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Research Article

(p,q)-Growth of an Entire GASHE Function and the Coefficient $\beta_n = |a_n|/\Gamma(n+\mu+1)$

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The main purpose of this paper is to extend the work concerning the measures of growth of an entire function solution of the generalized axially symmetric Helmholtz equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + (2\mu/y)(\partial u/\partial y) = 0$, $\mu > 0$, by studying the general measures of growth ((p,q)-order, lower (p,q)-order, (p,q)-type, and lower (p,q)-type) in terms of coefficients $|a_n|/\Gamma(n+\mu+1)$ and the ratios of these successive coefficients.

1. Introduction

The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \quad (1)$$

is called generalized axially symmetric Helmholtz equation (GASHE) and the solutions of (1) are called GASHE functions. The GASHE function u is regular about the origin and has the following Bessel-Gegenbauer series expansion:

$$u(r,\theta) = \Gamma(2\mu)(kr)^{-\mu}\sum_{n=0}^{\infty}\frac{a_n n!}{\Gamma(2\mu+n)} J_{\mu+n}(kr)\, C_n^{\mu}(\cos(\theta)), \quad (2)$$

where $x = r\cos(\theta)$ and $y = r\sin(\theta)$, $J_{\mu+n}$ are Bessel functions of first kind, C_n^{μ} are Gegenbauer polynomials, and $\Gamma(2\mu) = 2^{\mu-1}(\mu-1)!$.

When series (2) converges absolutely and uniformly on the compact subsets of the whole complex plane, then the GASHE function u is said to be entire. For u being entire, it is known [1, page 214] that

$$\limsup_{r\longrightarrow\infty}\left(\frac{|a_n|}{\Gamma(n+\mu+1)}\right)^{1/n} = 0. \quad (3)$$

The concept of order $\rho(f)$ and lower order $\lambda(f)$ of an entire function was introduced by R. P. Boas [2] as follows:

$$\begin{matrix}\rho(f) \\ \lambda(f)\end{matrix} = \lim_{r\longrightarrow+\infty}\begin{matrix}\sup \\ \inf\end{matrix}\frac{\log\log(M(r,f))}{\log(r)} \quad (4)$$

and the concept of type $T(f)$ and lower type $t(f)$ has been introduced to give more precise description of growth of entire functions when they have the same nonzero finite order. An entire function, of order ρ, $0<\rho<+\infty$, is said to be of type $T(f)$ and lower type $t(f)$ if

$$\begin{matrix}T(f) \\ t(f)\end{matrix} = \lim_{r\longrightarrow+\infty}\begin{matrix}\sup \\ \inf\end{matrix}\frac{\log(M(r,f))}{r^{\rho(f)}}, \quad 0<\rho(f)<\infty \quad (5)$$

where $M(r,f) = \max_{0\le\theta\le2\pi}|f(r,\theta)|$.

Gilbert and Howard [3] have studied the order $\rho(u)$ of an entire GASHE function u in terms of the coefficients a_n occurring in the series expansion (2) of u. McCoy [4] studied the rapid growth of entire function solution of Helmholtz equation using the concept of index. Kumar [5, 6] extended and improved this result and studied the growth using the concept of index pair. Khan and Ali [7] studied the generalized order and type of entire GASHE function. Kumar and Singh [8] have studied the lower order and lower type of entire GASHE function in terms of the coefficients in its

Bessel-Gegenbauer series expansion (2) when the order of u is a finite nonzero number. But, for the class of order $\rho(u) = 0$ and $\rho(u) = \infty$, we cannot define a type of u. For this reason, numerous attempts have been made to refine the concept of order and type. Therefore, the (p,q)-order and (p,q)-type of an entire function have been defined [9, 10]. In this paper, we extend the work of Kumar and Singh [8] to this new classification of entire function.

For $(p,q) \neq (1,1)$ and $p \geq q \geq 1$, we define the (p,q)-order and lower (p,q)-order as

$$\begin{matrix} \rho(p,q,u) \\ \lambda(p,q,u) \end{matrix} = \lim_{r\longrightarrow+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]}(M(r,u))}{\log^{[q]}(r)}, \tag{6}$$

where p and q are integers such that $b \leq \rho(p,q,u) \leq \infty$ where $b = 0$ if $p > q$ and $b = 1$ if $p = q$.

The (p,q)-type and lower (p,q)-type are defined as

$$\begin{matrix} T(p,q,u) \\ t(p,q,u) \end{matrix} = \lim_{r\longrightarrow+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]}(M(r,u))}{\log^{[q-1]}(r)^{\rho(p,q,u)}}, \tag{7}$$

and $\log^{[0]}(x) = x$ and $\log^{[m]}(x) = \log^{[m-1]}\log(x)$ for $m \geq 1$ and we use the notations

$$P(\alpha) = P(\alpha,p,q) = \begin{cases} \alpha & \text{if } p > q \\ 1+\alpha & \text{if } p = q = 2 \\ \max(1,\alpha) & \text{if } 3 \leq p = q < \infty \\ \infty & \text{if } p = q = \infty, \end{cases} \tag{8}$$

and

$$M(\alpha) = M(\alpha,p,q) = \begin{cases} \dfrac{1}{e.\alpha} & \text{if } (p,q) = (2,1) \\ \dfrac{(\alpha-1)^{(\alpha-1)}}{\alpha^{\alpha}} & \text{if } (p,q) = (2,2) \\ 1 & \text{if } p \geq 3, \end{cases} \tag{9}$$

$$N(\alpha) = N(\alpha,p,q) = \begin{cases} \dfrac{1}{\alpha} & \text{if } p = 2 \\ 1 & \text{if } p \neq 2. \end{cases}$$

We note that the smallest integer p is 2 ($p \geq 2$) since, for example, the order is given by $\rho = \inf\{\mu > 0 : |f(z)| = O(e^{|z|^{\mu}}), |z| \longrightarrow +\infty\}$.

To prove that $P(\alpha) = P(\beta)$, $M(\alpha) = M(\beta)$, or $N(\alpha) = N(\beta)$ we can prove that $\alpha = \beta$ for the different values of p and q. From [9], we define the relation between (p,q)-order, lower (p,q)-order, the coefficients of u, and the ratios of these successive coefficients as follows.

Theorem 1. *Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,u)$, and then*

$$\rho(p,q,u) = P(L(p,q)) \tag{10}$$

where

$$L(p,q) = \limsup_{n\longrightarrow+\infty} \frac{\log^{[p-1]}(n)}{\log^{[q-1]}(-(1/n)\log(|a_n|))} \tag{11}$$

Theorem 2. *Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,u)$, and then*

$$\rho(p,q,u) = P(L^*(p,q)) \tag{12}$$

where

$$L^*(p,q) = \limsup_{n\longrightarrow\infty} \frac{\log^{[p-1]}(n)}{\log^{[q]}(|a_n/a_{n+1}|)} \tag{13}$$

Theorem 3. *Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,u)$ and $(|a_n/a_{n+1}|)_n$ a nondecreasing function of n for $n > n_0$ and then*

$$\lambda(p,q,u) = P(l(p,q)), \tag{14}$$

where

$$l(p,q) = \liminf_{n\longrightarrow+\infty} \frac{\log^{[p-1]}(n)}{\log^{[q-1]}(-(1/n)\log(|a_n|))}. \tag{15}$$

Theorem 4. *Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,u)$ and $(|a_n/a_{n+1}|)$ a nondecreasing function of n for $n > n_0$ and then*

$$\lambda(p,q,u) = P(l^*(p,q)), \tag{16}$$

where

$$l^*(p,q) = \liminf_{n\longrightarrow\infty} \frac{\log^{[p-1]}(n)}{\log^{[q]}(|a_n/a_{n+1}|)}. \tag{17}$$

From [10], we define the relation between (p,q)-type, lower (p,q)-type, the coefficients of u, and the ratios of these successive coefficients as follows.

Theorem 5. *Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. The function $f(z)$ is of (p,q)-order $\rho(p,q,u)$ and (p,q)-type $T(p,q,u)$ if and only if $T = MV$, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$, and V is defined as*

$$V(p,q,u) = \limsup_{n\longrightarrow\infty} \frac{\log^{[p-2]}(n)}{\log^{[q-2]}(-(1/n)\log(|a_n|))^{\rho-C}} \tag{18}$$

with $C = 1$ if $(p,q) = (2,2)$ and $C = 0$ if $(p,q) \neq (2,2)$.

Theorem 6. *Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order ρ and lower (p,q)-type $t(p,q,u)$ and $(|a_n/a_{n+1}|)_n$ a nondecreasing function of n for $n > n_0$ and then $t = Mv$, where*

$$v(p,q,u) = \liminf_{n\longrightarrow\infty} \frac{\log^{[p-2]}(n)}{\log^{[q-2]}(-(1/n)\log(|a_n|))^{\rho-C}} \tag{19}$$

with $C = 1$ if $(p,q) = (2,2)$ and $C = 0$ if $(p,q) \neq (2,2)$.

Theorem 7. *Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,u)$ and lower (p,q)-type $t(p,q,u)$, and $|a_n/a_{n+1}|$ forms a nondecreasing function of k for $k > k_0$; then*

$$N * R \leq t \leq T \leq N * Q, \tag{20}$$

where

$$Q(p,q) = \limsup_{n\longrightarrow\infty} \frac{\log^{[p-2]}(n)}{\log^{[q-1]}\left(|a_n/a_{n+1}|\right)^{\rho-C}}, \tag{21}$$

and

$$R(p,q) = \liminf_{n\longrightarrow\infty} \frac{\log^{[p-2]}(n)}{\log^{[q-1]}\left(|a_n/a_{n+1}|\right)^{\rho-C}}, \tag{22}$$

with $C = 1$ *if* $(p,q) = (2,2)$ *and* $C = 0$ *if* $(p,q) \neq (2,2)$.

2. Auxiliary Results

Let f and g be two functions defined as

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} \frac{|a_n|}{n!}\left(\frac{k}{2}\right)^n z^n, \\ g(z) &= \sum_{n=0}^{+\infty} \frac{|a_n|}{n!}\left(\frac{k}{2r_*}\right)^n z^n. \end{aligned} \tag{23}$$

According to [3], we know that if u is an entire GASHE function, then f and g are also entire functions of the complex variable z, and

$$\frac{\Gamma(\mu+1/2)}{k_*\pi 2^{2-\mu}} m(r,g) \leq M(r,u) \leq kM(r,f) \tag{24}$$

where $m(r,g) = \max_{n\geq 0}[(|a_n|/n!)(k/2r_*)^n]$ and $M(r,f) = \max_{|z|\leq r}|f(z)|$. In this section, we shall prove some auxiliary results which will be used in the sequel.

Lemma 8. *Let f and g be entire functions of particular form defined above. Then the (p,q)-orders and the (p,q)-types of f and g, respectively, are identical.*

Proof. Let $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, and then, according to Theorem 1, the (p,q)-order of ϕ is given as

$$\rho(p,q,\phi) = \limsup_{n\longrightarrow+\infty} \frac{\log^{[p-1]}(n)}{\log^{[q-1]}\left(-(1/n)\log\left(|a_n|\right)\right)}, \tag{25}$$

and the (p,q)-type is defined in view of Theorem 5 as

$$T(p,q,\phi) = \limsup_{n\longrightarrow+\infty} \frac{\log^{[p-2]}(n)}{\log^{[q-2]}\left((-1/n)\log\left(|a_n|\right)\right)^{\rho(p,q,\phi)}}. \tag{26}$$

In the consequence of [3], we have

$$\begin{aligned} \rho(2,1,f) &= \rho(2,1,g), \\ T(2,1,f) &= T(2,1,g). \end{aligned} \tag{27}$$

Here we consider the case when $q \geq 2$.

We have

$$\begin{aligned} \frac{1}{\rho(2,1,f)} &= \liminf_{n\longrightarrow+\infty} \frac{(-1/n)\log\left((k/2)^n\left(|a_n|/n!\right)\right)}{\log(n)} \\ &= \liminf_{n\longrightarrow+\infty} \frac{\log\left(|a_n|^{-1}\right) + \log(n!) - \log(k/2)^n}{n\log(n)} \\ &= \liminf_{n\longrightarrow+\infty} \left(\frac{\log\left(|a_n|^{-1}\right)}{n\log(n)} + \frac{\log(n!)}{n\log(n)} - \frac{\log(k/2)^n}{n\log(n)}\right) \\ &= \liminf_{n\longrightarrow+\infty} \frac{\log\left(|a_n|^{-1}\right)}{n\log(n)} + 1, \end{aligned} \tag{28}$$

and then $1/\rho(2,1,f) \geq 1$ and $\rho(2,1,f) \leq 1$.

This implies that we will necessarily have $\rho(2,1,f) = 0$ to define $\rho(p,q,f)$. And we have

$$\begin{aligned} &\rho(2,1,f) = 0 \Longrightarrow \\ &\liminf_{n\longrightarrow+\infty} \frac{1}{\rho(2,1,f)} = +\infty \Longrightarrow \\ &\liminf_{n\longrightarrow+\infty} \frac{\log\left(|a_n|^{-1}\right)}{n\log(n)} = +\infty \Longrightarrow \\ &\limsup_{n\longrightarrow+\infty} \frac{n\log(n)}{\log\left(|a_n|^{-1}\right)} = 0 \Longrightarrow \\ &\lim_{n\longrightarrow+\infty} \frac{\log(n!)}{\log\left(|a_n|^{-1}\right)} = 0, \\ &|a_n|^{1/n} \longrightarrow 0. \end{aligned} \tag{29}$$

Hence, for the function f we have

$$\begin{aligned} \frac{1}{\rho(p,q,f)} &= \liminf_{n\longrightarrow+\infty} \frac{\log^{[q-1]}\left((-1/n)\log\left((k/2)^n\left(|a_n|/n!\right)\right)\right)}{\log^{[p-1]}(n)} \\ &= \liminf_{n\longrightarrow+\infty} \frac{\log^{[q-1]}\left(\log\left(|a_n|^{-1}\right)/n + \log(n!)/n - (1/n)\log(k/2)^n\right)}{\log^{[p-1]}(n)} \\ &= \liminf_{n\longrightarrow+\infty} \frac{1}{\log^{[p-1]}(n)} * \log^{[q-1]}\left(\frac{\log\left(|a_n|^{-1}\right)}{n}\left(1 + \frac{\log(n!)}{\log\left(|a_n|^{-1}\right)} - \frac{1}{\log\left(|a_n|^{-1}\right)}\log\left(\frac{k}{2}\right)^n\right)\right) \\ &= \liminf_{n\longrightarrow+\infty} \left(\frac{\log^{[q-1]}\left((-1/n)\log\left(|a_n|\right)\right)}{\log^{[p-1]}(n)} + \frac{\log(1+o(1))}{\log^{[p-1]}(n)}\right) = \liminf_{n\longrightarrow+\infty} \frac{\log^{[q-1]}\left((-1/n)\log\left(|a_n|\right)\right)}{\log^{[p-1]}(n)} \end{aligned} \tag{30}$$

and

$$\begin{aligned}\frac{1}{\rho(p,q,g)} &= \liminf_{n\longrightarrow+\infty}\frac{\log^{[q-1]}\left((-1/n)\log\left((k/2r_*)^n\left(|a_n|/n!\right)\right)\right)}{\log^{[p-1]}(n)}\\ &= \liminf_{n\longrightarrow+\infty}\frac{\log^{[q-1]}\left(\log\left(|a_n|^{-1}\right)/n+\log(n!)/n-(1/n)\log\left(k/2r_*\right)^n\right)}{\log^{[p-1]}(n)}\\ &= \liminf_{n\longrightarrow+\infty}\left(\frac{\log^{[q-1]}\left((-1/n)\log\left(|a_n|\right)\right)}{\log^{[p-1]}(n)}+\frac{\log(1+o(1))}{\log^{[p-1]}(n)}\right) = \liminf_{n\longrightarrow+\infty}\frac{\log^{[q-1]}\left((-1/n)\log\left(|a_n|\right)\right)}{\log^{[p-1]}(n)}\end{aligned} \tag{31}$$

Since f and g have the same (p,q)-order it follows that

$$\rho(p,q,f)=\rho(p,q,g)=\rho. \tag{32}$$

Now we will prove that f and g have the same (p,q)-type for $q=2$.

$$\begin{aligned}\frac{1}{T(p,2,f)} &= \liminf_{n\longrightarrow+\infty}\frac{\left((-1/n)\log\left((k/2)^n\left(|a_n|/n!\right)\right)\right)^\rho}{\log^{[p-2]}(n)}\\ &= \liminf_{n\longrightarrow+\infty}\frac{\left((-1/n)\log\left(|a_n|\right)+\log(n!)/n-(1/n)\log(k/2)^n\right)^\rho}{\log^{[p-2]}(n)}\\ &= \liminf_{n\longrightarrow+\infty}\frac{\left((-1/n)\log\left(|a_n|\right)\right)^\rho}{\log^{[p-2]}(n)}\\ &\quad *\left(1+\frac{\log(n!)}{\log\left(|a_n|\right)^{-1}}-\frac{1}{\log\left(|a_n|\right)^{-1}}\log\left(\frac{k}{2}\right)^n\right)^\rho\\ &= \liminf_{n\longrightarrow+\infty}\frac{\left((-1/n)\log\left(|a_n|\right)\right)^\rho}{\log^{[p-2]}(n)}(1+o(1))^\rho\\ &= \liminf_{n\longrightarrow+\infty}\frac{\left((-1/n)\log\left(|a_n|\right)\right)^\rho}{\log^{[p-2]}(n)}.\end{aligned} \tag{33}$$

In the same way we prove that

$$\frac{1}{T(p,2,g)}=\liminf_{n\longrightarrow+\infty}\frac{\left((-1/n)\log\left(|a_n|\right)\right)^\rho}{\log^{[p-2]}(n)}. \tag{34}$$

Now, for the case $q\geq 3$, we have

$$\begin{aligned}\frac{1}{T(p,q,f)} &= \liminf_{n\longrightarrow+\infty}\frac{\log^{[q-2]}\left((-1/n)\log\left((k/2)^n\left(|a_n|/n!\right)\right)\right)^\rho}{\log^{[p-2]}(n)}\\ &= \liminf_{n\longrightarrow+\infty}\frac{\log^{[q-2]}\left((-1/n)\log\left(|a_n|\right)+\log(n!)/n-(1/n)\log(k/2)^n\right)^\rho}{\log^{[p-2]}(n)}\\ &= \liminf_{n\longrightarrow+\infty}\frac{1}{\log^{[p-2]}(n)}\cdot\log^{[q-2]}\left(\frac{-\log\left(|a_n|\right)}{n}\left(1+\frac{\log(n!)}{\log\left(|a_n|\right)^{-1}}-\frac{\log(k/2)^n}{\log\left(|a_n|\right)^{-1}}\right)\right)^\rho\\ &= \liminf_{n\longrightarrow+\infty}\frac{\log^{[q-2]}\left((-1/n)\log\left(|a_n|\right)(1+o(1))\right)^\rho}{\log^{[p-2]}(n)} = \liminf_{n\longrightarrow+\infty}\frac{\left(\log^{[q-2]}\left((-1/n)\log\left(|a_n|\right)\right)+\log(1+o(1))\right)^\rho}{\log^{[p-2]}(n)}\\ &= \liminf_{n\longrightarrow+\infty}\frac{\log^{[q-2]}\left((-1/n)\log\left(|a_n|\right)\right)^\rho}{\log^{[p-2]}(n)}.\end{aligned} \tag{35}$$

The same is true for

$$\frac{1}{T(p,q,g)}=\liminf_{n\longrightarrow+\infty}\frac{\log^{[q-2]}\left((-1/n)\log\left(|a_n|/n!\right)\right)^\rho}{\log^{[p-2]}(n)} \tag{36}$$

since f and g have identical (p,q)-order and (p,q)-type. □

Lemma 9. *For an entire GASHE function u of (p,q)-order $\rho(p,q,u)$, lower (p,q)-order $\lambda(p,q,u)$, (p,q)-type $T(p,q,u)$, and lower (p,q)-type $t(p,q,u)$. If f and g are entire functions as defined above, then*

$$\rho(p,q,f)=\rho(p,q,u)=\rho(p,q,g), \tag{37}$$

$$\lambda(p,q,f)\leq\lambda(p,q,u)\leq\lambda(p,q,g), \tag{38}$$

$$T(p,q,f)=T(p,q,u)=T(p,q,g), \tag{39}$$

$$t(p,q,f)\leq t(p,q,u)\leq t(p,q,g). \tag{40}$$

Proof. Using (24), we get

$$\frac{\log^{[p]}m(r,g)}{\log^{[q]}(r)}\leq\frac{\log^{[p]}M(r,u)}{\log^{[q]}(r)}\leq\frac{\log^{[p]}M(r,f)}{\log^{[q]}(r)}. \tag{41}$$

From the above relation we obtain

$$\begin{gathered}\rho(p,q,f)\le\rho(p,q,u)\le\rho(p,q,g),\\ \lambda(p,q,f)\le\lambda(p,q,u)\le\lambda(p,q,g),\end{gathered}\tag{42}$$

and since $\rho(p,q,f)=\rho(p,q,g)$ it proves (37) and (38).

Denoting by ρ the common value of (p,q)-order of f, g, and u, we have from (24)

$$\begin{aligned}\frac{\log^{[p-1]}m(r,g)}{\log^{[q-1]}(r)^{\rho}}&\le\frac{\log^{[p-1]}M(r,u)}{\log^{[q-1]}(r)^{\rho}}\\ &\le\frac{\log^{[p-1]}M(r,f)}{\log^{[q-1]}(r)^{\rho}}.\end{aligned}\tag{43}$$

This proves (39) and (40). □

Before we start the next section, let us define $\beta_n=|a_n|/\Gamma(n+\mu+1)$, $\gamma_n=(|a_n|/n!)(k/2)^n$, and $\delta_n=(|a_n|/n!)(k/2r_*)^n$.

It is known, according to [3], that if (β_n/β_{n+1}) is a nondecreasing function of n then (Γ_n/Γ_{n+1}) and (δ_n/δ_{n+1}) also is a nondecreasing function of n.

3. Main Results

Theorem 10. *Let u be an entire GASHE function of (p,q)-order $\rho(p,q,u)$ and (p,q)-type $T(p,q,u)$. If (β_n/β_{n+1}) is a nondecreasing function of n for $n>n_0$, then*

$$\rho(p,q,u)=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))},\tag{44}$$

$$T(p,q,u)=\limsup_{n\to\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(\beta_n))^{\rho}}.\tag{45}$$

Proof. For an entire function $\phi(z)=\sum_0^\infty a_nz^n$ and according to Theorem 1, we have

$$\rho(p,q,\phi)=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(|a_n|))}.\tag{46}$$

We know that if (β_n/β_{n+1}) is a nondecreasing function of n for $n>n_0$, and then also (γ_n/γ_{n+1}) and (δ_n/δ_{n+1}).

Applying (46) to f, we get

$$\begin{aligned}&\rho(p,q,f)\\ &=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log((|a_n|/n!)(k/2)^n))}\\ &=\limsup_{n\to\infty}\log^{[p-1]}(n)\cdot\left(\log^{[q-1]}\left(\frac{-1}{n}\right.\right.\\ &\quad\left.\left.\cdot\log\left(\frac{|a_n|}{n!}\frac{\Gamma(n+\mu+1)}{\Gamma(n+\mu+1)}\left(\frac{k}{2}\right)^n\right)\right)\right)^{-1}\\ &=\limsup_{n\to\infty}\log^{[p-1]}(n)\cdot\left(\log^{[q-1]}\left(\frac{-1}{n}\right.\right.\\ &\quad\left.\left.\cdot\log\left(\beta_n(n+1)^{\mu}\left(\frac{k}{2}\right)^n\right)\right)\right)^{-1}\\ &=\limsup_{n\to\infty}\log^{[p-1]}(n)\cdot\left(\log^{[q-1]}\left(\frac{-1}{n}\right.\right.\\ &\quad\left.\left.\cdot\log(\beta_n)-\frac{\log((n+1)^{\mu}(k/2)^n)}{n}\right)\right)^{-1}\\ &=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))+o(1)}\\ &=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))}.\end{aligned}\tag{47}$$

Similarly, applying (46) to g, we prove

$$\begin{aligned}&\rho(p,q,g)\\ &=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}\left((-1/n)\log\left((|a_n|/n!)(k/2r_*)^n\right)\right)}\\ &=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))}.\end{aligned}\tag{48}$$

Then result (44) is found from the two relations $\rho(p,q,f)$ and $\rho(p,q,g)$ above and relation (37).

Let ρ be the common (p,q)-order of f and g.

The (p,q)-type of ϕ is defined according to Theorem 5 as

$$\begin{aligned}&T(p,q,\phi)\\ &=\limsup_{n\to\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(|a_n|))^{\rho(p,q,\phi)}},\end{aligned}\tag{49}$$

and we can easily prove that

$$T(p,q,f)=\limsup_{n\to\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(\beta_n))^{\rho}},\tag{50}$$

and

$$T(p,q,g)=\limsup_{n\to\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(\beta_n))^{\rho}}.\tag{51}$$

Equation (45) now follows in view of (37) and (39).

Hence the proof is completed. □

Theorem 11. *Let u be an entire GASHE function of (p,q)-order $\rho(p,q,u)$, and $|\beta_n/\beta_{n+1}|$ is a nondecreasing function of n for $n>n_0$. Then*

$$\rho(p,q,u)=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(\beta_n/\beta_{n+1})}.\tag{52}$$

Proof. For an entire function $\phi(z)=\sum_{n=0}^\infty a_nz^n$, according to Theorem 2,

$$\rho(p,q,\phi)=\limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(|a_n/a_{n+1}|)},\tag{53}$$

provided $|a_n/a_{n+1}|$ is a nondecreasing function of n for $n>n_0$.

Applying this equation on f we get

$$\begin{aligned}\rho(p,q,f) &= \limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(\gamma_n/\gamma_{n+1})} = \limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}((|a_n|/|a_{n+1}|)((n+1)!/n!)(2/k))} \\ &= \limsup_{n\to\infty}\log^{[p-1]}(n)\cdot\left(\log^{[q]}\left(\frac{|a_n|\cdot\Gamma(n+\mu+2)\cdot\Gamma(n+\mu+1)}{|a_{n+1}|\cdot\Gamma(n+\mu+1)\cdot\Gamma(n+\mu+2)}(n+1)\frac{2}{k}\right)\right)^{-1} \\ &= \limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}((\beta_n/\beta_{n+1})(\Gamma(n+\mu+1)/\Gamma(n+\mu+2))(n+1)(2/k))}.\end{aligned} \tag{54}$$

Since $(\Gamma(n+\mu+1)/\Gamma(n+\mu+2))(n+1)\approx(n+1)/(n+\mu+1)\approx 1$ as $n\longrightarrow\infty$ then

$$\begin{aligned}\rho(p,q,f) &= \limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}((\beta_n/\beta_{n+1})(2/k))} \\ &= \limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(\beta_n/\beta_{n+1})}.\end{aligned} \tag{55}$$

By the same way, we prove

$$\rho(p,q,g) = \limsup_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(\beta_n/\beta_{n+1})}. \tag{56}$$

Relation (52) now follows on using (37). Hence the proof is completed. □

Theorem 12. *Let u be an entire GASHE function of (p,q)-order $\rho(p,q,u)$, lower (p,q)-order $\lambda(p,q,u)$, and lower (p,q)-type $t(p,q,u)$ and let (β_n/β_{n+1}) be a nondecreasing function of n for $n>n_0$. Then*

$$\lambda(p,q,u) = \liminf_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))}, \tag{57}$$

$$t(p,q,u) = \liminf_{n\to\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(\beta_n))^{\rho}}. \tag{58}$$

Proof. For an entire function $\phi(z)=\sum_0^\infty a_n z^n$, and according to Theorem 3,

$$\lambda(p,q,\phi) = \liminf_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(|a_n|))}. \tag{59}$$

We know that if (β_n/β_{n+1}) form a nondecreasing function of n for $n>n_0$, then, also, (γ_n/γ_{n+1}) and (δ_n/δ_{n+1}).

Applying (59) to f, we get

$$\begin{aligned}&\lambda(p,q,f) \\ &= \liminf_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log((|a_n|/n!)(k/2)^n)))} \\ &= \liminf_{n\to\infty}\log^{[p-1]}(n) * \log^{[q-1]}\Bigg(\frac{-1}{n} \\ &\quad\cdot\log\left(\frac{|a_n|}{\Gamma(n+\mu+1)}\frac{\Gamma(n+\mu+1)}{n!}\left(\frac{k}{2}\right)^n\right)\Bigg)^{-1} \\ &= \liminf_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))+o(1)} \\ &= \liminf_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))}\end{aligned} \tag{60}$$

Similarly, applying (59) to g, we prove that

$$\begin{aligned}&\lambda(p,q,g) \\ &= \liminf_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log((|a_n|/n!)(k/2r_*)^n)))} \\ &= \liminf_{n\to\infty}\frac{\log^{[p-1]}(n)}{\log^{[q-1]}((-1/n)\log(\beta_n))}.\end{aligned} \tag{61}$$

Then result (57) is now followed by (38) and above by the two relations for $\lambda(p,q,f)$ and $\lambda(p,q,g)$.

If $\phi(z)=\sum_0^\infty a_n z^n$ is an entire function of (p,q)-order $\rho(p,q,\phi)$ and lower (p,q)-type $t(p,q,\phi)$, and if $|a_n/a_{n+1}|$ is a nondecreasing function of n for $n>n_0$, then, according to Theorem 6, we have

$$t(p,q,\phi) = \liminf_{n\to\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(|a_n|))^{\rho(p,q,\phi)}}. \tag{62}$$

We can easily prove that

$$t(p,q,f) = \liminf_{n\to\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(\beta_n))^{\rho(p,q,f)}}, \tag{63}$$

and

$$t(p,q,g)=\liminf_{n\longrightarrow\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-2]}((-1/n)\log(|\beta_n|))^{\rho(p,q,g)}}. \quad (64)$$

Equation (58) now follows in view of (37) and (40). Hence the proof is completed. □

Theorem 13. *Let u be an entire GASHE function of lower (p,q)-order $\lambda(p,q,u)$, and let (β_n/β_{n+1}) be a nondecreasing function of n for $n>n_0$. Then*

$$\lambda(p,q,u)=\liminf_{n\longrightarrow\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(\beta_n/\beta_{n+1})}. \quad (65)$$

Proof. Let $\phi(z)=\sum_0^\infty a_n z^n$ be an entire function and $|a_n/a_{n+1}|$ form a nondecreasing function of n for $n>n_0$, and, according to Theorem 4, we have

$$\lambda(p,q,\phi)=\liminf_{n\longrightarrow\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}|a_n/a_{n+1}|}. \quad (66)$$

Applying this on f and g, we can easily prove that

$$\lambda(p,q,f)=\liminf_{n\longrightarrow\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(\beta_n/\beta_{n+1})}, \quad (67)$$

and

$$\lambda(p,q,g)=\liminf_{n\longrightarrow\infty}\frac{\log^{[p-1]}(n)}{\log^{[q]}(\beta_n/\beta_{n+1})}. \quad (68)$$

Thus, we obtain relation (65) by using (38) and the two equalities above. □

Theorem 14. *Let u be an entire GASHE function of (p,q)-order ρ, $(0<\rho<\infty)$, (p,q)-type $T(p,q,u)$, and lower (p,q)-type $t(p,q,u)$. Then*

$$\begin{aligned}\liminf_{n\longrightarrow+\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\beta_n/\beta_{n+1})^{\rho}}&\le t(u,p,q)\\ \le T(u,p,q)\le\limsup_{n\longrightarrow+\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\beta_n/\beta_{n+1})^{\rho}}&.\end{aligned} \quad (69)$$

Proof. If $\phi(z)=\sum_{n=0}^\infty a_n z^n$ is an entire function of (p,q)-type $T(p,q,g)$ and lower (p,q)-type $t(p,q,g)$, then in view of Theorem 7 we have

$$\liminf_{n\longrightarrow\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(|a_n/a_{n+1}|)^{\rho-C}}\le t\le T. \quad (70)$$

$$T\le\limsup_{n\longrightarrow\infty}N(\rho)*\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(|a_n/a_{n+1}|)^{\rho-C}}. \quad (71)$$

Applying inequality (71) to $f(z)=\sum_0^\infty\gamma_n z^n$, we get

$$\begin{aligned}T(f)&\le\limsup_{n\longrightarrow\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\gamma_n/\gamma_{n+1})^{\rho-C}}\\&\le\limsup_{n\longrightarrow\infty}N(\rho)\\&\quad\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}((|a_n|/|a_{n+1}|)((n+1)!/n!)(2/k))^{\rho-C}}\\&\le\limsup_{n\longrightarrow\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]},(\beta_n/\beta_{n+1})^{\rho-C}},\end{aligned} \quad (72)$$

and then

$$\begin{aligned}T(u)&=T(f)\\&\le\limsup_{n\longrightarrow\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\beta_n/\beta_{n+1})^{\rho-C}}\end{aligned} \quad (73)$$

and applying (70) to the function $g(z)=\sum_0^\infty\delta_n z^n$ we get

$$\liminf_{n\longrightarrow+\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\delta_n/\delta_{n+1})^{\rho}}\le t(g)\le t(u), \quad (74)$$

and we can easily prove

$$\begin{aligned}&\liminf_{n\longrightarrow+\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\delta_n/\delta_{n+1})^{\rho}}\\&\quad=\liminf_{n\longrightarrow+\infty}\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\beta_n/\beta_{n+1})^{\rho}},\end{aligned} \quad (75)$$

and, thus,

$$\liminf_{n\longrightarrow+\infty}N(\rho)\cdot\frac{\log^{[p-2]}(n)}{\log^{[q-1]}(\beta_n/\beta_{n+1})^{\rho(p,q,f)}}\le t(u), \quad (76)$$

and thus the proof is completed. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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