Research Article

h -Adic Polynomials and Partial Fraction Decomposition of Proper Rational Functions over $\mathbb{R}$ or $\mathbb{C}$

Kwang Hyun Kim ¹ and Xin Zhang²

¹Departments of Mathematics and Computer Science, Queensborough Community College, 222-05 56th Avenue, Bayside, NY 11364, USA

²Zicklin School of Business, Baruch College, One Bernard Baruch Way, New York, NY 10010, USA

Correspondence should be addressed to Kwang Hyun Kim; kkim@qcc.cuny.edu

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The partial fraction decomposition technique is very useful in many areas including mathematics and engineering. In this paper we present a new and simple method on the partial fraction decomposition of proper rational functions which have completely factored denominators over $\mathbb{R}$ or $\mathbb{C}$. The method is based on a recursive computation of the h -adic polynomial in commutative algebra which is a generalization of the Taylor polynomial. Since its computation requires only simple algebraic operations, it does not require a computer algebra system to be programmed.

1. Introduction

Let F be $\mathbb{R}$ or $\mathbb{C}$ and $R = F[x]$ be a polynomial ring with the coefficients in F . We also assume the rational function to be proper (i.e., the degree of denominator is greater than the degree of numerator) with the denominator factored completely over F . Now, we will show how to apply our method to the following partial fraction decomposition over $F = \mathbb{C}$:

$$\frac{1}{x^3(x-1)^2} = \frac{p(x)}{x^3} + \frac{q(x)}{(x-1)^2}. \quad (1)$$

We multiply through by the least common denominator to clear the fractions:

$$1 = p(x)(x-1)^2 + q(x)x^3, \quad (2)$$

$$p(x) \equiv \frac{1}{(x-1)^2} \pmod{x^3}.$$

Since $R/\langle x^3 \rangle \cong F[[x]]/\langle x^3 \rangle$, we can replace $p(x)$ with a power series in $F[[x]]$. From Proposition 6(a) which we will prove later, we get

$$\frac{1}{(x-1)^2} = 1 + 2x + 3x^2 + \cdots \in F[[x]], \quad (3)$$

$$p(x) \equiv 1 + 2x + 3x^2 \pmod{x^3}.$$

Since we have $\deg(p) < \deg(x^3)$, $p(x)$ is equal to $1 + 2x + 3x^2$, which is the Taylor polynomial of order 2 for $1/(x-1)^2$ at $x = 0$. Similarly, $q(x)$ is the Taylor polynomial of order 1 for $1/x^3$ at $x = 1$ and we obtain

$$q(x) = 1 - 3(x-1). \quad (4)$$

Hence

$$\begin{aligned} \frac{1}{x^3(x-1)^2} &= \frac{1 + 2x + 3x^2}{x^3} + \frac{1 - 3(x-1)}{(x-1)^2} \\ &= \frac{1}{x^3} + \frac{2}{x^2} + \frac{3}{x} + \frac{1}{(x-1)^2} + \frac{-3}{(x-1)}. \end{aligned} \quad (5)$$

In [1], Ma et al. explained several recent approaches of pfd (partial fraction decomposition) and introduced a fast recursive method of pfd over $\mathbb{C}$. Our method is more algebraic and works for pfd of any proper rational functions over $\mathbb{R}$ or $\mathbb{C}$ by extending the concept of Taylor polynomials into h -adic polynomials.

2. h -Adic Polynomials and Main Theorem

In this section we introduce the completion of a commutative ring which is useful in commutative algebra and algebraic geometry. We also define the h -adic polynomial of order n .

Definition 1 (ch. 10 in [3]). For any commutative ring R with unity and a proper ideal I of R , the completion of R with respect to I is

$$\begin{aligned}\widehat{R}^I &:= \varprojlim \left(\frac{R}{I^i} \right) \\ &= \left\{ c = (c_0, c_1, \dots) \in \prod_{i \in \mathbb{N}} \frac{R}{I^i} \mid c_{i-1} = \pi_i(c_i) \right\}\end{aligned}\quad (6)$$

with the quotient maps $\pi_i : R/I^{i+1} \xrightarrow{\pi_i} R/I^i$. We also use $\widehat{R}^h$ if $I = \langle h \rangle$ with $h \in R$.

If $R = F[x]$ with a maximal ideal $\mathfrak{m} = \langle x \rangle$, then $\widehat{R}^x \cong F[[x]]$ is the ring of formal power series. For $R = \mathbb{Z}$ with a prime ideal $\mathfrak{p} = \langle p \rangle$, $\widehat{R}^p$ is $\mathbb{Z}_p$, the ring of p -adic integers. Now we will define the h -adic polynomial of order n , which is crucial for our method.

Lemma 2 (h -adic expansion and h -adic polynomial of order n). Let h be an irreducible polynomial in $R = F[x]$ with $\deg(h) \geq 1$ and $\langle h \rangle$ be the corresponding maximal ideal. Then, consider the following:

- (a) The natural map $R \hookrightarrow \widehat{R}^h$ via $r \mapsto \widehat{r} = (r, r, r, \dots)$ is injective and factors through the localization $R_{\langle h \rangle} = \{f/g \mid g \notin \langle h \rangle\}$ of R at the maximal ideal $\langle h \rangle$ to $\widehat{R}^h$.

$$\begin{array}{ccc} R & \xhookrightarrow{\quad} & \widehat{R}^h \\ & \searrow & \nearrow \\ & R_{\langle h \rangle} & \end{array}\quad (7)$$

- (b) $R \hookrightarrow \widehat{R}^h$ induces

$$\frac{R}{h^n R} \cong \frac{R_{\langle h \rangle}}{h^n R_{\langle h \rangle}} \cong \frac{\widehat{R}^h}{h^n \widehat{R}^h}, \quad \text{where } n \in \mathbb{N}. \quad (8)$$

- (c) For $c \in \widehat{R}^h$, there exists the unique h -adic expansion $P(c, h) \in \widehat{R}^h$ such that

$$c = P(c, h) := \left(a_0, a_0 + a_1 h, \dots, \sum_{i=0}^n a_i \cdot h^i, \dots \right) \in \widehat{R}^h, \quad (9)$$

where $a_i \in R$ is a polynomial with $\deg(a_i) < \deg(h)$.

Definition 3. For $c \in \widehat{R}^h$, $f/g \in R_{\langle h \rangle}$, and $r \in R$, we define the h -adic polynomial of order n for c , f/g , and r , respectively, as

$$\begin{aligned}P_n(c, h) &:= P(c, h)_n = \sum_{i=0}^n a_i \cdot h^i \in R, \\ P_n\left(\frac{f}{g}, h\right) &:= P_n\left(\widehat{\left(\frac{f}{g}\right)}, h\right), \\ P_n(r, h) &:= P_n(\widehat{r}, h).\end{aligned}\quad (10)$$

Proof. (a) is true by the Krull intersection theorem for Noetherian domains (Corollary 10.18 in [3]) and (22.13) in [4]. (b) is true from Proposition 10.15 in [3].

Case (c). For $c = (c_0, c_1, \dots) \in \widehat{R}^h \subset (R/h) \times (R/h^2) \times \dots$, let c'_n be a representative of c_n in R . Now we will use induction on n . For $n = 0$, let a_0 be the remainder of $c'_0 \div h$. Then a_0 is independent of choice of c'_0 , $c_0 \equiv a_0 \pmod{h}$, and $\deg(a_0) < \deg(h)$. Assume that there exists $\sum_{i=0}^n a_i h^i$ with $\deg(a_i) < \deg(h)$ such that $c_n \equiv \sum_{i=0}^n a_i \cdot h^i \pmod{h^{n+1}}$. The condition $\pi_{n+1}(c_{n+1}) = c_n$ implies that

$$c'_{n+1} - \sum_{i=0}^n a_i h^i = e_{n+1} \cdot h^{n+1} \quad (11)$$

with $e_{n+1} \in R$. Let a_{n+1} and q_{n+1} be the remainder and the quotient of e_{n+1}/h . Then a_{n+1} does not depend on choice of c'_{n+1} and we have $\deg(a_{n+1}) < \deg(h)$. Since $e_{n+1} = a_{n+1} + h \cdot q_{n+1}$,

$$\begin{aligned}c'_{n+1} - \sum_{i=0}^n a_i h^i &= (a_{n+1} + h \cdot q_{n+1}) h^{n+1} \\ &= a_{n+1} \cdot h^{n+1} + q_{n+1} \cdot h^{n+2}.\end{aligned}\quad (12)$$

Hence

$$c_{n+1} \equiv \sum_{i=0}^{n+1} a_i h^i \pmod{h^{n+2}}. \quad (13)$$

The uniqueness is clear by the construction. $\square$

If $h = x - a$ with $a \in F$, $(x - a)$ -adic polynomial of order n for f/g is the Taylor polynomial of order n for f/g at $x = a$. Here we present the following main theorem.

Theorem 4 (main theorem). For $f, g, p, q \in R = F[x]$ and $n \in \mathbb{N}$, let h be irreducible in R with $\deg(h) \geq 1$ and $\gcd(h(x), g(x)) \in F$. Assume that

$$\frac{f(x)}{g(x)h^n(x)} = \frac{p(x)}{h^n(x)} + \frac{q(x)}{g(x)} \quad (14)$$

with $\deg(p) < \deg(h^n)$. Then

$$p(x) = P_{n-1}\left(\frac{f}{g}, h\right). \quad (15)$$

Proof. By taking the modulus of h^n and applying Lemma 2(b), we have

$$p(x) \equiv \frac{f(x)}{g(x)} \equiv P_{n-1}\left(\frac{f}{g}, h\right) \pmod{h^n}. \quad (16)$$

$\deg(p(x) - P_{n-1}(f/g, h)) < \deg(h^n)$ implies that

$$p(x) - P_{n-1}\left(\frac{f}{g}, h\right) = 0, \quad (17)$$

from which we complete the proof. $\square$

In the next section we will explain how to compute h -adic polynomials with $\deg(h) \leq 2$.

3. Formulas of h -Adic Polynomials with $\deg(h) \leq 2$ and the Product of h -Polynomials

For $F = \mathbb{R}$ or $\mathbb{C}$, an irreducible polynomial h has at most degree 2. Since we can convert $x - a$ and $(x - a)^2 + c^2$ into t and $t^2 + c^2$, respectively, by replacing $x - a$ with t , we only present recursive formulas of (x) -adic polynomials and $(x^2 + c^2)$ -adic polynomials.

Lemma 5. For $p(\neq 0) \in R = F[x]$ with $\gcd(p, h) = 1$, let $y = p^{-m} \in R_{(h)}$ with $m \in \mathbb{Z}$. Then

$$m \cdot \frac{d}{dx}(p) \cdot y + p \cdot \frac{d}{dx}(y) = 0. \quad (18)$$

Proof. $p^m \cdot y = 1$ implies $m \cdot p^{m-1}(d/dx)(p) \cdot y + p^m \cdot (d/dx)(y) = p^{m-1}(m \cdot (d/dx)(p) \cdot y + p \cdot (d/dx)(y)) = 0$. $\square$

Proposition 6. We present two formulas for h -adic polynomials with $h = x$:

(a) If $f(x) = (x - a)^{-m} = \sum_{i=0}^{\infty} c_i x^i$ with $m \in \mathbb{Z}$ and $a \neq 0$, then $c_0 = (-a)^{-m}$ and

$$c_{k+1} = \frac{(m+k)}{a(k+1)} c_k \quad (k \geq 0). \quad (19)$$

(b) If $f(x) = [(x - a)^2 + b^2]^{-m} = \sum_{i=0}^{\infty} c_i x^i$ with $m \in \mathbb{Z}$ and $a^2 + b^2 \neq 0$, then $c_0 = (a^2 + b^2)^{-m}$, $c_{-1} = 0$, and

$$c_{k+1} = \frac{2a(m+k)}{(a^2 + b^2)(k+1)} c_k - \frac{(2m+k-1)}{(a^2 + b^2)(k+1)} c_{k-1} \quad (k \geq 0). \quad (20)$$

Proof.

Case (a). From Lemma 5 with $p = x - a$ and $y = \sum_{i=0}^{\infty} c_i x^i$, we have

$$m \cdot 1 \cdot \sum_{i=0}^{\infty} c_i x^i + (x - a) \sum_{i=1}^{\infty} i \cdot c_i x^{i-1} = 0. \quad (21)$$

By comparing the coefficients of x^k , one can find

$$[x^k] : m \cdot c_k + k \cdot c_k - a \cdot (k+1) \cdot c_{k+1} = 0. \quad (22)$$

Thus $c_{k+1} = ((m+k)/a(k+1))c_k$ ($k \geq 0$), where $c_0 = (-a)^m$.

Case (b). We can also prove part (b) by repeating the same argument with $p = (x - a)^2 + b^2$. $\square$

To compute the h -adic polynomial of a polynomial, we need formulas of the Taylor shift.

Definition 7 (coefficient vector of a h -adic polynomial of order n). For a given h -adic polynomial $f = \sum_{i=0}^n a_i h^i \in R$ of order

$n \geq 1$, we define the corresponding coefficient vectors $v_f(h)$, $v_f^{\text{odd}}(h)$, and $v_f^{\text{even}}(h)$, respectively, as

$$\begin{aligned} v_f(h, n) &:= (a_0, a_1, \dots, a_n)^t, \\ v_f^{\text{even}}(h, n) &:= (a_0, a_2, \dots, a_{2 \lfloor n/2 \rfloor})^t, \\ v_f^{\text{odd}}(h, n) &:= (a_1, a_3, \dots, a_{2 \lfloor (n-1)/2 \rfloor + 1})^t, \end{aligned} \quad (23)$$

where v^t is the transpose of a row vector v and $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$ (the floor function).

Definition 8 (the Taylor shift matrix). For $s \in F$ with $r, c \in \mathbb{N} \cup \{0\}$, $(r+1) \times (c+1)$ matrix $T_{r,c}(s)$ is

$$\begin{aligned} T_{r,c}(s) &= (t_{i,j}(s))_{0 \leq i \leq r, 0 \leq j \leq c} \\ \text{where } t_{i,j}(s) &= \begin{cases} \binom{j}{i} \cdot s^{j-i}, & i \leq j \\ 0, & i > j \end{cases}, \\ \binom{j}{i} &= \frac{j!}{(j-i)! \cdot i!} \text{ with } j \geq i \geq 0. \end{aligned} \quad (24)$$

From the binomial expansion, $\sum_{j=0}^n a_j (x+s)^j = \sum_{i=0}^n (\sum_{j=i}^n a_j \binom{j}{i} \cdot s^{j-i}) x^i$, we obtain the following Taylor shift formulas for the coefficient vectors.

Proposition 9 (Taylor shift). For a polynomial $f \in R$ with $\deg(f) = d$,

$$v_f(x-s, n) = T_{n,d}(s) \cdot v_f(x, d), \quad (25)$$

where $s \in F$ and $n \in \{0, 1, 2, \dots\}$.

Remark 10. To compute the Taylor shift matrix which is a Toeplitz matrix, we may use recursive formulas given by

$$\begin{aligned} t_{0,0} &= 1, \\ t_{0,j} &= s \cdot t_{0,j-1} \quad \text{for } j \geq 0, \\ t_{i,j} &= \frac{(j-i+1)}{s \cdot i} \cdot t_{i-1,j} \quad \text{for } j \geq i \geq 1. \end{aligned} \quad (26)$$

Proposition 11. We present three formulas for h -adic polynomials with $h = x^2 + c^2$:

(a) (h -Adic shift for $\deg(h) = 2$) for a polynomial $f \in R = F[x]$ with $\deg(f) = d \geq 1$, the corresponding h -adic polynomial of order n , $P_n(f, h) = \sum_{i=0}^n (\beta_i + \alpha_i x) h^i$ with $\alpha_i, \beta_i \in F$, is determined by

$$\begin{aligned} v_{\beta} &= (\beta_0, \beta_1, \dots, \beta_n)^t = T_{n, \lfloor d/2 \rfloor}(-c^2) \cdot v_f^{\text{even}}(x, d), \\ v_{\alpha} &= (\alpha_0, \alpha_1, \dots, \alpha_n)^t = T_{n, \lfloor (d-1)/2 \rfloor}(-c^2) \cdot v_f^{\text{odd}}(x, d). \end{aligned} \quad (27)$$

If $d = 0$, then $f \in F$ and

$$P_n(f, h) = f + \sum_{i=1}^n 0 \cdot h^i. \quad (28)$$

(b) For $a \neq 0$ with $m \in \mathbb{N}$,

$$(x-a)^{-m} = [h - (a^2 + c^2)]^{-m} \cdot (x+a)^m. \quad (29)$$

For $a = 0$, let $q = \lfloor (m+1)/2 \rfloor$ and $\delta = 2q - m$. Then $m = 2q - \delta$ with $\delta \in \{0, 1\}$ and

$$(x)^{-m} = [h - c^2]^{-q} \cdot (x^\delta). \quad (30)$$

(c) For $a \neq 0$, $b \neq 0$, and $m \in \mathbb{N}$,

$$\begin{aligned} [(x-a)^2 + b^2]^{-m} &= [(h-A)^2 + B^2]^{-m} \\ &\cdot [(x+a)^2 + b^2]^m \end{aligned} \quad (31)$$

$$\text{where } A = a^2 - b^2 + c^2, \quad B = 2ab.$$

For $a = 0$, $b \neq 0$, $b^2 \neq c^2$, and $m \in \mathbb{N}$,

$$[x^2 + b^2]^{-m} = [h - (-b^2 + c^2)]^{-m}. \quad (32)$$

Proof. Case (a). You can find a similar argument on p. 591 in [2]. For $d = 0$, it is clear. For $f(x) = \sum_{i=0}^d a_0 x^i \in R$ with $d \geq 1$, we have

$$\begin{aligned} f(x) &= \sum_{i=0}^{\lfloor d/2 \rfloor} a_{2i} x^{2i} + x \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} a_{2i+1} x^{2i} \\ &= \sum_{i=0}^{\lfloor d/2 \rfloor} a_{2i} (h - c^2)^i + x \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} a_{2i+1} (h - c^2)^i \quad (33) \\ &= \sum_{i=0}^{\lfloor d/2 \rfloor} \beta_i h^i + x \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \alpha_i h^i. \end{aligned}$$

Hence α_i and β_i can be computed by the Taylor shift (Proposition 9).

Cases (b) and (c). (b) and (c) are clear by direct computations. $\square$

To get h -adic polynomials of arbitrary rational functions, we just multiply simple h -adic polynomials which are computed using Propositions 6, 9, and 11. In general, the product of two h -adic polynomials may carry a term like

$$\begin{aligned} &[(1+x) + (1-x)(1+x^2)] \\ &\cdot [(1-x) + (1+x)(1+x^2)] = (1-x^2) \\ &+ 2(1+x^2)^2 + (1-x^2)(1+x^2)^2 = 2 \\ &- 1(1+x^2) + 4(1+x^2)^2 - 1(1+x^2)^3. \end{aligned} \quad (34)$$

But if the coefficients of one of two h -adic polynomials are in F , then the product is carryless multiplication and the classic Cauchy product (convolution) formula still holds.

Proposition 12 (h -adic Cauchy product formula). For $\widehat{r}_1, \widehat{r}_2 \in \widehat{R}^h$, let $P_n(\widehat{r}_1, h) = \sum_{i=0}^n a_i h^i$, $P_n(\widehat{r}_2, h) = \sum_{i=0}^n b_i h^i$, and $P_n(\widehat{r}_1 \cdot \widehat{r}_2, h) = \sum_{i=0}^n c_i h^i$ and assume $a_i \in F$. Then

$$c_k = \sum_{i=0}^k a_i \cdot b_{k-i}, \quad (35)$$

where $0 \leq k \leq n$.

4. Example

Example 1. We compute the following partial fraction decomposition:

$$\frac{x^2 + x}{(x-1)^3 (x^2 + 4)^2} = \frac{p(x)}{(x-1)^3} + \frac{q(x)}{(x^2 + 4)^2}. \quad (36)$$

(i) $h = x - 1$: from Theorem 4, we get

$$\begin{aligned} p(x) &= P_2 \left(\frac{x^2 + x}{(x^2 + 2)^2}, h \right) \\ &= P_2 \left(P_2(x^2 + x, h) \cdot P_2 \left(\frac{1}{(x^2 + 4)^2}, h \right), h \right). \end{aligned} \quad (37)$$

From the Taylor shift (Proposition 9), we have $P_2(x^2 + x, h) = 2 + 3h + h^2$.

From Proposition 6(b), we compute

$$\begin{aligned} P_2 \left(\frac{1}{(x^2 + 4)^2}, h \right) &= P_2 \left([(h+1)^2 + 2]^{-2}, h \right) \\ &= \frac{1}{25} - \frac{4}{125}h + \frac{2}{625}h^2. \end{aligned} \quad (38)$$

Hence

$$\begin{aligned} p(x) &= P_2 \left((2 + 3h + h^2) \cdot \left(\frac{1}{25} - \frac{4}{125}h + \frac{2}{625}h^2 \right), h \right) \\ &= \frac{2}{25} + \frac{7}{125}(x-1) - \frac{31}{625}(x-1)^2. \end{aligned} \quad (39)$$

(ii) $h = x^2 + 2^2$: using Proposition 11(b), we get $1/(x-1)^3 = (x+1)^3 \cdot (h-5)^{-3}$. From Theorem 4, we have

$$\begin{aligned} q(x) &= P_1 \left(\frac{x^2 + x}{(x-1)^3}, h \right) \\ &= P_1 \left((h-5)^{-3} \cdot (x^2 + x)(x+1)^3, h \right). \end{aligned} \quad (40)$$

From Proposition 6(a), we compute $P_1((h-5)^{-3}, h) = -1/125 - (3/625)h$.

Using Propositions 6(b) and 12, we get

$$\begin{aligned}(x^2 + x)(x + 1)^3 &= (0 + x + x^2)(1 + 3x + 3x^2 + x^3) \\ &= (0 + 4x^2 + 4x^4) \\ &\quad + (1x + 6x^3 + 1x^5).\end{aligned}\quad (41)$$

From Proposition 11(a), we compute

$$\begin{aligned}\begin{pmatrix} 1 & -4 & 16 \\ 0 & 1 & -8 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} &= \begin{pmatrix} 48 \\ -28 \end{pmatrix}, \\ \begin{pmatrix} 1 & -4 & 16 \\ 0 & 1 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} &= \begin{pmatrix} -7 \\ -2 \end{pmatrix} \Rightarrow\end{aligned}\quad (42)$$

$$P_1((x^2 + x)(x + 1)^3, h) = (48 - 7x) - (28 + 2x)h.$$

Using Proposition 12, we get

$$\begin{aligned}q(x) &= P_1\left(\left(-\frac{1}{125} - \frac{3}{625}h\right)\right. \\ &\quad \cdot ((48 - 7x) - (28 + 2x)h), h) \\ &= \frac{-48 + 7x}{125} + \frac{-4 + 31x}{625}h.\end{aligned}\quad (43)$$

Hence

$$\begin{aligned}\frac{x^2 + x}{(x - 1)^3 (x^2 + 4)^2} &= \frac{2}{25(x - 1)^3} + \frac{7}{125(x - 1)^2} \\ &\quad - \frac{31}{625(x - 1)} + \frac{-48 + 7x}{125(x^2 + 4)^2} \\ &\quad + \frac{-4 + 31x}{625(x^2 + 4)}.\end{aligned}\quad (44)$$

5. Main Algorithm (Algorithm 1)

For a given proper ($\deg(N) < \deg(D)$) fraction

$$\frac{N(x)}{D(x)} = \frac{N(x)}{\prod_{i=1}^{k_1} (x - a_i)^{m_i} \prod_{i=1}^{k_2} [(x - b_i)^2 + c_i^2]^{n_i}}, \quad (45)$$

where $(x - a_i)$ and $[(x - b_i)^2 + c_i^2]$ are relatively prime, respectively, we return $C_1, \dots, C_{k_1}$ ($(x - a_i)$ -adic polynomials of order $(m_i - 1)$) and $D_1, \dots, D_{k_2}$ ($[(x - b_i)^2 + c_i^2]$ -adic polynomial of order $(n_i - 1)$) such that

$$\begin{aligned}\frac{N(x)}{\prod_{i=1}^{k_1} (x - a_i)^{m_i} \prod_{i=1}^{k_2} [(x - b_i)^2 + c_i^2]^{n_i}} \\ = \sum_{i=1}^{k_1} \frac{C_i(x)}{(x - a_i)^{m_i}} + \sum_{i=1}^{k_2} \frac{D_i(x)}{[(x - b_i)^2 + c_i^2]^{n_i}}.\end{aligned}\quad (46)$$

From the computation of Theorems 3.1 and 6.1 in [2], the complexity of the main algorithm (Algorithm 1) with simple algebraic operations is given by $O(k \cdot n^2)$ with $M(n) = O(n^2)$, where $k = k_1 + k_2$, $n = \deg(D(x))$, and $M(n)$ is any upper bound on the number of operations needed to multiply two n -th degree polynomials. Since $F = \mathbb{R}$ or $\mathbb{C}$, if an FFT (fast Fourier transform) is used, we can reduce the complexity to $O(k \cdot (n \log n))$ since $M(n) = O(n \log n)$ with FFT (Ch. 2 in [5]). For $k_2 = 0$, the best known complexity is $O((\log k) \cdot (n \log n))$ in [2]. If k is not big enough, our algorithm is still effective since it does not require computing

$$R(x) = \sum_{i=1}^k \left(\prod_{j=1, j \neq i}^k R_j(x) \right) \quad (47)$$

which is an overhead of the algorithm in [2], where $R_j(x)$ is $(x - a_j)^{m_j}$ or $[(x - b_j)^2 + c_j^2]^{n_j}$.

6. Discussion

The main theorem and Propositions 6, 9, 11, and 12 still hold for any field with characteristic zero. Since an irreducible polynomial h over $\mathbb{R}$ or $\mathbb{C}$ has at most degree 2, our recursive formulas for $\deg(h) \leq 2$ cover all possible proper rational functions over $\mathbb{R}$ or $\mathbb{C}$. If $\deg(h) > 2$, we need to replace Propositions 6, 11(b), and 11(c) with the Extended Euclidean algorithm (Ch. 2 in [5]) which requires an algebra system to be implemented. To compute a h -adic polynomial $P_n(f, n) = \sum_{i=0}^n r_i h^i$ of a polynomial f , we use the repeated division of f by h instead of Propositions 9 and 11(c). Then we obtain $\{r_i\}$ as the remainders.

$$\begin{aligned}f &= q_1 h + r_0 \\ q_1 &= q_2 h + r_1 \\ &\vdots \\ q_n &= q_{n+1} h + r_n.\end{aligned}\quad (48)$$

We can compute it quickly by applying the divide-and-conquer method (Problem 3.5 of [6]).

7. Conclusion

This paper shows that the partial fraction of a proper rational function with denominator factored completely over $\mathbb{R}$ or $\mathbb{C}$ can be found by computing suitable h -adic polynomials with $\deg(h) \leq 2$. We also present recursive formulas (Propositions 6, 9, 11, and 12) to compute h -adic polynomials with $\deg(h) \leq 2$. Since this algorithm only requires simple arithmetic operations, it can be implemented easily without a complex computer algebra system.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Output: C_i for $i = 1, \dots, k_1$ and D_i for $i = 1, \dots, k_2$.

(1) **for** $i \leftarrow 1$ **to** k_1 **do**

(2) Let $h = x - a_i$ and find $\tilde{N}(h) = N(x)$ using the Taylor shift (Proposition 9). Compute C_i :

$$C_i = P_{m_i} \left(N(x) D(x)^{-1} (x - a_i)^{m_i}, x - a_i \right) = P_{m_i} \left(\tilde{N}(h) \prod_{j=1, j \neq i}^{k_1} (h - (a_j - a_i))^{-m_j} \prod_{j=1}^{k_2} \left[(h - (b_j - a_i))^2 + c_j^2 \right]^{-n_j}, h \right)$$

using a binary splitting scheme (Lemma 4.1 on [2]), Proposition 6 and the h -adic Cauchy product formula (Proposition 12).

(3) **end**

(4) **for** $i \leftarrow 1$ **to** k_2 **do**

(5) Let $t = x - b_i$ and $h = t^2 + c_i^2$. Using the Taylor shift (Proposition 9), find $\tilde{N}(t) = N(x)$. Using the h -adic shift (Proposition 11(a)), find

$$\tilde{M}(h) = \tilde{N}(t) \prod_{j=1, a_j \neq b_i}^{k_1} (t + (a_j - b_i))^{m_j} \prod_{j=1, a_j = b_i}^{k_1} t^{\delta_j} \prod_{j=1, b_j \neq b_i}^{k_2} \left[(t + (b_j - b_i))^2 + c_j^2 \right]^{n_j}$$

where $q_j = \lfloor (m_j + 1)/2 \rfloor$ and $\delta_j = 2q_j - m_j$ in Proposition 11(b). Compute D_i :

$$D_i = P_{n_i} \left(N(x) D(x)^{-1} \left[(x - b_i)^2 + c_i^2 \right]^{n_i}, (x - b_i)^2 + c_i^2 \right) = P_{n_i} \left(\tilde{M}(h) \cdot \prod_{j=1, a_j \neq b_i}^{k_1} \left(h - ((a_j - b_i)^2 + c_j^2) \right)^{-m_j} \prod_{j=1, a_j = b_i}^{k_1} (h - c_j^2)^{-q_j} \right. \\ \left. \cdot \prod_{j=1, b_j \neq b_i}^{k_2} \left[\left(h - ((b_j - b_i)^2 - c_j^2 + c_i^2) \right)^2 + (2(b_j - b_i)c_j)^2 \right]^{-n_j} \prod_{j=1, i \neq j, b_j = b_i}^{k_2} \left(h - (-c_j^2 + c_i^2) \right)^{-n_j}, h \right)$$

using a binary splitting scheme (Lemma 4.1 on [2]), Proposition 6 and the h -adic Cauchy product formula (Proposition 12).

(6) **end**

ALGORITHM 1: Main algorithm.

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