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Research Article

Fuzzy Triangular Aggregation Operators

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We present a new class of fuzzy aggregation operators that we call *fuzzy triangular aggregation operators*. To do so, we focus on the situation where the available information cannot be assessed with exact numbers and it is necessary to use another approach to assess uncertain or imprecise information such as fuzzy numbers. We also use the concept of triangular norms (t-norms and t-conorms) as pseudo-arithmetic operations. As a result, we get notably the fuzzy triangular weighted arithmetic (FTWA), the fuzzy triangular ordered weighted arithmetic (FTOWA), the fuzzy generalized triangular weighted arithmetic (FGTWA), the fuzzy generalized triangular ordered weighted arithmetic (FGTOWA), the fuzzy triangular weighted quasi-arithmetic (Quasi-FTWA), and the fuzzy triangular ordered weighted quasi-arithmetic (Quasi-FTOWA) operators. Main properties of these operators are discussed as well as their comparison with other existing ones. The fuzzy triangular aggregation operators not only cover a wide range of useful existing fuzzy aggregation operators but also provide new interesting cases. Finally, an illustrative example is also developed regarding the selection of strategies.

1. Introduction

The available information for the human knowledge is said to be precise (crisp information) or not (fuzzy information). The aggregation operators (resp., fuzzy aggregation operators) are useful models for combining and summarizing a finite set of numerical values (resp., fuzzy values) into a single numerical value (resp., fuzzy value). Such operators are essential to solving multicriteria and group decision-making (MCGDM) problems. Indeed, in MCGDM problems, one generally considers a finite set of alternatives $[m] = \{1, \dots, m\}$, from which we must select the best with respect to a finite set of criteria $[n] = \{1, \dots, n\}$. For each criterion $i \in [n]$ of alternative $j \in [m]$, an expert is consulted to assign a value (or score). This value can be seen as the expression of his/her opinion (or preference) with respect to degree of satisfaction of criterion i by alternative j. For each criterion, each expert expresses his/her evaluation on the same scale that can have a quantitative or a qualitative form. According to Marichal (1998) [1], the MCGDM procedure comprises three main steps: the modeling step, in which we look for appropriate models to represent available information (scores and weights); the aggregation step, in which we try to find an overall score for each alternative on the basis of the partial scores and the weights; the exploitation step, in which we transform the global information about the alternatives either into a complete ranking of the elements in $[m]$, or into a global choice of the best alternatives in $[m]$.

In the real world, we are sometimes confronted with situations where the available information cannot be assessed with exact numbers and it is necessary to use another approach to represent such information with high degree of uncertainty or imprecision. Several methods exist in this case, some of which include fuzzy numbers, interval numbers, and linguistic numbers. Details on these can be found in Merigó (2008) [2]. In this paper, as, for instance, in [S. J. Chen and S. M. Chen (2003) [3]], [Herrera et al. (2000) [4]], [Herrera and Martinez (2000) [5]], [Merigó (2008) [2], chapter 3], and [Merigó and Gil-Lafuente (2010) [6]], we deal with the situation where the scores $\widetilde{x}$ and the weights $\widetilde{w}$ belong to the set of fuzzy subsets of the unit interval $[0, 1]$, denoted by $\widetilde{\mathscr{P}}([0, 1])$. In the sequel, this set is called the set of normal/normalized fuzzy numbers (NFNs).

The aim of this paper is to develop a new class of fuzzy aggregation operators dealing with NFNs, which we call *fuzzy triangular aggregation operators*. For these ones, investigations will be done such as the main properties of operators and an illustrative example. In order to do so, the paper is organized as follows. In Section 2, we give some preliminary notions and present the main properties of fuzzy aggregations operators. In Section 3, we review some fuzzy aggregation operators. Section 4 develops our proposed fuzzy aggregation operators. Section 5 presents an illustrative example regarding the selection of investment strategies. Finally, we summarize the main conclusions in Section 6.

2. Preliminaries

2.1. Fuzzy Numbers. In this section, we briefly describe fuzzy numbers (FNs) and arithmetic operations related to it. The notion of FNs was originally introduced by Zadeh (1975) [7]. Since then, it has been studied and applied by a lot of authors, especially Dubois and Prade (1980) [8] and Kaufmann and Gupta (1985) [9]. Its main advantage is that it can represent, in a more complete way, information coming from human language. That is, it can consider the maximum and minimum values, and the possibility that the internal values may occur.

Definition 1 (Zadeh (1965, 1975) [7, 10]). A fuzzy number $\tilde{x}$ is a fuzzy set (the membership function of $\tilde{x}$ is denoted by $\mu_{\tilde{x}}$) of a universe of discourse (the real line $\mathbb{R}$) which is

(i) convex, that is, $\forall t_1, t_2 \in \mathbb{R}$ and $\alpha \in [0,1]$, $\mu_{\tilde{x}}(\alpha t_1 + (1-\alpha)t_2) \geq \min(\mu_{\tilde{x}}(t_1), \mu_{\tilde{x}}(t_2))$;

(ii) normalized, that is, $\sup\{\mu_{\tilde{x}}(t) \mid t \in \mathbb{R}\} = 1$.

In the literature, we find a wide range of FNs notably the L-R FNs [8]. For example, a trapezoidal FN (TpFN) $\tilde{x}$ can be characterized by a trapezoidal membership function defined by

$$\mu_{\tilde{x}}(t) = \begin{cases} 0 & \text{if } t \leq a, \\ \dfrac{t-a}{b-a} & \text{if } a \leq t \leq b, \\ 1 & \text{if } b \leq t \leq c, \\ \dfrac{t-d}{c-d} & \text{if } c \leq t \leq d, \\ 0 & \text{if } t \geq d, \end{cases} \tag{1}$$

where a, b, c, and d are the real parameters of $\tilde{x}$, with $a \leq b \leq c \leq d$. Note that if $b = c$, the FN is called triangular FN (TFN). If $a = b \leq c = d$, the FN is called an interval number (see [11]). Also, if $a = b = c = d$, the FN is reduced to a crisp value. Notice that we will denote the TpFN $\tilde{x}$ as (a, b, c, d).

Assume that there are two TpFNs, $\tilde{x} = (a_1, b_1, c_1, d_1)$ and $\tilde{y} = (a_2, b_2, c_2, d_2)$, with $0 \leq a_i \leq b_i \leq c_i \leq d_i$, for each $i = 1, 2$. That is, $\tilde{x}, \tilde{y} \in \widetilde{\mathscr{P}}(\mathbb{R}^+)$, the set of fuzzy subsets of $\mathbb{R}^+$ (positive real numbers). The pseudo-arithmetic operations $\{\oplus, \odot\}$ between $\tilde{x}$ and $\tilde{y}$ are defined as follows [Chen and Hwang (1992) [12], Kaufmann and Gupta (1985) [9]]:

$$\begin{aligned} &\tilde{x} \oplus \tilde{y} = \tilde{y} \oplus \tilde{x} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); \\ &\tilde{x} \odot \tilde{y} = \tilde{y} \odot \tilde{x} = (a_1 \cdot a_2, b_1 \cdot b_2, c_1 \cdot c_2, d_1 \cdot d_2); \\ &\tilde{x} \odot \lambda = \lambda \odot \tilde{x} = (\lambda \cdot a_1, \lambda \cdot b_1, \lambda \cdot c_1, \lambda \cdot d_1), \\ &\qquad \text{with } \lambda \geq 0. \end{aligned} \tag{2}$$

In order to rank FNs, a lot of methods exist in the literature. Nevertheless, in this paper, we recommend the use of the method mentioned in [Merigó and Casanovas (2010) [13], Merigó (2008) [2], p. 206], which consists of using the value found in the highest membership degree and, if it is an interval number, the middle value of the interval. In fact, in [2, 13], the authors recommend to rank $\tilde{x} = (a_1, b_1, c_1, d_1)$ and $\tilde{y} = (a_2, b_2, c_2, d_2)$ according to the following procedure:

(i) $\tilde{x} > \tilde{y} \Leftrightarrow (b_1 + c_1)/2 > (b_2 + c_2)/2$;

(ii) $\tilde{x} < \tilde{y} \Leftrightarrow (b_1 + c_1)/2 < (b_2 + c_2)/2$;

(iii) $\tilde{x} = \tilde{y} \Leftrightarrow (b_1 + c_1)/2 = (b_2 + c_2)/2$.

To aggregate FNs, a number of aggregation operators have been developed. Before briefly giving some well-known fuzzy aggregation operators, let us present their main properties.

2.2. Main Properties of Fuzzy Aggregation Operators. We present some properties that are generally considered as relevant for aggregation in a fuzzy environment. In a general way, let us consider fuzzy aggregation operators defined as $\tilde{F} : \widetilde{\mathscr{P}}(\mathbb{R})^n \to \widetilde{\mathscr{P}}(\mathbb{R})$, with $n \geq 2$ a natural number. So, the main mathematical properties of those operators are [Merigó (2008) [2]] as follows:

(**P1**) Boundary conditions:

$$\begin{aligned} &\tilde{F}(0, \ldots, 0) = 0, \\ &\tilde{F}(1, \ldots, 1) = 1. \end{aligned} \tag{3}$$

For this property, we assume that $\tilde{F}$ can be given as $\tilde{F} : \widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$

(**P2**) Monotonicity:

$$\begin{aligned} &\tilde{x}_i \leq \tilde{y}_i \Longrightarrow \\ &\tilde{F}(\tilde{x}_1, \ldots, \tilde{x}_n) \leq \tilde{F}(\tilde{y}_1, \ldots, \tilde{y}_n), \\ &\qquad \forall i \in [n]. \end{aligned} \tag{4}$$

(**P3**) Continuity:

$\tilde{F}$ is continuous if and only if the corresponding operator in the crisp case, $F : \mathbb{R}^n \to \mathbb{R}$, is continuous. In order words, $\tilde{F}$ is continuous if and only if it is a component-wise continuous operator.

(**P4**) Commutativity:

$$\widetilde{F}(\widetilde{x}_1,\ldots,\widetilde{x}_n)=\widetilde{F}(\widetilde{x}_{\pi(1)},\ldots,\widetilde{x}_{\pi(n)}),\quad \forall\pi=(\pi(1),\ldots,\pi(n)) \text{ permutation on } [n]. \tag{5}$$

(**P5**) Idempotency:

$$\widetilde{F}(\widetilde{u},\ldots,\widetilde{u})=\widetilde{u}. \tag{6}$$

(**P6**) Bounded:

$$\min(\widetilde{x}_1,\ldots,\widetilde{x}_n)\le\widetilde{F}(\widetilde{x}_1,\ldots,\widetilde{x}_n)\le\max(\widetilde{x}_1,\ldots,\widetilde{x}_n). \tag{7}$$

Remark 2. Notice that (**P1**) and (**P2**) are the two fundamental properties that characterize general fuzzy aggregation operators. They must not be violated.

3. Some Basic Fuzzy Aggregation Operators

In what follows, a vector $\mathbf{w}=(w_1,\ldots,w_n)$ is called a weighting vector if $w_i\in[0,1]$, $\forall i\in[n]=\{1,\ldots,n\}$, and $\sum_{i=1}^n w_i=1$. Moreover, without loss of generality, it is worth noting that our presentation deals with $\widetilde{\mathbf{x}}=(\widetilde{x}_1,\ldots,\widetilde{x}_n)\in\widetilde{\mathscr{P}}(\mathbb{R}^+)^n$, $\widetilde{x}_i=(a_i,b_i,c_i,d_i)$.

3.1. The Fuzzy Weighted Averaging Operator. Dong and Wong (1987) [14] are the first authors who investigated the WA (weighted averaging) operator when the available information cannot be assessed with exact numbers and it is necessary to use other techniques such as FNs.

Definition 3. The fuzzy weighted averaging operator, denoted by FWA, is the mapping FWA : $\widetilde{\mathscr{P}}(\mathbb{R}^+)^n\to\widetilde{\mathscr{P}}(\mathbb{R}^+)$, which has an associated weighting vector $(w_1,\ldots,w_n)$ such that,

$$\begin{aligned}\mathrm{FWA}(\widetilde{x}_1,\ldots,\widetilde{x}_n)&=\bigoplus_{i\in[n]}(w_i\odot\widetilde{x}_i)=\Bigg(\sum_{i=1}^n w_i\cdot a_i,\sum_{i=1}^n w_i\cdot b_i,\sum_{i=1}^n w_i\cdot c_i,\sum_{i=1}^n w_i\cdot d_i\Bigg)=(\mathrm{WA}(a_1,\ldots,a_n),\\&\quad\mathrm{WA}(b_1,\ldots,b_n),\mathrm{WA}(c_1,\ldots,c_n),\mathrm{WA}(d_1,\ldots,d_n)),\end{aligned} \tag{8}$$

where the operations $\oplus$ and $\odot$ are defined in (2).

Remark 4. It is easy to see that the FWA operator is an extension of the WA operator. Moreover, if $w_i=w=1/n$ for all $i\in[n]$, then the WA is reduced to the AA (arithmetic averaging) operator and the FWA is reduced to the FAA (fuzzy arithmetic averaging) operator, where $\mathrm{AA}(x_1,\ldots,x_n)=\sum_{i=1}^n w\cdot x_i$ and $\mathrm{FAA}(\widetilde{x}_1,\ldots,\widetilde{x}_n)=\bigoplus_{i\in[n]}(w\odot\widetilde{x}_i)$.

Following, for example, Merigó (2008) [2] and Merigó and Casanovas (2010) [13], we can state the following result.

Theorem 5. *The FWA operator satisfies the properties (**P1**), (**P2**), (**P3**), (**P4**) if all $w_i=w$, (**P5**), and (**P6**).*

Proof. Let $(\widetilde{x}_1,\ldots,\widetilde{x}_n),(\widetilde{y}_1,\ldots,\widetilde{y}_n)\in\widetilde{\mathscr{P}}(\mathbb{R}^+)^n$ and let $(w_1,\ldots,w_n)$ be a weighting vector.

(**P1**) Boundary conditions:

$$\begin{aligned}\mathrm{FWA}(0,\ldots,0)&=\bigoplus_{i\in[n]}(w_i\odot 0)=0+\cdots+0=0,\\ \mathrm{FWA}(1,\ldots,1)&=\bigoplus_{i\in[n]}(w_i\odot 1)=\sum_{i=1}^n w_i=1.\end{aligned} \tag{9}$$

(**P2**) Monotonicity: let $\widetilde{x}_i\le\widetilde{y}_i$, $\forall i\in[n]$.

$$\begin{gathered}\widetilde{x}_i\le\widetilde{y}_i\Longrightarrow\\ w_i\odot\widetilde{x}_i\le w_i\odot\widetilde{y}_i\Longrightarrow\\ (w_i\odot\widetilde{x}_i)\oplus(w_j\odot\widetilde{x}_j)\le(w_i\odot\widetilde{y}_i)\oplus(w_j\odot\widetilde{y}_j)\Longrightarrow\\ \bigoplus_{i\in[n]}(w_i\odot\widetilde{x}_i)\le\bigoplus_{i\in[n]}(w_i\odot\widetilde{y}_i)\Longrightarrow\\ \mathrm{FWA}(\widetilde{x}_1,\ldots,\widetilde{x}_n)\le\mathrm{FWA}(\widetilde{y}_1,\ldots,\widetilde{y}_n).\end{gathered} \tag{10}$$

(**P3**) Continuity: since WA is continuous, then FWA is continuous by definition.

(**P4**) Commutativity: let $(\widetilde{x}_{\pi(1)},\ldots,\widetilde{x}_{\pi(n)})$ be a permutation of $(\widetilde{x}_1,\ldots,\widetilde{x}_n)$.

$\mathrm{FWA}(\widetilde{x}_1,\ldots,\widetilde{x}_n)\ne\mathrm{FWA}(\widetilde{x}_{\pi(1)},\ldots,\widetilde{x}_{\pi(n)})$ in general. However, if $w_i=w=1/n$, then

$$\begin{aligned}\mathrm{FWA}(\widetilde{x}_{\pi(1)},\ldots,\widetilde{x}_{\pi(n)})&=\bigoplus_{i\in[n]}(w\odot\widetilde{x}_{\pi(i)})\\&=(w\odot\widetilde{x}_{\pi(1)})\oplus(w\odot\widetilde{x}_{\pi(2)})\oplus\cdots\oplus(w\odot\widetilde{x}_{\pi(n)})\\&=(w\odot\widetilde{x}_{\pi(n)})\oplus(w\odot\widetilde{x}_{\pi(n-1)})\oplus\cdots\oplus(w\odot\widetilde{x}_{\pi(1)})\\&=\cdots=\mathrm{FWA}(\widetilde{x}_1,\ldots,\widetilde{x}_n).\end{aligned} \tag{11}$$

(**P5**) Idempotency: assume that $\widetilde{x}_i=\widetilde{u}=(a,b,c,d)$ for all $i\in[n]$. Then

$$\begin{aligned}\mathrm{FWA}(\widetilde{u},\ldots,\widetilde{u})&=\bigoplus_{i\in[n]}(w_i\odot\widetilde{u})\\&=\left(\sum_{i=1}^n(w_i\cdot a),\sum_{i=1}^n(w_i\cdot b),\sum_{i=1}^n(w_i\cdot c),\sum_{i=1}^n(w_i\cdot d)\right)\\&=(a,b,c,d)=\widetilde{u}.\end{aligned} \tag{12}$$

(**P6**) Bounded: pose $\min(\tilde{x}_1, \ldots, \tilde{x}_n) = \tilde{x}_*$ and $\max(\tilde{x}_1, \ldots, \tilde{x}_n) = \tilde{x}^*$. Then

$$\begin{aligned}
\mathrm{FWA}\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) &= \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_i\right) \geq \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_*\right) \\
&\geq \tilde{x}_* \cdot \left(\sum_{i=1}^{n} w_i\right) \geq \tilde{x}_*, \\
\mathrm{FWA}\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) &= \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_i\right) \leq \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}^*\right) \\
&\leq \tilde{x}^* \cdot \left(\sum_{i=1}^{n} w_i\right) \leq \tilde{x}^*.
\end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}
\min\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) &\leq \mathrm{FWA}\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) \\
&\leq \max\left(\tilde{x}_1, \ldots, \tilde{x}_n\right).
\end{aligned} \tag{14}$$

□

3.2. The Fuzzy Ordered Weighted Averaging Operator. There are various versions of the fuzzy ordered weighted averaging (FOWA) operator. But today the formulation on this operator is attributed to S. J. Chen and S. M. Chen (2003) [3]. As the FWA operator, the reason for using the FOWA operator is that sometimes the available information cannot be assessed with exact numbers and it is necessary to use other techniques such as FNs.

Definition 6. The fuzzy ordered weighted averaging operator, denoted by FOWA, is the mapping FOWA : $\widetilde{\mathscr{P}}(\mathbb{R}^+)^n \to \widetilde{\mathscr{P}}(\mathbb{R}^+)$, which has an associated weighting vector $(w_1, \ldots, w_n)$, such that

$$\mathrm{FOWA}\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) = \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_{(i)}\right), \tag{15}$$

where the operations $\oplus$ and $\odot$ are defined in (2); $(\cdot)$ is a permutation on $[n]$ such that $\tilde{x}_{(1)} \leq \cdots \leq \tilde{x}_{(n)}$.

Remark 7. It is easy to see that the FOWA operator is an extension of the OWA operator. Moreover

(i) If $\mathbf{w} = (1, 0, \ldots, 0)$, then FOWA $= \tilde{x}_{(1)}$ (fuzzy mini mum)

(ii) If $\mathbf{w} = (0, \ldots, 0, 1)$, then FOWA $= \tilde{x}_{(n)}$ (fuzzy maxi mum)

(iii) If $w_i = w = 1/n$, for all i, then FOWA = FAA (fuzzy arithmetic averaging)

(iv) FOWA = FWA when the ranking of $\{\tilde{x}_i,\ i \in [n]\}$ matches with the ranking of $\{\tilde{x}_{(i)},\ i \in [n]\}$.

There are several other particular cases with respect to the analysis of the weighting vector $\mathbf{w}$. For more details see [Merigó (2008) [2], chapter 4].

In the same way, we can state the following result [Merigó (2008) [2], Merigó and Casanovas (2010) [13]].

Theorem 8. *The FOWA operator satisfies the properties (**P1**), (**P2**), (**P3**), (**P4**), (**P5**), and (**P6**).*

Proof. (**P1**), (**P2**), (**P3**), (**P5**), and (**P6**) are similar to Theorem 5.

(**P4**) Commutativity: let $(\tilde{x}_{\pi(1)}, \ldots, \tilde{x}_{\pi(n)})$ be a permutation of $(\tilde{x}_1, \ldots, \tilde{x}_n)$.

$$\begin{aligned}
&\mathrm{FOWA}\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) = \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_{(i)}\right), \\
&\qquad \tilde{x}_{(i)}\ i\text{th largest of } \left\{\tilde{x}_i,\ i \in [n]\right\}, \\
&\mathrm{FOWA}\left(\tilde{x}_{\pi(1)}, \ldots, \tilde{x}_{\pi(n)}\right) = \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}'_{\pi(i)}\right), \\
&\qquad \tilde{x}'_{\pi(i)}\ i\text{th largest of } \left\{\tilde{x}_{\pi(i)},\ i \in [n]\right\}.
\end{aligned} \tag{16}$$

Since $(\tilde{x}_{\pi(1)}, \ldots, \tilde{x}_{\pi(n)})$ is a permutation of $(\tilde{x}_1, \ldots, \tilde{x}_n)$, we have $\tilde{x}_{(i)} = \tilde{x}'_{\pi(i)}$, $\forall i \in [n]$. And then

$$\mathrm{FOWA}\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) = \mathrm{FOWA}\left(\tilde{x}_{\pi(1)}, \ldots, \tilde{x}_{\pi(n)}\right). \tag{17}$$

□

3.3. Quasi-Arithmetic Means, Generalized Means, and Fuzzy Aggregation Operators. The two well-known and most important fuzzy aggregation operators presented above have been generalized using the concepts of quasi-arithmetic means and generalized means. The fuzzy weighted quasi-arithmetic averaging (Quasi-FWA) is an aggregation operator that generalizes the FWA operator by using quasi-arithmetic means while the fuzzy ordered weighted quasi-arithmetic averaging (Quasi-FOWA) is an aggregation operator that generalizes the FOWA operator by using quasi-arithmetic means.

Definition 9 (Wang and Luo (2009) [15]). The fuzzy weighted quasi-arithmetic averaging operator, denoted by Quasi-FWA, is the mapping Quasi-FWA : $\widetilde{\mathscr{P}}(\mathbb{R}^+)^n \to \widetilde{\mathscr{P}}(\mathbb{R}^+)$, which has an associated weighting vector $(w_1, \ldots, w_n)$, such that

$$\text{Quasi-FWA}\left(\tilde{x}_1, \ldots, \tilde{x}_n\right) = f^{-1}\left(\bigoplus_{i\in[n]} \left[w_i \odot f\left(\tilde{x}_i\right)\right]\right), \tag{18}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly continuous monotone function and the arithmetic operations $\oplus$ and $\odot$ are defined in (2). f is called a generating function of the Quasi-FWA operator.

Theorem 10. *The Quasi-FWA operator satisfies the properties (**P1**), (**P2**), (**P3**), (**P4**) if all $w_i = w$, (**P5**), and (**P6**).*

Definition 11 (Merigó and Casanovas (2010) [13]). The fuzzy ordered weighted quasi-arithmetic averaging operator, denoted by Quasi-FOWA, is the mapping Quasi-FOWA: $\widetilde{\mathscr{P}}(\mathbb{R}^+)^n \to \widetilde{\mathscr{P}}(\mathbb{R}^+)$, which has an associated weighting vector $(w_1, \ldots, w_n)$, such that

$$\begin{aligned}&\text{Quasi-FOWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right)\\&=f^{-1}\left(\bigoplus_{i\in[n]}\left[w_i\odot f\left(\tilde{x}_{(i)}\right)\right]\right),\end{aligned}\tag{19}$$

where $f:\mathbb{R}^+\to\mathbb{R}^+$ is a strictly continuous monotone function, the arithmetic operations $\oplus$ and $\odot$ are defined in (2), and $(\cdot)$ is a permutation on $[n]$ such that $\tilde{x}_{(1)}\le\cdots\le\tilde{x}_{(n)}$. f is called a generating function of the Quasi-FOWA operator.

Theorem 12. *The Quasi-FOWA operator satisfies the properties (**P1**), (**P2**), (**P3**), (**P4**), (**P5**), and (**P6**).*

Remark 13. In (18) and (19), note the following:

(i) For any TpFN $\tilde{x}=(a,b,c,d)$, $f(\tilde{x}):=(f(a),f(b),f(c),f(d))$ is also a TpFN.

(ii) If $f(x)=x^\lambda$, with $\lambda\in\mathbb{R}^*$, the Quasi-FWA becomes the fuzzy generalized weighted averaging (FGWA) and the Quasi-FOWA becomes the fuzzy generalized ordered weighted averaging (FGOWA) operators. In particular, when $f(x)=x$ or $\lambda=1$, we obtain the FWA and the FOWA operators, respectively.

See Merigó (2008) [2] for further details on fuzzy aggregation operators presented above.

4. Fuzzy Triangular Aggregation Operators

In this section, we present a new method to construct fuzzy aggregation operators based on triangular norms (t-norms and t-conorms). This new class of aggregation operators is called *fuzzy triangular aggregation operators*. We deal with the situation where the values to be aggregated $\tilde{x}=(a,b,c,d)$ are expressed as NFNs.

4.1. Operational Laws on NFNs. The operational laws that we use are based on triangular norms (t-norms and t-conorms). t-norms and t-conorms were introduced in the present form in [16]. They are appropriate extensions of logical connectives AND and OR in the case when the valuation set is the unit interval $[0,1]$ rather than $\{0,1\}$. So, t-norms (resp., t-conorms) provide an important class of aggregation operators widely used to define the conjunction (resp., disjunction) in fuzzy set theory but also they are often used to implement the multivalued logical AND (resp., OR) operation.

Definition 14 (Dubois and Prade (1985) [17]). (i) A t-norm, denoted by $\odot$, is a binary aggregation operator such that, $\forall x,y,z\in[0,1]$,

(a) $x\odot y=y\odot x$

(b) $x\odot(y\odot z)=(x\odot y)\odot z$

(c) $y\le z\Rightarrow x\odot y\le x\odot z$

(d) $x\odot 1=x$ (1 is the neutral element).

(ii) A t-conorm, denoted by $\oplus$, is a binary aggregation operator which has the same properties as t-norm, unless the neutral element is 0; that is, $x\oplus 0=x$.

Example 15. Several important nonparametrized and parametrized families of t-norms and t-conorms exist but the three prototypical and most used examples are given as follows [17–21]:

(1) $\odot_Z(x,y):=\min(x,y)$; $\oplus_Z(x,y):=\max(x,y)$; (Zadeh (1965) [10])

(2) $\odot_P(x,y):=x\cdot y$; $\oplus_P(x,y):=x+y-x\cdot y$, deduced to probabilistic theory

(3) $\odot_L(x,y):=\max(0,x+y-1)$; $\oplus_L(x,y):=\min(1,x+y)$. (Giles (1976) [22]).

Motivated by the arithmetic operations investigated in [S.-M. Chen and J.-H. Chen (2009) [23], Kaufmann and Gupta (1985) [9], Xu (2007) [24], Zhao et al. (2010) [25]], we introduce below three new operational laws on NFNs, which will be very useful in the sequel of this paper.

Definition 16. Let $\tilde{w}=(\alpha,\beta,\gamma,\delta)$ and $\tilde{x}=(a,b,c,d)$ be two trapezoidal NFNs (TpNFNs). Let $\oplus$ be a t-conorm and $\odot$ be a t-norm. Then

$$\begin{aligned}\tilde{w}\oplus\tilde{x}&:=(\alpha\oplus a,\beta\oplus b,\gamma\oplus c,\delta\oplus d);\\\tilde{w}\odot\tilde{x}&:=(\alpha\odot a,\beta\odot b,\gamma\odot c,\delta\odot d);\\\tilde{x}^{\tilde{w}}&:=\left(a^\alpha,b^\beta,c^\gamma,d^\delta\right).\end{aligned}\tag{20}$$

Definition 17. Let $\oplus$ be a t-conorm. A vector $\mathbf{w}=(w_1,\ldots,w_n)$, with $w_i\in[0,1]$, is called a weighting vector associated with $\oplus$ if and only if the equality (H) is verified:

$$(H):\ w_1\oplus\cdots\oplus w_n=1.\tag{21}$$

Remark 18. It is important to stress that

(i) If $\oplus=\oplus_Z$, then $(H)\Leftrightarrow\bigvee_{i\in[n]}w_i=1$;

(ii) If $\oplus=\oplus_P$, then $(H)\Leftrightarrow\prod_{i=1}^n(1-w_i)=0$;

(iii) If $\oplus=\oplus_L$, then $(H)\Leftrightarrow\sum_{i=1}^n w_i\ge 1$.

Note that, in this paper, we do not enter in the problem of using FNs in the weighting vector. Nevertheless, if the weighting vector is presented with NFNs, then, instead of converting these fuzzy weights into representative exact numbers by using a method for doing so as recommended by some authors, our recommendation is, for example, to use the fuzzy weighting vector model defined as follows: let $\oplus$ be a t-conorm. A fuzzy vector $\tilde{\mathbf{w}}=(\tilde{w}_1,\ldots,\tilde{w}_n)$ with $\tilde{w}_i=(\alpha_i,\beta_i,\gamma_i,\delta_i)\in\widetilde{\mathscr{P}}([0,1])$ is called a fuzzy weighting vector associated with $\oplus$ if and only if the equality $(\widetilde{H})$ is verified:

$$\begin{aligned}\left(\widetilde{H}\right):\ \bigoplus_{i\in[n]}\tilde{w}_i&=\left(\bigoplus_{i\in[n]}\alpha_i,\bigoplus_{i\in[n]}\beta_i,\bigoplus_{i\in[n]}\gamma_i,\bigoplus_{i\in[n]}\delta_i\right)\\&=(1,1,1,1)=1.\end{aligned}\tag{22}$$

Contrary to other approaches, our method is more informative since it uses all the information and therefore leads to complete results. However, the main disadvantage of this approach is that it is not easy to use in practice.

Before closing this section, let us also recall the following result.

Theorem 19 (Grabisch et al. (2009) [21], section 3.9). *A continuous t-norm $\odot$ is restrictedly distributive over a continuous t-conorm $\oplus$ if and only if either $\oplus = \oplus_Z$ (and $\odot$ is arbitrary), or there exists a value $a \in [0,1]$, a strict t-norm $\odot_*$, and a nilpotent t-conorm $\oplus_*$, such that the additive generator g of $\oplus_*$ satisfying $g(1) = 1$ is also a multiplicative generator of $\odot_*$, and $\odot$ can be written as an ordinal sum as follows:*

$$\odot = (\langle 0, a, \odot_1 \rangle, \langle a, 1, \odot_* \rangle), \tag{23}$$

where $\odot_1$ is an arbitrary continuous t-norm and $\oplus = (\langle a, 1, \oplus_* \rangle)$.

4.2. The Fuzzy Triangular Weighted Arithmetic Operator. The fuzzy triangular weighted arithmetic (FTWA) operator is an extension of the TWA operator (see Remark 21 below) for situations where the available information is uncertain and it is necessary to use other techniques such as NFNs. Its main advantage is that it represents the information in a more complete way because it considers the maximum and the minimum result that may occur in the uncertain environment and the possibility that the internal values will occur. It can be defined as follows.

Definition 20. The fuzzy triangular weighted arithmetic operator, denoted by FTWA, is the mapping FTWA: $\widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$, such that

$$\begin{aligned} \mathrm{FTWA}(\tilde{x}_1, \ldots, \tilde{x}_n) &= \bigoplus_{i\in[n]} (w_i \odot \tilde{x}_i) \\ &= \Big(\bigoplus_{i\in[n]} (w_i \odot a_i), \bigoplus_{i\in[n]} (w_i \odot b_i), \bigoplus_{i\in[n]} (w_i \odot c_i), \\ &\quad \bigoplus_{i\in[n]} (w_i \odot d_i) \Big) \\ &= (\mathrm{TWA}(a_1, \ldots, a_n), \mathrm{TWA}(b_1, \ldots, b_n), \\ &\quad \mathrm{TWA}(c_1, \ldots, c_n), \mathrm{TWA}(d_1, \ldots, d_n)), \end{aligned} \tag{24}$$

where $\oplus$ is a t-conorm, $\odot$ a t-norm, and $(w_1, \ldots, w_n)$ a weighting vector associated with $\oplus$.

Remark 21. The fuzzy triangular weighted arithmetic (FTWA) operator provides a parameterized family of aggregation operators that include the triangular weighted arithmetic (TWA) and the fuzzy weighted averaging (FWA) operators among others.

Proof. When $\tilde{x} = x$, we get FTWA = TWA. On the other hand, let $\oplus \in \{\oplus_Z, \oplus_P, \oplus_L\}$ and $\odot \in \{\odot_Z, \odot_P, \odot_L\}$. We have the following:

(1) $\oplus = \oplus_Z$ and $\odot = \odot_Z$:

$$\begin{aligned} \mathrm{FTWA}(\tilde{x}_1, \ldots, \tilde{x}_n) &= \Big(\bigvee_{i\in[n]} (w_i \wedge a_i), \bigvee_{i\in[n]} (w_i \wedge b_i), \\ &\quad \bigvee_{i\in[n]} (w_i \wedge c_i), \bigvee_{i\in[n]} (w_i \wedge d_i) \Big) \\ &= (W\max(a_1, \ldots, a_n), W\max(b_1, \ldots, b_n), \\ &\quad W\max(c_1, \ldots, c_n), W\max(d_1, \ldots, d_n)) \\ &=: FW\max(\tilde{x}_1, \ldots, \tilde{x}_n). \end{aligned} \tag{25}$$

(2) $\oplus = \oplus_Z$ and $\odot = \odot_P$:

$$\begin{aligned} \mathrm{FTWA}(\tilde{x}_1, \ldots, \tilde{x}_n) = \Big(&\bigvee_{i\in[n]} (w_i \cdot a_i), \bigvee_{i\in[n]} (w_i \cdot b_i), \\ &\bigvee_{i\in[n]} (w_i \cdot c_i), \bigvee_{i\in[n]} (w_i \cdot d_i) \Big). \end{aligned} \tag{26}$$

(3) $\oplus = \oplus_Z$ and $\odot = \odot_L$:

$$\begin{aligned} \mathrm{FTWA}(\tilde{x}_1, \ldots, \tilde{x}_n) = \Big(&\bigvee_{i\in[n]} [0 \vee (w_i + a_i - 1)], \\ &\bigvee_{i\in[n]} [0 \vee (w_i + b_i - 1)], \ldots, \\ &\bigvee_{i\in[n]} [0 \vee (w_i + d_i - 1)] \Big). \end{aligned} \tag{27}$$

(4) $\oplus = \oplus_P$ and $\odot = \odot_Z$:

$$\begin{aligned} \mathrm{FTWA}(\tilde{x}_1, \ldots, \tilde{x}_n) = \Big(&1 - \prod_{i=1}^{n} (1 - w_i \wedge a_i), 1 \\ &- \prod_{i=1}^{n} (1 - w_i \wedge b_i), \ldots, 1 - \prod_{i=1}^{n} (1 - w_i \wedge d_i) \Big). \end{aligned} \tag{28}$$

(5) $\oplus = \oplus_P$ and $\odot = \odot_P$:

$$\begin{aligned} \mathrm{FTWA}(\tilde{x}_1, \ldots, \tilde{x}_n) = \Big(&1 - \prod_{i=1}^{n} (1 - w_i \cdot a_i), 1 \\ &- \prod_{i=1}^{n} (1 - w_i \cdot b_i), \ldots, 1 - \prod_{i=1}^{n} (1 - w_i \cdot d_i) \Big). \end{aligned} \tag{29}$$

(6) $\oplus = \oplus_P$ and $\odot = \odot_L$:

$$\begin{aligned} \mathrm{FTWA}(\tilde{x}_1, \ldots, \tilde{x}_n) = \Big(&1 \\ &- \prod_{i=1}^{n} [1 - (0 \vee (w_i + a_i - 1))], \ldots, 1 \\ &- \prod_{i=1}^{n} [1 - (0 \vee (w_i + d_i - 1))] \Big). \end{aligned} \tag{30}$$

(7) $\oplus = \oplus_L$ and $\odot = \odot_Z$:

$$\mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) = \left(\sum_{i=1}^{n} w_i \wedge a_i, \sum_{i=1}^{n} w_i \wedge b_i, \sum_{i=1}^{n} w_i \wedge c_i, \sum_{i=1}^{n} w_i \wedge d_i\right). \tag{31}$$

(8) $\oplus = \oplus_L$ and $\odot = \odot_P$:

$$\mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) = \left(\sum_{i=1}^{n} w_i \cdot a_i, \sum_{i=1}^{n} w_i \cdot b_i, \sum_{i=1}^{n} w_i \cdot c_i, \sum_{i=1}^{n} w_i \cdot d_i\right) = \left(\mathrm{WA}\left(a_1,\ldots,a_n\right), \mathrm{WA}\left(b_1,\ldots,b_n\right), \mathrm{WA}\left(c_1,\ldots,c_n\right), \mathrm{WA}\left(d_1,\ldots,d_n\right)\right) = \mathrm{FWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right). \tag{32}$$

(9) $\oplus = \oplus_L$ and $\odot = \odot_L$:

$$\mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) = \left(\min\left[1, \sum_{i=1}^{n}\left(0 \vee \left(w_i + a_i - 1\right)\right)\right], \ldots, \min\left[1, \sum_{i=1}^{n}\left(0 \vee \left(w_i + d_i - 1\right)\right)\right]\right). \tag{33}$$

□

Regarding its properties, we have the following result.

Theorem 22. *The FTWA operator satisfies the properties (**P1**), (**P2**), (**P3**), (**P4**) if all $w_i = w$, (**P5**) if and only if $\odot$ is restrictively distributive over $\oplus$, and (**P6**) if and only if $\odot$ is restrictively distributive over $\oplus$.*

Proof. Let $(\tilde{x}_1,\ldots,\tilde{x}_n)$, $(\tilde{y}_1,\ldots,\tilde{y}_n)$ be two NFNs and let $(w_1,\ldots,w_n)$ be a weighting vector (Definition 17).

(**P1**) Boundary conditions:

$$\begin{aligned}\mathrm{FTWA}(0,\ldots,0) &= \bigoplus_{i\in[n]}\left(w_i \odot 0\right) = 0,\\ \mathrm{FTWA}(1,\ldots,1) &= \bigoplus_{i\in[n]}\left(w_i \odot 1\right) = 1.\end{aligned} \tag{34}$$

(**P2**) Monotonicity: assume that $\tilde{x}_i \leq \tilde{y}_i$, $\forall i \in [n]$.

$$\begin{gathered}\tilde{x}_i \leq \tilde{y}_i \Longrightarrow\\ w_i \odot \tilde{x}_i \leq w_i \odot \tilde{y}_i \Longrightarrow\\ \left(w_i \odot \tilde{x}_i\right) \oplus \left(w_j \odot \tilde{x}_j\right) \leq \left(w_i \odot \tilde{y}_i\right) \oplus \left(w_j \odot \tilde{y}_j\right) \Longrightarrow\\ \bigoplus_{i\in[n]}\left(w_i \odot \tilde{x}_i\right) \leq \bigoplus_{i\in[n]}\left(w_i \odot \tilde{y}_i\right) \Longrightarrow\\ \mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) \leq \mathrm{FTWA}\left(\tilde{y}_1,\ldots,\tilde{y}_n\right).\end{gathered} \tag{35}$$

(**P3**) Continuity: FTWA is continuous because it is component-wise continuous operators.

(**P4**) Commutativity: let $(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)})$ be a permutation of $(\tilde{x}_1,\ldots,\tilde{x}_n)$.

$\mathrm{FTWA}(\tilde{x}_1,\ldots,\tilde{x}_n) \neq \mathrm{FTWA}(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)})$ in general. However, if $w_i = w = 1/n$, then

$$\begin{aligned}\mathrm{FTWA}\left(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)}\right) &= \bigoplus_{i\in[n]}\left(w \odot \tilde{x}_{\pi(i)}\right)\\ &= \left(w \odot \tilde{x}_{\pi(1)}\right) \oplus \left(w \odot \tilde{x}_{\pi(2)}\right) \oplus \cdots \oplus \left(w \odot \tilde{x}_{\pi(n)}\right)\\ &= \left(w \odot \tilde{x}_{\pi(n)}\right) \oplus \left(w \odot \tilde{x}_{\pi(n-1)}\right) \oplus \cdots \oplus \left(w \odot \tilde{x}_{\pi(1)}\right)\\ &= \cdots = \mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right).\end{aligned} \tag{36}$$

(**P5**) Idempotency: assume that $\odot$ is restrictively distributive with respect to $\oplus$. Then

$$\begin{aligned}\mathrm{FTWA}(\tilde{u},\ldots,\tilde{u}) &= \bigoplus_{i\in[n]}\left(w_i \odot \tilde{u}\right)\\ &= \left(w_1 \odot \tilde{u}\right) \oplus \left(w_2 \odot \tilde{u}\right) \oplus \cdots \oplus \left(w_n \odot \tilde{u}\right)\\ &= \left(w_1 \oplus w_2 \oplus \cdots \oplus w_n\right) \odot \tilde{u} = 1 \odot \tilde{u}\\ &= \tilde{u}.\end{aligned} \tag{37}$$

(**P6**) Bounded: assume that $\odot$ is restrictively distributive with respect to $\oplus$ and pose $\min(\tilde{x}_1,\ldots,\tilde{x}_n) = \tilde{x}_*$, $\max(\tilde{x}_1,\ldots,\tilde{x}_n) = \tilde{x}^*$. Then

$$\begin{aligned}&\tilde{x}_* \leq \tilde{x}_i \Longrightarrow\\ &w_i \odot \tilde{x}_* \leq w_i \odot \tilde{x}_i \quad (1),\\ &\tilde{x}_* \leq \tilde{x}_j \Longrightarrow\\ &w_j \odot \tilde{x}_* \leq w_j \odot \tilde{x}_j \quad (2),\\ &(1)\text{-}(2) \Longrightarrow\\ &\left(w_i \odot \tilde{x}_*\right) \oplus \left(w_j \odot \tilde{x}_*\right) \leq \left(w_i \odot \tilde{x}_i\right) \oplus \left(w_j \odot \tilde{x}_j\right) \Longrightarrow\\ &\bigoplus_{i\in[n]}\left(w_i \odot \tilde{x}_*\right) \leq \bigoplus_{i\in[n]}\left(w_i \odot \tilde{x}_i\right) \Longrightarrow\\ &\left(w_1 \oplus w_2 \oplus \cdots \oplus w_n\right) \odot \tilde{x}_* \leq \mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) \Longrightarrow\\ &\tilde{x}_* \leq \mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right).\end{aligned} \tag{38}$$

In a similar way, we show that $\mathrm{FTWA}(\tilde{x}_1,\ldots,\tilde{x}_n) \leq \tilde{x}^*$. Therefore,

$$\min\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) \leq \mathrm{FTWA}\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) \leq \max\left(\tilde{x}_1,\ldots,\tilde{x}_n\right). \tag{39}$$

□

4.3. The Fuzzy Triangular Ordered Weighted Arithmetic Operator. The fuzzy triangular ordered weighted arithmetic (FTOWA) operator is an extension of the TOWA operator (see Remark 24 below) for situations where the available information is uncertain and it is necessary to use other techniques such as NFNs. It can be defined as follows.

Definition 23. The fuzzy triangular ordered weighted arithmetic operator, denoted by FTOWA, is the mapping FTOWA : $\widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$, such that

$$\mathrm{FTOWA}\left(\tilde{x}_1,\dots,\tilde{x}_n\right) = \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_{(i)}\right), \tag{40}$$

where $\oplus$ is a t-conorm, $\odot$ is a t-norm, $(w_1,\dots,w_n)$ is a weighting vector associated with $\oplus$, and $(\cdot)$ is a permutation on $[n]$ such that $\tilde{x}_{(1)} \le \cdots \le \tilde{x}_{(n)}$.

Remark 24. As the FTWA operator, the fuzzy triangular ordered weighted arithmetic (FTOWA) operator generalizes several usual aggregation operators, namely, the fuzzy ordered weighted averaging (FOWA) and the triangular ordered weighted arithmetic (TOWA) operators among others.

Proof. The proof is similar to Remark 21. □

Regarding its properties, we have the following result.

Theorem 25. *The FTOWA operator satisfies the properties **(P1)**, **(P2)**, **(P3)**, **(P4)**, **(P5)** if and only if $\odot$ is restrictively distributive over $\oplus$, and **(P6)** if and only if $\odot$ is restrictively distributive over $\oplus$.*

Proof. (**P1**), (**P2**), (**P3**), (**P5**), and (**P6**) are similar to Theorem 22.

(**P4**) Commutativity: let $(\tilde{x}_{\pi(1)},\dots,\tilde{x}_{\pi(n)})$ be a permutation of $(\tilde{x}_1,\dots,\tilde{x}_n)$.

$$\begin{aligned}
\mathrm{FTOWA}\left(\tilde{x}_1,\dots,\tilde{x}_n\right) &= \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_{(i)}\right), \\
&\qquad \tilde{x}_{(i)} \text{ ith largest of } \left\{\tilde{x}_i,\ i \in [n]\right\}, \\
\mathrm{FTOWA}\left(\tilde{x}_{\pi(1)},\dots,\tilde{x}_{\pi(n)}\right) &= \bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}'_{\pi(i)}\right), \\
&\qquad \tilde{x}'_{\pi(i)} \text{ ith largest of } \left\{\tilde{x}_{\pi(i)},\ i \in [n]\right\}.
\end{aligned} \tag{41}$$

Since $(\tilde{x}_{\pi(1)},\dots,\tilde{x}_{\pi(n)})$ is a permutation of $(\tilde{x}_1,\dots,\tilde{x}_n)$, we have $\tilde{x}_{(i)} = \tilde{x}'_{\pi(i)}$, $\forall i \in [n]$. And then

$$\mathrm{FTOWA}\left(\tilde{x}_1,\dots,\tilde{x}_n\right) = \mathrm{FTOWA}\left(\tilde{x}_{\pi(1)},\dots,\tilde{x}_{\pi(n)}\right). \tag{42}$$

□

4.4. Fuzzy Generalized Triangular Weighted Aggregation Operators. We focus now on generalizations of the FTWA operator by using generalized and quasi-arithmetic means. We start with the fuzzy generalized triangular weighted arithmetic (FGTWA) operator, which is defined as follows.

Definition 26. The fuzzy generalized triangular weighted arithmetic operator, denoted by FGTWA, is the mapping FGTWA : $\widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$, such that

$$\mathrm{FGTWA}\left(\tilde{x}_1,\dots,\tilde{x}_n\right) = \left(\bigoplus_{i\in[n]} \left(w_i \odot \tilde{x}_i^{\lambda}\right)\right)^{1/\lambda}, \tag{43}$$

where $\oplus$ is a t-conorm, $\odot$ is a t-norm, $(w_1,\dots,w_n)$ is a weighting vector associated with $\oplus$, and $\lambda > 0$ is a parameter.

As we can see, the FGTWA operator includes in the same formulation some usual fuzzy aggregation operators. Notably, if $\lambda = 1$, we get the FTWA operator. If $\lambda = 2$, we get the fuzzy triangular weighted quadratic (FTWQ) operator. Note that it is possible to further generalize the FGTWA operator by using quasi-arithmetic means. The result is the fuzzy triangular weighted quasi-arithmetic (Quasi-FTWA) operator.

Pose $\mathscr{F} = \{f : [0,1] \to [0,1],\ f(0) = 0, f(1) = 1,$ f strictly increasing and continuous function$\}$. However, it is worth noting that if f is such that $\{f(0), f(1)\} \neq \{0,1\}$, it suffices to consider the function $g \in \mathscr{F}$ given by

$$g(x) = \frac{f(x) - f(0)}{f(1) - f(0)}. \tag{44}$$

Definition 27. The fuzzy triangular weighted quasi-arithmetic operator, denoted by Quasi-FTWA, is the mapping Quasi-FTWA: $\widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$, such that

$$\begin{aligned}
&\text{Quasi-FTWA}\left(\tilde{x}_1,\dots,\tilde{x}_n\right) \\
&\quad = f^{-1}\left(\bigoplus_{i\in[n]} \left(w_i \odot f\left(\tilde{x}_i\right)\right)\right),
\end{aligned} \tag{45}$$

where $f \in \mathscr{F}$, $\oplus$ is a t-conorm, $\odot$ is a t-norm, and $(w_1,\dots,w_n)$ is a weighting vector associated with $\oplus$. f is called a generating function of the Quasi-FTWA operator.

Theorem 28. *The Quasi-FTWA operator satisfies the properties **(P1)**, **(P2)**, **(P3)**, **(P4)** if all $w_i = w$, **(P5)** if and only if $\odot$ is restrictively distributive over $\oplus$, and **(P6)** if and only if $\odot$ is restrictively distributive over $\oplus$.*

Proof. The proof is similar to the proof of Theorem 22 and follows from the fact that $f \in \mathscr{F}$. □

Remark 29. For $f \in \mathscr{F}$, the Quasi-FWA (see (18)) is a special case of the Quasi-FTWA operator.

Proof. It suffices to take $(\oplus,\odot) = (\oplus_L,\odot_P)$. □

4.5. Fuzzy Generalized Triangular Ordered Weighted Aggregation Operators. Another way to generalize the model is the use of the FTOWA operator.

Definition 30. The fuzzy generalized triangular ordered weighted arithmetic operator, denoted by FGTOWA, is the mapping FGTOWA : $\widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$, such that

$$\mathrm{FGTOWA}(\tilde{x}_1,\ldots,\tilde{x}_n)=\left(\bigoplus_{i\in[n]}\left(w_i\odot\tilde{x}_{(i)}^{\lambda}\right)\right)^{1/\lambda}, \quad (46)$$

where $\oplus$ is a t-conorm, $\odot$ is a t-norm, $(w_1,\ldots,w_n)$ is a weighting vector associated with $\oplus$, $(\cdot)$ is a permutation on $[n]$ such that $\tilde{x}_{(1)}\leq\cdots\leq\tilde{x}_{(n)}$, and $\lambda>0$ is a parameter.

As we can see, the FGTOWA operator includes in the same formulation some usual aggregation operators. Notably, if $\lambda=1$, we get the FTOWA operator. If $\lambda=2$, we get the fuzzy triangular ordered weighted quadratic (FTOWQ) operator. Note that it is possible to further generalize the FGTOWA operator by using quasi-arithmetic means. The result is the fuzzy triangular ordered weighted quasi-arithmetic (Quasi-FTOWA) operator.

Definition 31. The fuzzy triangular ordered weighted quasi-arithmetic operator, denoted by Quasi-FTOWA, is the mapping Quasi-FTOWA : $\widetilde{\mathscr{P}}([0,1])^n\to\widetilde{\mathscr{P}}([0,1])$, such that

$$\begin{aligned}&\text{Quasi-FTOWA}(\tilde{x}_1,\ldots,\tilde{x}_n)\\&\quad=f^{-1}\left(\bigoplus_{i\in[n]}\left(w_i\odot f\left(\tilde{x}_{(i)}\right)\right)\right),\end{aligned} \quad (47)$$

where $f\in\mathscr{F}$, $\oplus$ is a t-conorm, $\odot$ is a t-norm, $(w_1,\ldots,w_n)$ is a weighting vector associated with $\oplus$, and is $(\cdot)$ a permutation on $[n]$ such that $\tilde{x}_{(1)}\leq\cdots\leq\tilde{x}_{(n)}$. f is called a generating function of the Quasi-FTOWA operator.

Theorem 32. *The Quasi-FTOWA operator satisfies the properties **(P1)**, **(P2)**, **(P3)**, **(P4)**, **(P5)** if and only if $\odot$ is restrictively distributive over $\oplus$, and **(P6)** if and only if $\odot$ is restrictively distributive over $\oplus$.*

Proof. The proof is similar to the proof of Theorem 25 and follows from the fact that $f\in\mathscr{F}$. □

Remark 33. For $f\in\mathscr{F}$, the Quasi-FOWA (see (19)) is a special case of the Quasi-FTOWA operator.

Proof. It suffices to take $(\oplus,\odot)=(\oplus_L,\odot_P)$. □

4.6. Fuzzy Triangular Geometric Aggregation Operators. A further type of aggregation operators that could be used in the model is the fuzzy triangular geometric aggregation operators. In this section, we consider both the fuzzy triangular weighted geometric (FTWG) and the fuzzy triangular ordered weighted geometric (FTOWG) operators.

4.6.1. The Fuzzy Triangular Weighted Geometric Operator. The fuzzy triangular weighted geometric (FTWG) operator is an extension of the TWG operator (see Remark 35) for situations where the available information is uncertain and it is necessary to use other techniques such as NFNs. It can be defined as follows.

Definition 34. The fuzzy triangular weighted geometric operator, denoted by FTWG, is the mapping FTWG: $\widetilde{\mathscr{P}}([0,1])^n\to\widetilde{\mathscr{P}}([0,1])$, such that

$$\begin{aligned}\mathrm{FTWG}(\tilde{x}_1,\ldots,\tilde{x}_n)&=\bigodot_{i\in[n]}\tilde{x}_i^{w_i}\\&=\left(\bigodot_{i\in[n]}a_i^{w_i},\bigodot_{i\in[n]}b_i^{w_i},\bigodot_{i\in[n]}c_i^{w_i},\bigodot_{i\in[n]}d_i^{w_i}\right)\\&=\left(\mathrm{TWG}(a_1,\ldots,a_n),\mathrm{TWG}(b_1,\ldots,b_n),\right.\\&\qquad\left.\mathrm{TWG}(c_1,\ldots,c_n),\mathrm{TWG}(d_1,\ldots,d_n)\right),\end{aligned} \quad (48)$$

where $\odot$ is a t-norm and $(w_1,\ldots,w_n)$ is a weighting vector (Definition 17) associated with $\oplus$, a t-conorm corresponding to $\odot$.

Remark 35. As we can see, when $\tilde{x}=x$, the fuzzy triangular weighted geometric (FTWG) operator is nothing but the triangular weighted geometric (TWG) operator.

Theorem 36. *The FTWG operator satisfies the properties **(P1)**, **(P2)**, **(P3)**, **(P4)** if all $w_i=w$, **(P5)** if and only if $x^w\odot x^v=x^{w\oplus v}$ for all $x,w,v\in[0,1]$, and **(P6)** if and only if $x^w\odot x^v=x^{w\bigoplus v}$ for all $x,w,v\in[0,1]$.*

Proof. Let $(\tilde{x}_1,\ldots,\tilde{x}_n)$, $(\tilde{y}_1,\ldots,\tilde{y}_n)$ be two NFNs and let $(w_1,\ldots,w_n)$ be a weighting vector (Definition 17).

(P1) Boundary conditions: they are straightforward.

(P2) Monotonicity: assume that $\tilde{x}_i\leq\tilde{y}_i$, $\forall i\in[n]$.

$$\begin{aligned}&\tilde{x}_i\leq\tilde{y}_i\Longrightarrow\\&\tilde{x}_i^{w_i}\leq\tilde{y}_i^{w_i}\Longrightarrow\\&\tilde{x}_i^{w_i}\odot\tilde{x}_j^{w_j}\leq\tilde{y}_i^{\tilde{w}_i}\odot\tilde{y}_j^{w_j}\Longrightarrow\\&\bigodot_{i\in[n]}\tilde{x}_i^{w_i}\leq\bigodot_{i\in[n]}\tilde{y}_i^{w_i}\Longrightarrow\\&\mathrm{FTWG}(\tilde{x}_1,\ldots,\tilde{x}_n)\leq\mathrm{FTWG}(\tilde{y}_1,\ldots,\tilde{y}_n).\end{aligned} \quad (49)$$

(P3) Continuity: FTWG is continuous since it is a component-wise continuous operator.

(P4) Commutativity: let $(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)})$ be a permutation of $(\tilde{x}_1,\ldots,\tilde{x}_n)$.

FTWG$(\tilde{x}_1,\ldots,\tilde{x}_n)\neq$ FTWG$(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)})$ in general. However, if $w_i=w=1/n$, we have

$$\begin{aligned}\mathrm{FTWG}(\tilde{x}_1,\ldots,\tilde{x}_n)&=\bigodot_{i\in[n]}\tilde{x}_i^{w}=\tilde{x}_1^w\odot\tilde{x}_2^w\odot\cdots\odot\tilde{x}_n^w\\&=\tilde{x}_n^w\odot\tilde{x}_{n-1}^w\odot\cdots\odot\tilde{x}_1^w=\cdots\\&=\mathrm{FTWG}(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)}).\end{aligned} \quad (50)$$

(**P5**) Idempotency: assume that $x^w \odot x^v = x^{w\oplus v}$ for all $x, w, v \in [0,1]$. Then it is easy to see that $\tilde{x}^w \odot \tilde{x}^v = \tilde{x}^{w\oplus v}$ for all $\tilde{x} \in \widetilde{\mathscr{P}}([0,1])$. Thus

$$\begin{aligned} \mathrm{FTWG}(\tilde{u},\ldots,\tilde{u}) &= \bigodot_{i\in[n]} \tilde{u}^{w_i} = \tilde{u}^{w_1} \odot \tilde{u}^{w_2} \odot \cdots \odot \tilde{u}^{w_n} \\ &= \tilde{u}^{w_1\oplus w_2\oplus\cdots\oplus w_n} = \tilde{u}. \end{aligned} \tag{51}$$

(**P6**) Bounded: assume that $x^w \odot x^v = x^{w\oplus v}$ for all $x, w, v \in [0,1]$. Then it is easy to see that $\tilde{x}^w \odot \tilde{x}^v = \tilde{x}^{w\oplus v}$ for all $\tilde{x} \in \widetilde{\mathscr{P}}([0,1])$. Pose $\min(\tilde{x}_1,\ldots,\tilde{x}_n) = \tilde{x}_*$, $\max(\tilde{x}_1,\ldots,\tilde{x}_n) = \tilde{x}^*$. Thus

$$\begin{gathered} \tilde{x}_* \le \tilde{x}_i \Longrightarrow \\ \tilde{x}_*^{w_i} \le \tilde{x}_i^{w_i} \quad (1), \\ \tilde{x}_* \le \tilde{x}_j \Longrightarrow \\ \tilde{x}_*^{w_j} \le \tilde{x}_j^{w_j} \quad (2), \\ (1)\text{-}(2) \Longrightarrow \\ \tilde{x}_*^{w_i} \odot \tilde{x}_*^{w_j} \le \tilde{x}_i^{w_i} \odot \tilde{x}_j^{w_j} \Longrightarrow \\ \bigodot_{i\in[n]} \tilde{x}_*^{w_i} \le \bigodot_{i\in[n]} \tilde{x}_i^{w_i} \Longrightarrow \\ \tilde{x}_*^{(\tilde{w}_1\oplus\tilde{w}_2\oplus\cdots\oplus\tilde{w}_n)} \le \mathrm{FTWG}(\tilde{x}_1,\ldots,\tilde{x}_n) \Longrightarrow \\ \tilde{x}_* \le \mathrm{FTWG}(\tilde{x}_1,\ldots,\tilde{x}_n). \end{gathered} \tag{52}$$

In the same way, we show that $\mathrm{FTWG}(\tilde{x}_1,\ldots,\tilde{x}_n) \le \tilde{x}^*$. Therefore,

$$\begin{aligned} \min(\tilde{x}_1,\ldots,\tilde{x}_n) &\le \mathrm{FTWG}(\tilde{x}_1,\ldots,\tilde{x}_n) \\ &\le \max(\tilde{x}_1,\ldots,\tilde{x}_n). \end{aligned} \tag{53}$$

□

4.6.2. The Fuzzy Triangular Ordered Weighted Geometric Operator. By analogy, the fuzzy triangular ordered weighted geometric (FTOWG) operator is an extension of the TOWG operator (see Remark 38) for situations where the available information is uncertain and it is necessary to use other techniques such as NFNs. It can be defined as follows.

Definition 37. The fuzzy triangular ordered weighted geometric operator, denoted by FTOWG, is the mapping FTOWG: $\widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$, such that

$$\mathrm{FTOWG}(\tilde{x}_1,\ldots,\tilde{x}_n) = \bigodot_{i\in[n]} \tilde{x}_{(i)}^{w_i}, \tag{54}$$

where $\odot$ is a t-norm, $(w_1,\ldots,w_n)$ is a weighting vector (Definition 17) associated with $\oplus$, a t-conorm corresponding to $\odot$, and $(\cdot)$ is a permutation on $[n]$ such that $\tilde{x}_{(1)} \le \cdots \le \tilde{x}_{(n)}$.

Remark 38. When $\tilde{x} = x$, the fuzzy triangular ordered weighted geometric (FTOWG) operator is nothing but the triangular ordered weighted geometric (TOWG) operator.

Regarding their properties, we can also state the following theorem.

Theorem 39. *The FTOWG operator satisfies the properties (P1), (P2), (P3), (P4), (P5) if and only if $x^w \odot x^v = x^{w\oplus v}$ for all $x, w, v \in [0,1]$, and (P6) if and only if $x^w \odot x^v = x^{w\oplus v}$ for all $x, w, v \in [0,1]$.*

Proof. (**P1**), (**P2**), (**P3**), (**P5**), and (**P6**) are Similar to Theorem 36.

(**P4**) Commutativity: let $(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)})$ be a permutation of $(\tilde{x}_1,\ldots,\tilde{x}_n)$.

$$\begin{gathered} \mathrm{FTOWG}(\tilde{x}_1,\ldots,\tilde{x}_n) = \bigodot_{i\in[n]} \tilde{x}_{(i)}^{w_i}, \\ \tilde{x}_{(i)} \text{ } i\text{th largest of } \{x_i,\ i \in [n]\}, \\ \mathrm{FTOWG}(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)}) = \bigodot_{i\in[n]} \left(\tilde{x}'_{\pi(i)}\right)^{w_i}, \\ \tilde{x}'_{\pi(i)} \text{ } i\text{th largest of } \{x_{\pi(i)},\ i \in [n]\}. \end{gathered} \tag{55}$$

Since $(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)})$ is a permutation of $(\tilde{x}_1,\ldots,\tilde{x}_n)$, we have $\tilde{x}_{(i)} = \tilde{x}'_{\pi(i)}$, $\forall i \in [n]$. And then

$$\mathrm{FTOWG}(\tilde{x}_1,\ldots,\tilde{x}_n) = \mathrm{FTOWG}(\tilde{x}_{\pi(1)},\ldots,\tilde{x}_{\pi(n)}). \tag{56}$$

□

Remark 40. We can easily prove that (straightforward since it is a matter of calculation with $\odot = \odot_P$ and $\oplus = \oplus_L$) the fuzzy ordered weighted geometric averaging (FOWGA) [Xu (2002) [26]] is a special case of the FTOWG operator, where FOWGA: $\widetilde{\mathscr{P}}([0,1])^n \to \widetilde{\mathscr{P}}([0,1])$ such that

$$\mathrm{FOWGA}(\tilde{x}_1,\ldots,\tilde{x}_n) = \prod_{i=1}^{n} \tilde{x}_{(i)}^{w_i}, \quad \text{with } \sum_{i=1}^{n} w_i = 1. \tag{57}$$

5. Numerical Example

In this section, we are going to develop a brief example where we will see the applicability of the new approach. We will focus on a decision-making problem about selection of investment strategies. Let us consider the example investigated in [2, 13].

(1) Assume a company that operates in North America and Europe is analyzing the general policy for the next year. The group of experts of the company considers five possible strategies to follow.

(i) A_1 = Invest to the Asian market
(ii) A_2 = Invest to the South American market
(iii) A_3 = Invest to the African market
(iv) A_4 = Invest to the three continents
(v) A_5 = Do not develop any investment.

Table 1: Fuzzy decision matrix of expert 1.

	A_1	A_2	A_3	A_4	A_5
C_1	(.6, .7, .8)	(.7, .8, .9)	(.3, .4, .5)	(.1, .2, .3)	(.5, .6, .7)
C_2	(.7, .8, .9)	(.7, .8, .9)	(.3, .4, .5)	(.2, .3, .4)	(.6, .7, .8)
C_3	(.3, .4, .5)	(.7, .8, .9)	(.6, .7, .8)	(.4, .5, .6)	(.4, .5, .6)
C_4	(.4, .5, .6)	(.5, .6, .7)	(.4, .5, .6)	(.7, .8, .9)	(.7, .8, .9)
C_5	(.5, .6, .7)	(.4, .5, .6)	(.8, .9, 1)	(.8, .9, 1)	(.8, .9, 1)

Table 2: Fuzzy decision matrix of expert 2.

	A_1	A_2	A_3	A_4	A_5
C_1	(.6, .7, .8)	(.1, .2, .3)	(.4, .5, .6)	(.4, .5, .6)	(.5, .6, .7)
C_2	(.7, .8, .9)	(.1, .2, .3)	(.4, .5, .6)	(.5, .6, .7)	(.6, .7, .8)
C_3	(.8, .9, 1)	(.4, .5, .6)	(.7, .8, .9)	(.4, .5, .6)	(.4, .5, .6)
C_4	(.4, .5, .6)	(.8, .9, 1)	(.8, .9, 1)	(.6, .7, .8)	(.7, .8, .9)
C_5	(.7, .8, .9)	(.8, .9, 1)	(.3, .4, .5)	(.8, .9, 1)	(.6, .7, .8)

In order to evaluate these strategies, the group of experts considers that the key factor is the economic situation of the next year. Thus, depending on the situation, the expected benefits will be different. The experts have considered five possible situations for the next year.

(i) C_1 = Very bad
(ii) C_2 = Bad
(iii) C_3 = Regular
(iv) C_4 = Good
(v) C_5 = Very good.

The group of experts of the company is constituted by three persons that give their own opinion about the uncertain expected results that may occur in the future. The expected results depending on the situation C_i and the alternative A_j are shown in Tables 1, 2, and 3. Note that the results are triangular NFNs (TNFNs).

(2) In this problem, the decision-maker assigns the weighting vector $\mathbf{w}_E = (.3, .3, .4)$ of experts, which represents the importance of each expert in the analysis. With this information, we can aggregate expert opinions in order to obtain a fuzzy payoff matrix. The results are shown in Table 4. To obtain this matrix, we have used the FWA operator, a special case of the FTWA operator.

(3) In the same way, we assume that the criteria C_i have the weighting vector $\mathbf{w}_C = (.1, .2, .2, .2, .3)$. Then, we can aggregate criteria evaluations in order to have a global score for each strategy. In Table 5, we show the different results obtained by using some particular cases of FTWA presented in Remark 21. Note that configuration of the weighting vector $\mathbf{w}_C$ indicates that the underlying t-conorm is $\oplus_L$.

(4) At the end of the aggregation step, the system must be able to select the best strategy based on fuzzy overall scores given in Table 5. This can be done, for example, by ranking strategies according to their overall score. The results are shown in Table 6. As we can see, depending on the aggregation operator and the ranking method used, the ordering of strategies is different. Therefore, depending on the aggregation operator and the ranking FNs method used, the results may lead to different decisions.

6. Conclusion

We have introduced the class of fuzzy triangular aggregation operators. As we have seen, the fuzzy triangular aggregation operators are nothing but triangular aggregation operators in the situation where the available information cannot be assessed with exact numbers belonging to the unit interval $[0, 1]$ and it is necessary to use another approach like NFNs to assess such uncertain or imprecise information. Their construction is based on triangular norms (t-norms and t-conorms) and more especially on the three special triangular norms: Zadeh's triangular norms, probabilistic triangular norms, and Lucasiewicz's triangular norms.

We have also presented an application of this new approach to a decision-making problem regarding the selection of investment strategies. Depending on the particular aggregation operator used, the results and decisions may be different.

The main advantage of the model is that it makes it possible to consider a wide range of situations depending on interests of the decision-maker. It allows not only generalizing some useful existing aggregation operators, but also highlighting some interesting operators and results. The flexibility of that class of fuzzy aggregation operators allows combining pragmatic approaches (which consist of testing a large number of aggregation operators) and axiomatic approaches (which consist of defining mathematical properties that must be satisfied by aggregation operators) in the choice of the best aggregation technique for a given MCGDM problem.

Table 3: Fuzzy decision matrix of expert 3.

	A_1	A_2	A_3	A_4	A_5
C_1	(.5, .6, .7)	(.6, .7, .8)	(.4, .5, .6)	(.6, .7, .8)	(.5, .6, .7)
C_2	(.7, .8, .9)	(.1, .2, .3)	(.4, .5, .6)	(.3, .4, .5)	(.6, .7, .8)
C_3	(.6, .7, .8)	(.8, .9, 1)	(.7, .8, .9)	(.6, .7, .8)	(.4, .5, .6)
C_4	(.4, .5, .6)	(.5, .6, .7)	(.8, .9, 1)	(.5, .6, .7)	(.7, .8, .9)
C_5	(.7, .8, .9)	(.4, .5, .6)	(.3, .4, .5)	(.7, .8, .9)	(.7, .8, .9)

Table 4: Fuzzy payoff matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	(.56, .66, .76)	(.48, .58, .68)	(.37, .47, .57)	(.39, .49, .59)	(.5, .6, .7)
C_2	(.7, .8, .9)	(.28, .38, .48)	(.37, .47, .57)	(.33, .43, .53)	(.6, .7, .8)
C_3	(.57, .67, .77)	(.65, .75, .85)	(.67, .77, .87)	(.48, .58, .68)	(.4, .5, .6)
C_4	(.4, .5, .6)	(.59, .69, .79)	(.68, .78, .88)	(.59, .69, .79)	(.7, .8, .9)
C_5	(.64, .74, .84)	(.52, .62, .72)	(.45, .55, .65)	(.76, .86, .96)	(.7, .8, .9)

Table 5: Fuzzy global scores for each strategy.

	A_1	A_2	A_3	A_4	A_5
$\oplus_L, \odot_Z$	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)
$\oplus_L, \odot_P$	(.58, .68, .78)	(.51, .61, .71)	(.52, .62, .72)	(.55, .65, .75)	(.6, .7, .8)
$\oplus_L, \odot_L$	(0, .04, .24)	(0, 0, .07)	(0, 0, .15)	(.06, .16, .26)	(0, .1, .3)

Table 6: Ranking of strategies.

	Ranking according to method of [2, 13]
$\oplus_L, \odot_Z$	$A_1 = A_2 = A_3 = A_4 = A_5$
$\oplus_L, \odot_P$	$A_2 < A_3 < A_4 < A_1 < A_5$
$\oplus_L, \odot_L$	$A_2 = A_3 < A_1 < A_5 < A_4$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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