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Research Article

Arithmetical Functions Associated with the k-ary Divisors of an Integer

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The k-ary divisibility relations are a class of recursively defined relations beginning with standard divisibility and culminating in the so-called infinitary divisibility relation. We examine the summatory functions corresponding to the k-ary analogues of various popular functions in number theory, proving various results about the structure of the k-ary divisibility relations along the way.

1. Introduction

Let n be a positive integer and denote the set of divisors of n by $D(n)$. The set of unitary divisors of n, denoted by $D^1(n)$, are the divisors d of n which satisfy $D(d) \cap D(n/d) = \{1\}$; in other words, $(d, n/d) = 1$. The biunitary divisors of n are the divisors d of n which satisfy $D^1(d) \cap D^1(n/d)$. This differs from some definitions of biunitary divisibility in the literature (e.g., [1]) but is consistent with others (e.g., [2]). In general, we may define the k-ary divisors of an integer n to be the set

$$\begin{aligned} D^k(n) &:= \left\{d \in D(n) \mid D^{k-1}(d) \cap D^{k-1}\left(\frac{n}{d}\right) = \{1\}\right\} \\ &= \left\{d \in D(n) \mid \left(d, \frac{n}{d}\right)_{k-1} = 1\right\}, \end{aligned} \tag{1}$$

where we define the greatest common k-ary divisor of m and n by

$$(m, n)_k := \max\left\{D^k(m) \cap D^k(n)\right\}. \tag{2}$$

We write $d|_k n$ if $d \in D^k(n)$.

The k-ary divisibility relations as defined above were first introduced by Cohen [3] and have been studied more recently by Haukkanen [2] and Steuding et al. [4]. An alternative definition can be seen in Suryanarayana [5].

One easily verifies the following basic properties:

(i) $1 \in D^k(n)$ and $n \in D^k(n)$ for all n.

(ii) If n and m are coprime, then $D^k(nm) = D^k(n) \cdot D^k(m)$, where $A \cdot B := \{ab \mid a \in A, b \in B\}$.

(iii) If $d \in D^k(n)$, then $n/d \in D^k(n)$.

For example, the set of unitary divisors of a prime power p^a are $D^1(p^a) = \{1, p^a\}$. On the other hand, the biunitary divisors of a prime power p^a are given by $D(p^a)$ when a is odd and $D(p^a) \setminus \{p^{a/2}\}$ when a is even. We may then form the unitary and biunitary divisors of a positive integer n by "multiplying" the prime-power divisor sets that form the prime decomposition of n.

By viewing the sets D^k as representing some of the A-convolutions of Narkiewicz [6], we may define the k-ary convolution of arithmetic functions f and g:

$$(f \star_k g)(n) := \sum_{d|_k n} f(d)\, g\left(\frac{n}{d}\right). \tag{3}$$

The following properties of k-ary convolution can be found in [2]:

(i) The k-ary convolution is commutative.

(ii) The function $\iota(n)$, which takes on value of 1 if $n = 1$ and 0 otherwise, is the identity under k-ary convolution.

(iii) If an arithmetic function f satisfies $f(1) \neq 0$, then f possesses a unique inverse under k-ary convolution.

(iv) If f and g are multiplicative functions, then $f \star_k g$ is multiplicative as well.

By choosing f and g appropriately, we may obtain multiplicative k-ary analogues to the following classical functions from number theory in terms of the k-ary convolution:

(i) Let $f(n) = g(n) = 1$ for all n. Then $\tau_k(n) := (f \star_k g)(n)$ is the number of k-ary divisors of n.

(ii) Let $f(n) = n$ and $g(n) = 1$ for all n. Then $\sigma_k(n) := (f \star_k g)(n)$ is the sum of the k-ary divisors of n.

(iii) Let $f(n) = n$ and $g(n) = \mu_k(n)$ for all n, where μ_k is the unique inverse of the function $U(n) = 1$ under the k-ary convolution. Then $\varphi_k(n) := (f \star_k g)(n)$ is the analogue of the Euler totient function.

(iv) Let $f(n) = n$ and $g(n) = |\mu_k(n)|$. Then $\psi_k(n) := (f \star_k g)(n)$ is the analogue of the Dedekind ψ-function.

Note that while $\varphi(n)$ counts the totatives of n, in general φ_k does not count the k-ary totatives of n.

In this paper, we prove results concerning the structure of k-ary divisibility relations and use that to obtain formulae for the number of integers m less than or equal to n which satisfy $(m, n)_k = 1$. We then apply this result to obtain asymptotics for the summatory functions of k-ary generalizations of the classical functions mentioned above.

2. The Behavior of k-Ary Divisibility Relations

Let $D^\infty(n)$ be the set of infinitary divisors of n introduced and studied by Cohen [3, 7]. The infinitary divisibility relation can be thought of as the end behavior of the recursion defining the k-ary divisibility relations. It satisfies

(i) All properties of k-ary divisibility relations listed above

(ii) $d \in D^\infty(n)$ if and only if $D^\infty(d) \cap D^\infty(n/d) = \{1\}$

(iii) D^∞ being transitive; that is, for all n, if $d \in D^\infty(n)$, then $D^\infty(d) \subseteq D^\infty(n)$.

Additionally, the following reformulation of Theorem 1 from [3] characterizes in what sense k-ary divisibility relations "approach" the infinitary divisibility relation as k increases.

Theorem 1. *Let $k \geqslant 0$ be given, and suppose that $n \in \mathbb{N}$ is such that, for every prime p, $\nu_p(n) \leqslant k + 1$, where $\nu_p(n)$ is the exponent of the prime p in the prime decomposition of n (0 if p does not divide n). Then $D^k(n) = D^\infty(n)$.*

Proof. We proceed by induction on k. We need to only show the result for prime powers p^a, since by the second property listed above we obtain our theorem by multiplicative construction, akin to the treatment of multiplicative arithmetical functions. Therefore, we may speak of a set prime p and consider p^a, with $a \leqslant k+1$. For $k = 0$, we have $D^\infty(p) = D(p)$ and $D^\infty(1) = D(1)$.

Now assume that the result holds up to some $K - 1$. Then consider $D^K(p^a) = \{d \in D(p^a) \mid D^{K-1}(d) \cap D^{K-1}(p^a/d) = \{1\}\}$, with a such that $a \leqslant K + 1$. Notice that all divisors d of p^a, except p^a itself, satisfy $d = p^b$, with $b \leqslant K - 1$. Then, for $d \neq p^a$, we have $D^{K-1}(d) = D^\infty(d)$ and, for $d \neq 1$, we have $D^{K-1}(p^a/d) = D^\infty(p^a/d)$. So, for d not equal to 1 or p^a, $d \in D^K(p^a) \iff D^\infty(d) \cap D^\infty(p^a/d)$. But 1 and p^a are in $D^K(p^a)$ as well, so $D^K(p^a) = D^\infty(p^a)$, and by the second property of D^k and D^∞ listed above, we are done. □

Additionally, we observe the following.

Theorem 2. *For all $n \in \mathbb{N}$, $k \geqslant 0$, $D^\infty(n) \subseteq D^{2k+2}(n) \subseteq D^{2k}(n)$, and $D^{2k+1}(n) \subseteq D^{2k+3}(n) \subseteq D^\infty(n)$.*

Proof. We will again use induction. One can immediately verify that $D^\infty(n) \subseteq D^2(n) \subseteq D(n)$ and $D^1(n) \subseteq D^3(n) \subseteq D^\infty(n)$; $D^3(p^a)$ only differs from $D^1(p^a)$ at $a = 3$ and $a = 6$, where we have $D^3(p^3) = D^\infty(p^3)$ and $D^3(p^6) = D^\infty(p^6)$. Assuming that the theorem holds up to some K, observe that for each divisor d of n the condition $D^{2K+3}(d) \cap D^{2K+3}(n/d) = \{1\}$ implies that $D^{2K+1}(d) \cap D^{2K+1}(n/d) = \{1\}\}$ on account of $D^{2K+1}(d) \subseteq D^{2K+3}(d)$ for each d. Then $D^{2K+4}(n) \subseteq D^{2K+2}(n)$. But then, by the same reasoning, $D^{2K+3}(n) \subseteq D^{2K+5}(n)$.

To see that $D^\infty(n)$ is ordered according to the theorem, consider the statement $D^\infty(n) \subseteq D(n)$ for all n. Using the argument from above and the fact that $D^\infty(n) = \{d \in D(n) \mid D^\infty(d) \cap D^\infty(n/d) = \{1\}\}$, we conclude that, for all k, $D^{2k+1}(n) \subseteq D^\infty(n) \subseteq D^{2k}(n)$ and hence our relations hold. □

We observe that when looking at the k-ary divisors of the powers of a specific prime number, there is always an integer after which, for each a, the k-ary divisors of p^a will be either $D^2(p^a)$ or $D^1(p^a)$.

Theorem 3. *Let $k > 2$ be given. Then there is an integer N_k such that, for all $a > N_k$ and all primes p, $D^k(p^a) = D^1(p^a)$ if k is odd, and $D^k(p^a) = D^2(p^a)$ if k is even.*

Proof. We note first that, for $k = 1$ and $k = 2$, we have that $N_1 = 1$ and $N_2 = 3$ trivially suffice for bounds. Now assume that such an N_k exists for all $k = M - 1$. We consider the cases even M and odd M, respectively.

For even M, take $a > 2N_{M-1}$ and let $b \leqslant a/2$. Then $D^k(p^{a-b}) \cap D^k(p^b) = \{1\}$, since $D^k(p^{a-b}) = D^1(p^{a-b})$ and $b < a - b$. For $2b = a$, we have $p^b \notin D^M(p^a)$ as desired. Since k-ary divisions are symmetric, this argument holds for $b' = a - b$ as well. We see then that we may take $N_M = 2N_{M-1}$ for even M.

For odd M, take $a > 3N_{M-1}$. Let b be such that $0 \leqslant b \leqslant (2/3)a$. Then $D^{M-1}(p^{a-b}) \cap D^{M-1}(p^b) = \{1\}$ if $p^b \in D^{M-1}(p^{a-b})$. This occurs for all b and a satisfying $a \neq 3b$. If $a = 3b$, then $b > N_{M-1}$, so $D^k(p^b) = D^2(p^b)$. In either case, $D^{M-1}(p^{a-b}) \cap D^{M-1}(p^b) = \{1\}$, so that $D^{M-1}(p^a) = D^1(p^a)$. We then see that we may take $N_M > 3N_{M-1}$ for odd M. □

Definition 4. We denote by $N_k^\star$ the least N_k for a given k.

Our next section concerns itself with k-ary analogues of some classical results on summatory functions.

3. Summatory Functions

Let $k > 0$ be given and let $f(n)$ be an arithmetical function constructed as follows:

$$f(n) := \sum_{d \in D^k(n)} g(d) \left(\frac{n}{d}\right)^r, \tag{4}$$

where r is a positive integer and $g(n)$ is a function such that $g(n) = \mathcal{O}(n^{r-1})$. We wish to explore the end behavior of the summatory function of f:

$$S(f)(x) := \sum_{n \leqslant x} f(n) \tag{5}$$

We will employ techniques already used in [7, 8] to derive the result for the infinitary and unitary cases, respectively.

Definition 5. Let $k \geqslant 0$ and $r > 0$. We introduce the following function:

$$\varphi_{k,r}(x, n) := \sum_{\substack{m \leqslant x \\ (m,n)_k = 1}} m^r. \tag{6}$$

For $r = 0$, this function counts the number of integers m that are less than x and k-ary coprime to n. It is known that, for $k = 0$, $\varphi_{0,0}(x, n) = x(\varphi(n)/n) + \mathcal{O}(\tau(n))$. The summatory functions for $\varphi_{k,0}$ may be broken into the case of even k or odd k, in accordance with whether $D^k(n) \subseteq D^\infty(n)$ or $D^\infty(n) \subseteq D^k(n)$.

Theorem 6. *Let $k > 0$ be an integer. Then, for even k, $\varphi_{k,0}(x, n) = x(\varphi(n)/n)K_k(n) + \mathcal{O}(\tilde{\tau}_k(n)\tau(n))$, with*

$$K_k(n) = \prod_{p \mid n} \left(\sum_{\substack{b=0 \\ (p^b, p^{\nu_p(n)})_k = 1}}^{\infty} \frac{1}{p^b} \right) \tag{7}$$

and

$$\tilde{\tau}_k(n) = \prod_{p \mid n} \left(\sum_{\substack{b=0 \\ (p^b, p^{\nu_p(n)})_k = 1}}^{\infty} 1 \right); \tag{8}$$

for odd k,

$$\varphi_{k,0}(x, n) = x \left(1 - \sum_{d \in D^1(n)} \frac{\varphi(d)\mu_1(d)}{d} K_k(d) \right) + \mathcal{O}(\tilde{\tau}_k(n)\tau_1(n)\tau(n)), \tag{9}$$

with the following:

(i) *$\tau_1(n)$ is the number of elements (divisors) in $D^1(n)$.*

(ii) *$\mu_1(n)$ is the Möbius function corresponding to D^1: $\mu_1(n) = (-1)^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n, counted without multiplicity.*

(iii)

$$K_k(n) = \prod_{p \mid n} \left(\sum_{\substack{b=0 \\ (p^b, p^{\nu_p(n)})_k > 1}} \frac{1}{p^b} \right). \tag{10}$$

(iv)

$$\tilde{\tau}_k(n) = \prod_{p \mid n} \left(\sum_{\substack{b=0 \\ (p^b, p^{\nu_p(n)})_k > 1}} 1 \right). \tag{11}$$

Proof. First note that $K_k(n)$ and $\tilde{\tau}_k(n)$ are well defined: by Theorem 3, for even k, the number of integers m satisfying the condition $(m, n)_k = 1$ for a given integer n must be finite, whereas for odd k and for each maximal prime power dividing n, the number of integers satisfying $(p^b, p^{\nu_p(n)})_k > 1$ must be finite, and hence the product over sums of prime powers k-ary-coprime to n must be finite. Therefore, the sums are finite. We will prove the result for even k first.

Let k be even and consider

$$\varphi_{k,0}(x, n) = \sum_{\substack{m \leqslant x \\ (m,n)_k = 1}} 1 = \sum_{\substack{m \leqslant x \\ m = m_1 m_2 \\ (m_1, n) = 1 \\ (m_2, n)_k = 1 \\ \mathrm{core}(m_2) \mid n}} 1, \tag{12}$$

where

$$\mathrm{core}(m_2) := \prod_{p \mid m_2} p \tag{13}$$

is the square-free part of the integer m_2, from [8]. Here we have split each m uniquely into a part that has no common divisor with n and a part whose prime decomposition uses only the primes of n (note that there is no restriction on the prime powers used; e.g., $m_2 = n^2$ may appear in this decomposition for large enough x).

We proceed:

$$\begin{aligned} \sum_{\substack{m \leqslant x \\ m = m_1 m_2 \\ (m_1, n) = 1 \\ (m_2, n)_k = 1 \\ \mathrm{core}(m_2) \mid n}} 1 &= \sum_{\substack{m_2 : \mathrm{core}(m_2) \mid n \\ (m_2, n)_k = 1}} \sum_{\substack{m_1 \leqslant x/m_2 \\ (m_1, n) = 1}} 1 \\ &= \sum_{\substack{m_2 : \mathrm{core}(m_2) \mid n \\ (m_2, n)_k = 1}} \varphi_{0,0}\left(\frac{x}{m_2}, n\right) \\ &= \sum_{\substack{m_2 : \mathrm{core}(m_2) \mid n \\ (m_2, n)_k = 1}} \left(x \frac{\varphi(n)}{n m_2} + \mathcal{O}(\tau(n)) \right), \end{aligned} \tag{14}$$

using the fact that the behavior of $\varphi_{0,0}(x, n)$ is known. Pulling out the constants with respect to the sum then immediately gives us our result.

For odd k, we proceed in a different manner:

$$\sum_{\substack{m\leqslant x\\(m,n)_k=1}} = [x] - \sum_{\substack{m\leqslant x\\(m,n)_k>1}} 1. \tag{15}$$

We then analyze the term

$$\sum_{\substack{m\leqslant x\\(m,n)_k>1}} 1: \tag{16}$$

We wish to split m into m_1m_2 as before. However, this should be done in such a way as to be both unique and useful in dealing with the requirement that $(m,n)_k > 1$. For each divisor d of n, let $m = m_1m_2$, with $\mathrm{core}(m_2) \mid d \mid n$, $(m_1,n) = 1$, and $(m_2, p^{\nu_p(d)})_k > 1$ for each $p \mid d$. We invoke the principle of inclusion-exclusion, enabling us to write

$$\begin{aligned}
\sum_{\substack{m\leqslant x\\(m,n)_k>1}} &= \sum_{\substack{m\leqslant x\\ m=m_1m_2\\ (m_1,p_1^{\nu_{p_1}(n)})=1\\ \mathrm{core}(m_2)\mid p_1^{\nu_{p_1}(n)}\\ (m_2,p_1^{\nu_{p_1}(n)})_k>1}} 1 + \sum_{\substack{m\leqslant x\\ m=m_1m_2\\ (m_1,p_2^{\nu_{p_2}(n)})=1\\ \mathrm{core}(m_2)\mid p_2^{\nu_{p_2}(n)}\\ (m_2,p_2^{\nu_{p_2}(n)})_k>1}} 1 + \cdots \\
&+ \sum_{\substack{m\leqslant x\\ m=m_1m_2\\ (m_1,p_s^{\nu_{p_s}(n)})=1\\ \mathrm{core}(m_2)\mid p_s^{\nu_{p_s}(n)}\\ (m_2,p_s^{\nu_{p_s}(n)})_k>1}} 1 - \left(\sum_{\substack{m\leqslant x\\ m=m_1m_2\\ (m_1,p_1^{\nu_{p_1}(n)}p_2^{\nu_{p_2}(n)})=1\\ \mathrm{core}(m_2)\mid p_1^{\nu_{p_1}(n)}p_2^{\nu_{p_2}(n)}\\ (m_2,p_1^{\nu_{p_1}(n)})_k>1\\ (m_2,p_2^{\nu_{p_2}(n)})_k>1}} 1 \right. \\
&+ \sum_{\substack{m\leqslant x\\ m=m_1m_2\\ (m_1,p_1^{\nu_{p_1}(n)}p_3^{\nu_{p_3}(n)})=1\\ \mathrm{core}(m_2)\mid p_1^{\nu_{p_1}(n)}p_3^{\nu_{p_3}(n)}\\ (m_2,p_1^{\nu_{p_1}(n)})_k>1\\ (m_2,p_3^{\nu_{p_3}(n)})_k>1}} 1 + \cdots \\
&+ \left. \sum_{\substack{m\leqslant x\\ m=m_1m_2\\ (m_1,p_{s-1}^{\nu_{p_{s-1}}(n)}p_s^{\nu_{p_s}(n)})=1\\ \mathrm{core}(m_2)\mid p_{s-1}^{\nu_{p_{s-1}}(n)}p_s^{\nu_{p_s}(n)}\\ (m_2,p_{s-1}^{\nu_{p_{s-1}}(n)})_k>1\\ (m_2,p_s^{\nu_{p_s}(n)})_k>1}} 1 \right) + \cdots + (-1)^s \\
&\cdot \left(\sum_{\substack{m\leqslant x\\ m=m_1m_2\\ (m_1,n)=1\\ \mathrm{core}(m_2)\mid n\\ (m_2,p_1^{\nu_{p_1}(n)})_k>1\\ (m_2,p_2^{\nu_{p_2}(n)})_k>1\\ \vdots\\ (m_2,p_s^{\nu_{p_s}(n)})_k>1}} 1 \right),
\end{aligned} \tag{17}$$

where

$$n = \prod_{l=1}^{s} p_l^{\nu_{p_l}(n)}, \tag{18}$$

with p_l being an appropriately indexed set of primes, and we use the fact that, for d = 1, the sum is 0.

This simplifies to

$$\begin{aligned}
&- \sum_{d\in D^1(n)} \mu_1(d) \sum_{\substack{m_2:\mathrm{core}(m_2)\mid d\\ \forall p\mid d,(m_2,p^{\nu_p(d)})_k>1}} \sum_{\substack{m_1\leqslant x/m_2\\ (m_1,d)=1}} 1 \\
&= - \sum_{d\in D^1(n)} \mu_1(d) \sum_{\substack{m_2:\mathrm{core}(m_2)\mid d\\ \forall p\mid d,(m_2,p^{\nu_p(d)})_k>1}} \varphi_{0,0}\left(\frac{x}{m_2}, d\right) == \\
&- \sum_{d\in D^1(n)} \mu_1(d) \\
&\cdot \sum_{\substack{m_2:\mathrm{core}(m_2)\mid d\\ \forall p\mid d,(m_2,p^{\nu_p(d)})_k>1}} \left(x\frac{\varphi(d)}{dm_2} + \mathcal{O}(\tau(d)) \right) == \\
&- x \sum_{d\in D^1(n)} \mu_1(d) \frac{\varphi(d)}{d} \sum_{\substack{m_2:\mathrm{core}(m_2)\mid d\\ \forall p\mid d,(m_2,p^{\nu_p(d)})_k>1}} \frac{1}{m_2} \\
&+ \mathcal{O}\left(\sum_{d\in D^1(n)} \mu_1(d)\tau(d)\frac{\varphi(d)}{d} \right. \\
&\cdot \left. \sum_{\substack{m_2:\mathrm{core}(m_2)\mid d\\ \forall p\mid d,(m_2,p^{\nu_p(d)})_k>1}} 1 \right) == \\
&- \sum_{d\in D^1(n)} \frac{\varphi(d)\mu_1(d)}{d} K_k(d) + \mathcal{O}\left(\tilde{\tau}_k(n)\tau_1(n)\tau(n)\right),
\end{aligned} \tag{19}$$

and our result follows. □

We let $L_k(n)$ be the coefficient appearing in front of the "x" term in $\varphi_{k,0}(x,n)$ and let $E_k(n)$ be the function in the error term, so that $\varphi_{k,0}(x,n) = xL_k(n) + \mathcal{O}(E(n))$.

Remark 7. Regarding the function $\widetilde{\tau}_k(n)$, we may estimate that $\widetilde{\tau}_k(n) \leqslant N_k^\star$ through the following reasoning: for even k, consider $a \leqslant N_k^\star$ (see Definition 4). If $b \leqslant N_k^\star$, then $(p^b, p^a)_k = 1$ for at most $N_k^\star - 1$ such b (excluding $b = a$). If $b > N_k^\star$, then $(p^b, p^a)_k = 1$ for at most 1 such b (the case of $D^k(p^a) = \{1, p^a\}$ and $b = 2a > N_k^\star$, as here $D^k(p^b) = D^2(p^b)$). Thus, $\widetilde{\tau}_k(p^a) \leqslant N_k^\star$ for $a \leqslant N_k^\star$. For $a > N_k^\star$, $D^k(p^a) = D^2(p^a)$, and so $(p^b, p^a)_k = 1$ for at most 2 such b by the above comments.

For odd k, a similar argument gives $(p^b, p^a)_k > 1$ for at most $N_k^\star$ choices of b when $a \leqslant N_k^\star$, and precisely 2 choices of b when $a > N_k^\star$; namely, $b = 0$ and $b = a$. So $N_k^\star$ bounds $\widetilde{\tau}_k(p^a)$ for all a.

Now, $\tau_k(p) = 2$ for all k, and for some $B_k \leqslant N_k^\star$, $2^{B_k} \geqslant N_k^\star \geqslant \widetilde{\tau}_k(n)$. Thus, since $\tau_k(p^a) \geqslant 2$ for all $a > 0$, we have that $\tau_k(n)^{B_k} \geqslant \widetilde{\tau}_k(n)$. In particular, there is a least B_k such that, for all n, $\tau_k(n)^{B_k} \geqslant \widetilde{\tau}_k(n)$. We will use this B_k in our asymptotic estimates.

We immediately get the following result as a consequence of Theorem 6.

Corollary 8. $\varphi_{k,r}(x,n) = (x^r/(r+1))\varphi_{k,0}(x,n)$ *for* $r \in \mathbb{N}$.

Proof. The case $r = 0$ is trivially true. We prove for each $r > 0$ using Stieltjes Integration. Then

$$\begin{aligned}
&\left(\sum_{\substack{m\leqslant x\\(m,n)_k=1}} m^r\right) + \mathcal{O}(x^r)\\
&\quad= r\int_0^x \left(\sum_{\substack{m\leqslant y\\(m,n)_k=1}} m^{r-1}\right) dy\\
&\quad= r\int_0^x \left(y^r \frac{\varphi(n)}{n} K_k(n) + \mathcal{O}\left(y^{r-1}E_k(n)\right)\right) dy\\
&\quad= \frac{x^{r+1}}{r+1}\frac{\varphi(n)}{n}K_k(n) + \mathcal{O}\left(x^r E_k(n)\right)\\
&\quad= \frac{x^r}{r+1}\varphi_{k,0}(x,n),
\end{aligned} \tag{20}$$

where the error term from the integral is absorbed by $\mathcal{O}(x^r E_k(n))$. □

Theorem 9. *Let* $k \geqslant 0$. *Suppose that an arithmetical function* f *is of the form*

$$f(n) = \sum_{d|_k n} g(d)\left(\frac{n}{d}\right)^r, \tag{21}$$

with $k > 0$ *and* $r \in \mathbb{N}$ *and* $g(n)$ *is* $\mathcal{O}(n^s)$, *with* $s \leqslant r-1$. *Then*

$$\sum_{n\leqslant x} f(n) = \frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{g(n)L_k(n)}{n^{r+1}} + \mathcal{O}\left(x^r(\log x)^{B_k+2}\right) \tag{22}$$

Proof. Let k, r, and s be given. Then

$$\begin{aligned}
\sum_{n\leqslant x} f(n) &= \sum_{n\leqslant x}\sum_{d|_{k+1}n} g(d)\left(\frac{n}{d}\right)^r\\
&= \sum_{n\leqslant x}\sum_{\substack{dd'=n\\(d,d')_k=1}} g(d)\left(\frac{n}{d}\right)^r\\
&= \sum_{\substack{dd'\leqslant x\\(d,d')_k=1}} g(d)\left(\frac{n}{d}\right)^r\\
&== \sum_{d\leqslant x} g(d) \sum_{\substack{d'\leqslant x/d\\(d,d')_k=1}} (d')^r\\
&= \sum_{d\leqslant x} g(d)\varphi_{k,r}\left(\frac{x}{d}, d\right)\\
&= \sum_{d\leqslant x} g(d)\varphi_{k,r}\left(\frac{x}{d}, d\right)\\
&== \frac{x^{r+1}}{r+1}\sum_{d\leqslant x}\frac{g(d)L_k(d)}{d^{r+1}}\\
&\quad + \mathcal{O}\left(\sum_{d\leqslant x} x^r g(d)\frac{E_k(d)}{d^r}\right).
\end{aligned} \tag{23}$$

By our remark above, we may find B_k such that $\widetilde{\tau}_k(n) \leqslant (\tau_k(n))^{B_k}$ for all n, which enables us to estimate

$$\begin{aligned}
&\mathcal{O}\left(x^r\sum_{n\leqslant x}\frac{g(n)E_k(n)}{n^r}\right) \leqslant \mathcal{O}\left(x^r\sum_{n\leqslant x}\frac{(\tau(n))^{B_k+2}}{n}\right)\\
&\quad= \mathcal{O}\left(x^r(\log x)^{B_k+2}\right),
\end{aligned} \tag{24}$$

where we use the fact that g is $\mathcal{O}(n^s)$ with $s \leqslant r-1$. Also,

$$\begin{aligned}
\frac{x^{r+1}}{r+1}\sum_{n\leqslant x}\frac{g(n)L_k(n)}{n^{r+1}} &= \frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{g(n)L_k(n)}{n^{r+1}}\\
&\quad - \frac{x^{r+1}}{r+1}\sum_{n>x}\frac{g(n)L_k(n)}{n^{r+1}},
\end{aligned} \tag{25}$$

and since g is $\mathcal{O}(n^s)$ and $L_k(n)$ is bounded, the infinite sums converge, but

$$\frac{x^{r+1}}{r+1}\sum_{n>x}\frac{g(n)L_k(n)}{n^{r+1}} = \mathcal{O}\left(x^{s+1}\right), \tag{26}$$

and $s < r$, so this is absorbed into our error term and we have our result. □

Note that, for each k and for all $\epsilon > 0$, we may state our error term for the summatory function as $\mathcal{O}(x^{r+\epsilon})$, where the multiplicative constant implied by the Big-Oh notation depends only on ϵ and k. This enables us to achieve roughly the same error as Cohen [7], albeit not as asymptotically strong as $x^r(\log x)^{B_k}$. However, as k tends to infinity, our error becomes unbounded, and so we cannot achieve Cohen's result for the infinitary case.

We recall that μ_k, the k-ary analogue of the Möbius function, is defined recursively via

$$\mu_k\left(p^0\right) = 1, \qquad \sum_{p^b \|_k p^a} \mu_k\left(p^b\right) = 0, \tag{27}$$

which is extended to all n by making μ_k multiplicative. We then have the following.

Lemma 10. *Let k be given. Then there is a constant C_k depending only on k such that, for each k, $|\mu_k(n)| \leqslant \tau(n)^{C_k}$.*

Proof. By Theorem 3, for each prime p and each k, there is an $N_k^\star$ that ensures $D^k(p^a) = D^1(p^a)$ or $D^2(p^a)$, depending on the parity of k, for all $a > N_k^\star$. The values of $|\mu_k(p^a)|$ for $a \leqslant 2N_k^\star$ are finite, being generated from a finite recursion. For odd k with $a > N_k^\star$, $\mu_k(p^a) = -1$. For even k and even a, with $a > 2N_k^\star + 1$,

$$\begin{aligned} -\mu_k\left(p^a\right) &= \sum_{\substack{p^b \|_k p^a \\ b \neq a}} \mu_k\left(p^b\right) \\ &= \sum_{p^b \|_k p^{a-1}} \mu_k\left(p^b\right) - \mu_k\left(p^{a/2}\right), \end{aligned} \tag{28}$$

since $D^k(p^a) = D^2(p^a)$ and $D^k(p^{a-1}) = D^2(p^{a-1}) = D(p^{a-1})$, as $a - 1$ is odd. But

$$\sum_{p^b \|_k p^{a-1}} \mu_k\left(p^b\right) = 0, \tag{29}$$

so $\mu_k(p^a) = \mu_k(p^{a/2})$. Also,

$$\begin{aligned} -\mu_k\left(p^{a+1}\right) &= \sum_{\substack{p^b \|_k p^{a+1} \\ b \neq a+1}} \mu_k\left(p^b\right) \\ &= \sum_{p^b \|_k p^a} \mu_k\left(p^b\right) + \mu_k\left(p^{a/2}\right) = \mu_k\left(p^{a/2}\right), \end{aligned} \tag{30}$$

so

$$\mu_k\left(p^a\right) = \mu_k\left(p^{a/2}\right) = -\mu_k\left(p^{a+1}\right) \tag{31}$$

for even a. Hence, $\mu_k(p^a)$ is bounded in absolute value for each k—call this bound 2^{C_k}—and so $|\mu_k(n)| \leqslant 2^{C_k \omega(n)} = \tau_1(n)^{C_k} \leqslant \tau(n)^{C_k}$ and we are done. □

We will analyze the summatory functions for the k-ary analogues of several well-known families of arithmetical functions:

(i) The k-ary divisor sum functions:

$$\sigma_{k,r}(n) := \sum_{d \|_k n} d^r \tag{32}$$

(ii) The k-ary Jordan totient functions:

$$J_{k,r}(n) := \sum_{d \|_k n} \mu_k\left(\frac{n}{d}\right) d^r \tag{33}$$

(iii) The k-ary Dedekind functions:

$$\psi_{k,r}(n) := \sum_{d \|_k n} \left|\mu_k\left(\frac{n}{d}\right)\right| d^r \tag{34}$$

Here r denotes a positive integer. By Lemma 10, we may apply Theorem 9 to the Jordan and Dedekind functions of order $r > 0$ without issue, since $\mu_k(n)$ is logarithmic in n; the summatory functions for the divisor sum functions carry no special restriction on r aside from it being a positive integer:

(i)

$$S\left(\sigma_{k,r}\right)(x) = \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{L_k(n)}{n^{r+1}} + \mathcal{O}\left(x^r (\log x)^{B_k}\right) \tag{35}$$

(ii)

$$\begin{aligned} S\left(J_{k,r}\right)(x) = {} & \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{\mu_k(n) L_k(n)}{n^{r+1}} \\ & + \mathcal{O}\left(x^r (\log x)^{B_k}\right) \end{aligned} \tag{36}$$

(iii)

$$\begin{aligned} S\left(\psi_{k,r}\right)(x) = {} & \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{\left|\mu_k(n)\right| L_k(n)}{n^{r+1}} \\ & + \mathcal{O}\left(x^r (\log x)^{B_k}\right) \end{aligned} \tag{37}$$

By Theorems 2 and 3, for each n, the sequence $\{L_{2k}(n)\}_{k=0}^{\infty}$ (resp., $\{L_{2k+1}(n)\}_{k=0}^{\infty}$) is monotonically increasing (resp., decreasing). Both sequences must have the same limit, $L_\infty(n)$, which one can identify with the function $K_\infty(n)$ from Cohen's manuscript. However, we cannot obtain the function $\varphi_{\infty,0}(x,n)$ via a limit as k tends to infinity of $\varphi_{k,0}(x,n)$, as the error term grows without bound in k. A new approach will likely be needed in order to unify the infinite case with the finite cases.

Data Availability

All data used either was obtained via the cited sources or is derived explicitly from first principles.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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