \frac{-2}{n} \sum_{1 \leq i < j \leq n} D(x_1, \cdots, D(x_{j-1}), x_j, x_{j+1}, \cdots, x_n).
\]

(5)

(6)

**Proof.** If \( D \) is an involutive derivation on \( A \), then, for all \( x_1, \cdots, x_n \in A \),

\[
[x_1, \cdots, x_n] = D^2([x_1, \cdots, x_n]) = \frac{-2}{n} \sum_{1 \leq i < j \leq n} [x_1, \cdots, D(x_i), \cdots, x_n]
\]

\[
= n [x_1, \cdots, x_n]
\]

\[
+ 2 \sum_{1 \leq i < j \leq n} [x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n].
\]

Equation (5) follows. Equation (6) follows from (4) and \( D^2 = Id \). \( \square \)

**Theorem 3.** Let \( A \) be a finite dimensional \( n \)-Lie algebra with \( n = 2s, s \geq 1 \). Then there is an involutive derivation \( D \) on \( A \) if and only if \( A \) is abelian.

**Proof.** If \( A \) is abelian, then the result is trivial.

Conversely, let \( D \) be an involutive derivation on \( A \). By Lemma 1, \( A = A_1 + A_{-1} \). Then, for any \( i \in \mathbb{Z}, 1 \leq i \leq n, x_1, \cdots, x_n \in A_{1}, \) and \( y_1, \cdots, y_n \in A_{-1}, \)

\[
D([x_1, \cdots, x_i, y_1, \cdots, y_{n-i}])
\]

\[
= i [x_1, \cdots, x_i, y_1, \cdots, y_{n-i}]
\]

\[
- (n - i) [x_1, \cdots, x_i, y_i, \cdots, y_{n-i}]
\]

\[
= (2i - 2s) [x_1, \cdots, x_i, y_1, \cdots, y_{n-i}] \in A_{2i-2s}.
\]

(8)

\[
D([x_1, \cdots, x_n]) = 2s [x_1, \cdots, x_n],
\]

\[
D([y_1, \cdots, y_n]) = -2s [y_1, \cdots, y_n].
\]

Thanks to \( \pm 2s \neq \pm 1 \) and \( 2i - 2s \neq \pm 1, A_{2i-2s} = A_{s+2s} = 0. \) Therefore, \( A \) is abelian. \( \square \)

**Theorem 4.** Let \( A \) be a finite dimensional \( n \)-Lie algebra with \( n = 2s+1, s \geq 1, \) and \( D \) be an involutive derivation on \( A \). Then \( A_1 \) and \( A_{-1} \) are abelian subalgebras, and

\[
\begin{bmatrix}
A_1, & \cdots, & A_1, & A_{-1}, & \cdots, & A_{-1}
\end{bmatrix}_{\frac{2s+1}{i}} = 0,
\]

\( \forall 1 \leq i \leq 2s, i \neq s, s + 1, \)

(9)

\[
\begin{bmatrix}
A_1, & \cdots, & A_1, & A_{-1}, & \cdots, & A_{-1}
\end{bmatrix}_{\frac{s+1}{i}} \subseteq A_{-1},
\]

(10)

\[
\begin{bmatrix}
A_1, & \cdots, & A_1, & A_{-1}, & \cdots, & A_{-1}
\end{bmatrix}_{\frac{s+1}{i}} \subseteq A_1.
\]

(11)

**Proof.** Since \( D \in \text{Der}(A), [A_1, \cdots, A_1, A_{-1}, \cdots, A_{-1}] \subseteq A_{2i-2s-1}, 0 \leq i \leq 2s + 1. \)

If \( [A_1, \cdots, A_1, A_{-1}, \cdots, A_{-1}] \neq 0, \) then \( 2i - 2s - 1 = \pm 1, \) that is, \( i = s + 1, \) or \( i = s. \) Therefore, \( [A_1, \cdots, A_1] = [A_{-1}, \cdots, A_{-1}] = 0. \) The result follows. \( \square \)

**Theorem 5.** Let \( A \) be an \( m \)-dimensional \( n \)-Lie algebra with \( n = 2s+1, s \geq 1. \) Then there is an involutive derivation on \( A \) if and only if \( A \) has the decomposition \( A = B + C \) (as direct sum of subspaces), and

\[
\begin{bmatrix}
B_1, & \cdots, & B_1, & C_1, & \cdots, & C
\end{bmatrix}_{\frac{2s+1}{i}} = 0,
\]

\( 0 \leq i \leq 2s + 1, i \neq s, s + 1, \)

(12)

\[
\begin{bmatrix}
B_1, & \cdots, & B_1, & C_1, & \cdots, & C
\end{bmatrix}_{\frac{s+1}{i}} \subseteq C,
\]

(13)

\[
\begin{bmatrix}
B_1, & \cdots, & B_1, & C_1, & \cdots, & C
\end{bmatrix}_{\frac{s+1}{i}} \subseteq B.
\]
Proof. If there is an involutive derivation $D$ on $A$, then, by Theorem 4, $B = A_1$ and $C = A_{-1}$ satisfy (12) and (13).

Conversely, define an endomorphism $D$ of $A$ by $D(x) = x$, $D(y) = -y$, $\forall x, y \in B, y \in C$. Then $D^2 = 1_d$, $B = A_1$ and $C = A_{-1}$. By (12) and (13), $D$ is a derivation. □

Corollary 6. Let $A$ be a $(2s + 1)$-dimensional $(2s + 1)$-Lie algebra with the multiplication $\{e_1, \ldots, e_{2s+1}\} = e_1$, where $\{e_1, \ldots, e_{2s+1}\}$ is a basis of $A$. Then the linear mapping $D : A \longrightarrow A$ defined by $D(e_j) = e_j$, $1 \leq i \leq s + 1$, $D(e_j) = -e_j$, $s + 2 \leq j \leq 2s + 1$ is an involutive derivation on $A$.

Proof. The result follows from a direct computation. □

3. Involutive Derivations on $(n + 1)$-Dimensional $n$-Lie Algebras with $n = 2s + 1$

In this section, we study involutive derivations on $(n + 1)$-dimensional $n$-Lie algebras over $F$. From Theorem 3, we only need to discuss the case of $n = 2s + 1, s \geq 1$.

Lemma 7 (see [5]). Let $A$ be an $(n + 1)$-dimensional non-abelian $n$-Lie algebra over $F$, $n \geq 3$. Then up to isomorphisms $A$ is one and only one of the following possibilities:

\begin{align*}
(b_1) \quad & \{e_2, e_3, \ldots, e_{n+1}\} = e_1, \\
(b_2) \quad & \{e_1, e_2, \ldots, e_n\} = e_1, \\
(c_1) \quad & \{e_2, \ldots, e_{n+1}\} = e_1, \\
(c_2) \quad & \{e_1, e_3, \ldots, e_{n+1}\} = e_2, \\
(d_1) \quad & \{e_1, \ldots, e_{n+1}\} = e_i, \quad 1 \leq i \leq r, \\
(d_2) \quad & \{e_1, e_3, \ldots, e_{n+1}\} = e_2, \\
(d_3) \quad & \{e_1, \ldots, e_{n+1}\} = e_i, \quad 1 \leq i \leq r,
\end{align*}

where $\{e_1, \ldots, e_{n+1}\}$ is a basis of $A$, $3 \leq r \leq n + 1$, and $\bar{e}_i$ means that $e_i$ is omitted.

Now we discuss the case $\dim A^1 = r \geq s + 3$. Let $\{e_1, \ldots, e_{2s+2}\}$ be a basis of $A$ and the multiplication in the basis be as follows:

\[ e^i = (-1)^{2s+i} \sum_{l=1}^{2s+2} \beta_{ij} e_l, \]

where $\beta_{ij} \in F, 1 \leq i \leq 2s + 2$.

Corollary 6. Let $A$ be a $(2s + 1)$-dimensional $(2s + 1)$-Lie algebra over $F$ with a basis $\{e_1, \ldots, e_{2s+3}\}$. Then $A$ is isomorphic to one and only one of the following possibilities:

3. Involutive Derivations on $(n + 2)$-Dimensional $n$-Lie Algebras with $n = 2s + 1$

By Theorem 3, we only need to discuss the case where $n$ is odd. So we suppose that $A$ is a $(2s + 3)$-dimensional $(2s + 1)$-Lie algebra over $F$, $s \geq 1$, and that $E_0 = \text{Diag}(1, \ldots, 1)$ is the $(t \times t)$-unit matrix.

Lemma 9 (see [6]). Let $A$ be a $(2s+3)$-dimensional $(2s+1)$-Lie algebra over $F$ with a basis $\{e_1, \ldots, e_{2s+3}\}$. Then $A$ is isomorphic to one and only one of the following possibilities:

Proof. If $\dim A^1 = r \leq s + 2$, then, by Lemma 7, and a direct computation, the linear mapping $D : A \longrightarrow A$ defined by $D(e_j) = e_j, D(e_j) = -e_j, 1 \leq i \leq s + 2, s + 3 \leq j \leq 2s + 2$, is an involutive derivation on $A$. □
(a) A is an abelian.

(b) \( \dim A^1 = 1: \)

\[
\begin{align*}
(b^1) \quad [e_2, \cdots, e_{2s+2}] &= e_1; \\
(b^2) \quad [e_1, e_2, \cdots, e_{2s+1}] &= e_1.
\end{align*}
\]

(c) \( \dim A^1 = 2: \)

\[
\begin{align*}
(c^1) \quad \begin{bmatrix} e_2, \cdots, e_{2s+2} \\ e_3, \cdots, e_{2s+3} \end{bmatrix} &= e_1; \\
(c^2) \quad \begin{bmatrix} e_2, e_4, \cdots, e_{2s+2} \\ e_1, e_4, \cdots, e_{2s+3} \end{bmatrix} &= e_1; \\
(c^3) \quad \begin{bmatrix} e_2, \cdots, e_{2s+2} \end{bmatrix} &= \alpha e_1 + e_2; \\
(c^4) \quad \begin{bmatrix} e_2, e_4, \cdots, e_{2s+2} \end{bmatrix} &= e_2; \\
(c^5) \quad \begin{bmatrix} e_2, \cdots, e_{2s+2} \end{bmatrix} &= e_1; \\
(c^6) \quad \begin{bmatrix} e_2, e_4, \cdots, e_{2s+2} \end{bmatrix} &= e_2; \\
\end{align*}
\]

\[
\begin{align*}
(e^1) \quad \begin{bmatrix} e_2, \cdots, e_{2s+2} \end{bmatrix} &= \alpha e_1 + e_2, \\
(e^2) \quad [e_1, e_3, \cdots, e_{2s+2}] &= e_2; \\
(e^3) \quad [e_1, e_3, \cdots, e_{2s+2}] &= e_2; \\
\end{align*}
\]

(d) \( \dim A^1 = 3: \)

\[
\begin{align*}
(d^1) \quad [e_2, \cdots, e_{2s+2}] &= e_1; \\
(d^2) \quad [e_2, e_4, \cdots, e_{2s+3}] &= e_2; \\
(d^3) \quad [e_3, \cdots, e_{2s+3}] &= e_3; \\
(d^4) \quad [e_2, e_4, \cdots, e_{2s+3}] &= e_3; \\
(d^5) \quad [e_1, e_3, \cdots, e_{2s+3}] &= e_3; \\
\end{align*}
\]

And \( n \)-Lie algebras corresponding to the case \((d^5)\) with coefficients \( s, t, u \) and \( s', t', u' \) are isomorphic if and only if there exists a nonzero element \( \lambda \in \mathbb{F} \) such that \( s = \lambda^3 s', t = \lambda^2 t', u = \lambda u', s', t', t, u, u' \in \mathbb{F} \).

\[
\begin{align*}
(r) \quad \dim A^1 &= r, \\
\end{align*}
\]

4 \( \leq r < 2s + 3 \), for \( 2 \leq j \leq r, 1 \leq i \leq r, \)

\[
\begin{align*}
(r^1) \quad \begin{bmatrix} e_2, \cdots, e_{2s+2} \end{bmatrix} &= e_1; \\
(r^2) \quad \begin{bmatrix} e_1, e_3, \cdots, e_{2s+2} \end{bmatrix} &= e_2; \\
\end{align*}
\]

Theorem 10. If \( A \) is a \((2s + 3)\)-dimensional \((2s + 1)\)-Lie algebra over \( \mathbb{F} \) with \( \dim A^1 = r < s + 3 \), then there are involutive derivations on \( A \).

Proof. Define linear mappings \( D_j : A \to A, 1 \leq j \leq 6 \) by

\[
\begin{align*}
D_1 (e_i) &= \begin{cases} 
  e_i, & 1 \leq i \leq s + 2, \text{ or } i = 2s + 3, \\
  -e_i, & \text{otherwise};
\end{cases} \\
D_2 (e_i) &= \begin{cases} 
  e_i, & 1 \leq i \leq s + 2, \\
  -e_i, & \text{otherwise};
\end{cases}
\end{align*}
\]
\[ D_3(e_i) = \begin{cases} -e_i, & s + 2 \leq i \leq 2s + 1, \\ e_i, & \text{otherwise}; \end{cases} \]
\[ D_4(e_i) = \begin{cases} e_i, & 1 \leq i \leq s + 1, \text{ or } i = 2s + 2, \\ -e_i, & \text{otherwise}; \end{cases} \]
\[ D_5(e_i) = \begin{cases} e_i, & 1 \leq i \leq s + 1, \text{ or } i = 2s + 3, \\ -e_i, & \text{otherwise}; \end{cases} \]
\[ D_6(e_i) = \begin{cases} e_i, & 1 \leq i \leq s + 3, \\ -e_i, & \text{otherwise}. \end{cases} \] (19)

Next, we discuss the case of \( \dim A^1 = r \geq s + 3 \). Let \( D \) be an endomorphism of \( A \),
\[ De_i = \sum_{j=1}^{2s+3} b_{ij} e_j, \quad b_{ij} \in F, \quad 1 \leq i \leq 2s + 3, \] (20) and \( B = (b_{ij}) \) be the \( (2s + 3) \times (2s + 3) \)-matrix. Then
\[ D(e_1, \ldots, e_{2s+3})^T = B(e_1, \ldots, e_{2s+3})^T \]
\[ = \begin{pmatrix} B_1 & B_0 \\ B_2 & B_3 \end{pmatrix} (e_1, \ldots, e_{2s+3})^T, \] (21) where \( \begin{pmatrix} B_1 & B_0 \\ B_2 & B_3 \end{pmatrix} \) is the block matrix of \( B \). First we discuss \((2s + 3)\)-dimensional \((2s + 1)\)-Lie algebras of the case \((r^1)\) in Lemma 9.

**Lemma 11.** If \( A \) is a \((2s + 3)\)-dimensional \((2s + 1)\)-Lie algebra of the case \((r^1)\) with \( \dim A^1 = r \geq s + 3, s \geq 1 \). Then the linear mapping \( D \) is an involutive derivation on \( A \) if and only if the block matrix \( B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \) satisfies that \( B_0 = O_{r(2s+3-r)} \) (which is the zero \((r \times (2s+3-r))\)-matrix), and
\[ B_1^2 = E_r, \]
\[ B_2^2 = E_{2s+3-r}, \] (22) and
\[ B_2B_1 + B_3B_2 = 0, \] (23)
\[ \sum_{j=2}^{2s+2} b_{jj} = b_{11}, \]
\[ \sum_{j=2}^{2s+3} b_{jj} = b_{ii}, \quad 2 \leq i \leq r, \] (24)
\[ b_{2s+3, i} = (-1)^{i+1} b_{1,i}, \quad 2 \leq i \leq r, \]
\[ b_{ij} = (-1)^{j-i-1} b_{ij}, \quad 2 \leq i, j \leq r, i \neq j. \]

Therefore, matrix \( B \) satisfies (23) and \( B_0 = O_{r(2s+3-r)} \). And \( D^2 = Id \) if and only if
\[ B^2 = \begin{pmatrix} B_1 & B_2 B_1 + B_3 B_2 \\ B_2 B_1 + B_3 B_2 & B_0 + B_4 \end{pmatrix} \]
\[ = \begin{pmatrix} E_r & O \\ O & E_{2s+3-r} \end{pmatrix}. \] (25)

Thanks to \( B_0 = O_{r(2s+3-r)} \), (22) holds. 

**Theorem 12.** Let \( A \) be a \((2s + 3)\)-dimensional \((2s + 1)\)-Lie algebra of the case \((r^1)\) with \( \dim A^1 = r \geq s + 3, s \geq 1 \). If \( r \) is odd, then there are involutive derivations on \( A \).

**Proof.** Let \( r = 2t + 1 \geq s + 3 \). Then \( t \geq 2 \) and \( r \geq 5 \). Suppose \( D \) is an endomorphism of \( A \) and the matrix of \( D \) with respect to the basis \( \{e_1, \ldots, e_{2s+3}\} \) is \( B = (b_{ij}) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \) which satisfies (22) and (23), and \( B_0 = O_{r(2s+3-r)} \). Then
\[ B_1 = \begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2,r-1} & b_{2,r} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3,r-1} & b_{3,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{r-1,1} (-1)^{r-1} b_{r-1,r-1} & \cdots & b_{r-1,r-1} b_{r-1,r} & \cdots & b_{r-1,r-1} b_{r-1,r} \\ b_{r,1} (-1)^{r+1} b_{r,r} & \cdots & (-1)^r b_{r,r} & \cdots & (-1)^{r-1} b_{r,r} \\ b_{r+1,1} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2s+1,1} & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \]
Since $\sum_{j=2}^{2s+3} b_{ji} = b_{11}, \sum_{j=3}^{2s} b_{ji} = b_{ji}, 2 \leq i \leq r$, we have $-b_{11} + 2b_{22} - b_{2s+3,2s+3} = 0, (r-3)b_{22} + \sum_{i=3}^{r+1} b_{ji} = 0, b_{22} = b_{ji}, 3 \leq i \leq r$. Therefore,

$$b_1 = \frac{-1}{r-3} \left( (r-1)k_1 + 2 \sum_{j=2}^{2s+3-r} k_j \right),$$

$$b_{ii} = \frac{-1}{r-3} \sum_{j=i}^{2s+3-r} k_j, \quad 2 \leq i \leq r,$$

$$b_{jj} = k_{2s+3-j+1}, \quad r+1 \leq j \leq 2s+3, \quad k_{2s+3-j+1} \in \mathbb{F}.$$

Suppose

$$B_2^2 = \begin{pmatrix}
    c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1r} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ir} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    c_{j1} & c_{j2} & \cdots & c_{jj} & \cdots & c_{jr} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    c_{r1} & c_{r2} & \cdots & c_{rr} & \cdots & c_{rr}
\end{pmatrix}$$

By (23),

$$c_{ii} = b_{ii}^2, \quad 2 \leq i \leq r,$$

$$c_{ij} = \sum_{l=1}^{j} (-1)^{i+l-1} b_{li}^2 + \sum_{l=i+1}^{r} (-1)^{i+l-1} b_{lj}^2, \quad 2 \leq i \leq r,$$

Thanks to (22), $b_{2s+3,2s+3} = b_{11}^2 = k_1^2 = 1$. Therefore, $(-2s+1)k_1/(2s-1)^2 = ((2s+1)/(2s-1))^2 = 1$, which is a contradiction.

Now we discuss case $(r^2)$. 

**Theorem 13.** Let $A$ be a $(2s+3)$-dimensional $(2s+1)$-Lie algebra of the case $(r^2)$ with $\dim A^1 = r = 2s + 2 (s \geq 1)$, then there does not exist an involutive derivation on $A$.

**Proof.** If $D$ is an involutive derivation on $A$, then, by Lemma 7 and (23),

$$b_{11} = \frac{-(2s+1)k_1}{2s-1},$$

$$b_{2s+3,2s+3} = k_1 = \frac{-k_1}{2s-1}, 2 \leq i \leq r, \quad k_1 \in \mathbb{F}.$$

Therefore, the endomorphisms $D$ of $A$, which are defined by

$$De_1 = e_1,$$

or $De_1 = -e_1,$

$$De_2 = \sum_{k=3}^{r} e_k, \quad 3 \leq i \leq r,$$

$$De_j = (-1)^{i-1} e_2 + \sum_{k=3}^{i-1} (-1)^i e_k + (-1)^{i-1} \sum_{k=i+1}^{r} e_k, \quad 3 \leq i \leq r,$$

are involutive derivations on $A$.

**Theorem 14.** Let $A$ be a $(2s+3)$-dimensional $(2s+1)$-Lie subalgebra of the case $(r^2)$ with $\dim A^1 = r = 2s + 3$. Then there exist involutive derivations on $A$ if and only if $r$ is even.

**Proof.** By Lemma 7, $A = A_1 + Fe_{2s+3}$, where $e_{2s+3} \in Z(A)$, and $A_1$ is a $(2s+2)$-dimensional $(2s+1)$-Lie subalgebra
of $A$ with $\dim A^1 = \dim A_1^1 = r$. Then there exist involutive derivations on $A$ if and only if there exist involutive derivations on $A_1$.

By Theorem 3 in [1], there is a basis $\{e_1, \cdots, e_{2r+2}\}$ of $A_1$ such that
\[ [e_t, \cdots, \hat{e}_j, \cdots, e_{2r+2}] = 0, \quad t < j \leq 2s + 3 - t, \]
\[ [e_t, \cdots, \hat{e}_j, \cdots, e_{2r+2}] = (-1)^t e_{2s+3-j}, \quad 1 \leq i \leq t, \quad \text{or} \quad 2s + 3 - t < i \leq 2s + 2. \]
(32)

If $r$ is even, then $r = 2t \geq 4$. By Theorem 8 and (32), endomorphism $D_1$ of $A_1$ defined by
\[ D_1(e_i) = \begin{cases} e_i, & i = 1, \cdots, s + 1, \\ -e_i, & i = s + 2, \cdots, 2s + 2 \end{cases} \]
is an involutive derivation on $A_1$. Therefore, the endomorphism $D$ of $A$ defined by
\[ D(e_i) = e_i, \quad 1 \leq i \leq s + 1, \]
\[ D(e_j) = -e_j, \quad s + 2 \leq j \leq 2s + 2, \]
\[ D(e_{2s+3}) = \pm e_{2s+3} \]
is involutive derivation on $A$.

If $\dim A^1 = r$ is odd and endomorphism $D$ of $A$ is an involutive derivation on $A$, then $r = 2t + 1 > 4$. Suppose $D(e_i) = \pm \sum_{j=1}^{2s+3} a_{ij} e_j, 1 \leq i \leq 2s + 3$. Then
\[ [D(e_{2s+3}), e_{i_1}, \cdots, e_{i_t}] = D\left[ e_{2s+3}, e_{i_1}, \cdots, e_{i_t} \right] - \sum_{j=1}^{2s+3} e_{2s+3}, \]
\[ \cdots \]
\[ = 0. \]
(35)
We get $D(e_{2s+3}) \notin Z(A) = \mathbb{F} e_{2s+3}$. Since $D^2 = Id, D(e_{2s+3}) = \pm e_{2s+3}$. By (32), $A^1 = \mathbb{F} e_1 + \cdots + \mathbb{F} e_t + \mathbb{F} e_{2s+1} + \cdots + \mathbb{F} e_{2s+3}$, and
\[ (-1)^t D(e_{2s+3-j}) = \sum_{j=1}^{2s+3} e_{2s+3-j}. \]
(36)
where $1 \leq i \leq t$, and $2s + 3 - t < i \leq 2s + 2$. Then $a_{2s+3} = 0$, for $1 \leq i \leq t$, or $2s + 3 - t < i \leq 2s + 2$, and $DA^1 \subseteq A_1$. Then the endomorphism $D_2$ of $A_1$ defined by
\[ D_2(e_i) = D(e_i), \quad 1 \leq i \leq t, \]
\[ D_2(e_j) = D(e_j) - a_{j,2s+3} e_{2s+3} = \sum_{j=1}^{2s+2} e_{j,2s+3} e_{2s+3} = a_{j,2s+3} e_{2s+3}, \]
(37)
is an involutive derivation on the $(2s+2)$-dimensional $(2s+1)$-Lie algebra $A_1$, contradiction (Theorem 8). Therefore, there does not exist involutive derivation on $A$.

5. Structure of 3-Lie Algebras with Involutive Derivations

Let $(L, [ , , ])$ be a Lie algebra over $\mathbb{F}$, and $p$ be an element which is not contained in $L$. Then $L = L + \mathbb{F} p$ is a 3-Lie algebra in the multiplication
\[ [x, y, z] = 0, \]
\[ [p, x, y] = [x, y] \]
(38)
for all $x, y, z \in L$.

And the 3-Lie algebra $(A, [ , , ])$ is called one-dimensional extension of $L$.

Theorem 15. Let $A$ be a 3-Lie algebra, then $A$ is one-dimensional extension of a Lie algebra if and only if there exists an involutive derivation $D$ on $A$ such that $\dim A^1 = 1$, or $\dim A_{-1} = 1$.

Proof. If $A$ is an one-dimensional extension of a Lie algebra $L$, then $A = L + \mathbb{F} p$. Define the endomorphism $D$ of $A$ by $D(p) = -p$ (or $p$), and $D(x) = x$ (or $-x$), $\forall x \in L$. Thanks to (38), $D^2 = Id$, and $D([x, y, z]) = 0 = [Dx, y, z] + [x, Dy, z] + [x, y, Dz]$. $D([p, x, y]) = [p, x, y] = [Dp, x, y] + [p, Dx, y] + [p, x, Dy]$, for all $x, y, z \in L$. Therefore, $D$ is an involutive derivation on $A$, and $\dim A_{-1} = 1$ (or $\dim A_{-1} = 1$).

Conversely, let $D$ be an involutive derivation on a 3-Lie algebra $A$, and $\dim A_{-1} = 1$ (or $\dim A_{-1} = 1$). Let $A_{-1} = \mathbb{F} p$, and $A_1 = L$ (or $A_{-1} = L, A_1 = \mathbb{F} p$), where $p \in A - L$. Thanks to Theorem 3, $L$ is a Lie algebra with the multiplication $[x, y] = [p, x, y]$, for all $x, y \in L$, and $A$ is one-dimensional extension of $L$.

Let $(L, [ , , ])$ and $(L, [ , , ])$ be Lie algebras and $\{x_1, \cdots, x_m\}$ be a basis of $L$. For convenience, denote Lie algebras $(L, [ , ])$ by $L_k, k = 1, 2$, respectively. Suppose $p_1$ and $p_2$ are two distinct elements which are not contained in $L$, and 3-Lie algebras $(B, [ , , ])$ and $(C, [ , , ])$ are one-dimensional extensions of Lie algebras $L_1$ and $L_2$, respectively, where $B = L + \mathbb{F} p_1, C = L + \mathbb{F} p_2$. Then $Der(L_1)$ and $Der(L_2)$ be subalgebras of $gl(L)$.

Definition 16. Let $L_1 = (L, [ , ])$ and $L_2 = (L, [ , ])$ be two Lie algebras, and $p_1, p_2$ be two distinct elements which are not contained in $L$, and $A = L + \mathbb{F} p_1 + \mathbb{F} p_2$. Then 3-algebra $(A, [ , ])$ is called a two-dimensional extension of Lie algebras $L_k, k = 1, 2$, where $[,] : A \wedge A \wedge A \rightarrow A$ defined by
\[ [x, y, p_1] = [x, y]_1, \]
\[ [x, y, p_2] = [x, y]_2, \]
\[ [x, y, z] = 0, \]
(39)
\[ [p_1, p_2, x] = \lambda x p_1 + \mu x p_2, \]
\[ \forall x, y, z \in L, \lambda x, \mu x \in \mathbb{F}. \]
If $A$ is a 3-Lie algebra, then $A$ is called a two-dimensional extension 3-Lie algebra of Lie algebras $L_k, k = 1, 2$. 

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Let \( A = L + W \) be a two-dimensional extension of Lie algebras \( L_k, k = 1, 2 \), where \( W = \mathbb{F} p_1 + \mathbb{F} p_2 \). Define linear mappings \( D_1, D_2 : L \rightarrow \text{End}(L) \) and \( D : L \rightarrow W \) by
\[
D_1 (x) = ad(p_1, x),
D_2 (x) = ad(p_2, x),
D (x) = ad(p_1, p_2) (x),
\]
(40)
for all \( x \in L \).

That is, for all \( y \in L \), \( D_1 (x)(y) = [p_1, x, y] = [x, y]_1 \), \( D_2 (x)(y) = [p_2, x, y] = [x, y]_2 \), \( D(x) = [p_1, p_2, x] \). We have the following result.

**Theorem 17.** Let 3-algebra \( A \) be a two-dimensional extension of Lie algebras \( L_1 \) and \( L_2 \). Then \( A \) is a 3-Lie algebra if and only if linear mappings \( D_1, D_2 \), and \( D \) satisfy that \( D_1 : L_1 \rightarrow \text{Der}(L_1), D_2 : L_2 \rightarrow \text{Der}(L_2) \) are Lie homomorphisms, and
\[
D_1 (x_1) ([x_1, x_2, x_3]) = [D_1 (x_1) (x_1), x_2, x_3] + [x_1, D_1 (x_1) (x_2)]_2 - \lambda_{x_1} [x_1, x_2, x_3] - \mu_{x_1} [x_1, x_2, x_3]_2,
\]
(41)
for all \( x_1, x_2, x_3 \in L, \lambda, \mu \in \mathbb{R} \).

**Proof.** It is a straightforward computation.

**Theorem 18.** Let \( (A, [\cdot, \cdot]) \) be a 3-Lie algebra. Then \( A \) is a two-dimensional extension 3-Lie algebra of Lie algebras if and only if there is an involutive derivation \( T \) on \( A \) such that \( \dim A_{=2} = 2 \) or \( \dim A_{=1} = 2 \).

**Proof.** It is a straightforward computation.
$(L, [, ,])$ are Lie algebras, where $[x, y]_1 = [x, y, p_1]$, $[x, y]_2 = [x, y, p_2], \forall x, y \in L$. Thanks to Theorem 17, the $3$-Lie algebra $A$ is a two-dimensional extension $3$-Lie algebra of Lie algebras $L_1$ and $L_2$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

The first named author was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2018201126).

**References**


