

Research Article

Treating Transcendental Functions in Partial Differential Equations Using the Variational Iteration Method with Bernstein Polynomials

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Received 25 November 2018; Revised 3 January 2019; Accepted 11 February 2019; Published 3 March 2019

Academic Editor: Shyam L. Kalla

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We aim through this paper to present an improved variational iteration method (VIM) based on Bernstein polynomials (BP) approximations to be used with transcendental functions. The key benefits gained from this modification are to reach stable and fairly accurate results and, at the same time, to expand the unknown function's domain in partial differential equations (PDEs). The proposed approach introduces the Bernstein polynomials in the transcendental functions of nonlinear PDEs. A number of examples were included in order to expound the method's capacity and reliability. From the results, we conclude that the VIM with BP is a powerful mathematical tool that can be applied to solve nonlinear PDEs.

1. Introduction

It is common knowledge that a large number of phenomena are in essence nonlinear and therefore can be modeled using nonlinear differential equations [1]. However, it is not possible to find exact solutions to nonlinear differential equations. Therefore, the approach to tackle this shortcoming was to apply different numerical methods suffice to reach approximate solutions including the methods of Adomian decomposition [2, 3], homotopy perturbation [4–6], differential transform [7–9], and variational iteration [10, 11].

He [12] introduced the VIM in 1998, and since then it was applied by many scientists as an approach to solve mathematical and physical problems for its capability to simplify a difficult problem into a readily solvable problem. The VIM is considered a capable tool that can be used to solve functional equations. Commonly used numerical approaches such as finite difference or characteristics methods require tedious computations. These methods are often less accurate in their results because of their round-off error. Solutions for Schrodinger equations based on the widely used analytical methods are very constrained and can be only applied

in certain scenarios while they are inapplicable to obtain solutions for a large number of equations that model real life scenarios. The VIM is a modified Lagrange multiplier method with promising results in solving numerous classes of nonlinear problems with approximations that converge to the correct solution within a short number of iterations.

Bernstein polynomials' role is prominent in countless mathematics applications as they are utilized in solving differential equations in addition to their contribution to the approximation theory. Bézier curves, which are Bernstein polynomials restricted to the interval $x \in [a, b]$, played a more prominent role with the development of computer graphics [13].

A novel iterative approach based on the VIM is applied to differential equations by [14]. Olayiwola Mo [15] applied it to solve the Convection-Diffusion equation and for the class of fractional Convection-Diffusion equation [16]. The modified variational iteration method for sine-Gordon equation is suggested in [17] using Chebyshev polynomials and He's polynomials [18].

In this paper, we adjusted the VIM to utilize the Bernstein polynomials so as to approximate the correction functions'

nonlinear terms. The solutions obtained using our modified method are comparable to VIM. In our approach, we applied the Bernstein polynomial approximations to nonlinear functions in the correction functions before iterating the numerical solution of PDEs. The main objective is to achieve reliable accurate PDEs solutions if using VIM leads to unstable solutions. The feasibility of our approach is expounded through a number of examples.

The rest of the paper is divided into five sections. Section 2 introduces brief ideas of VIM. Section 3 presents one- and two-dimensional Bernstein polynomials. In Section 4, an analysis of the convergence is presented. Section 5 elucidates three examples that expound the feasibility of using VIM with the Bernstein polynomials. Section 6 is dedicated for the conclusions.

2. VIM Basics

The concept of VIM is based on the general Lagrange’s multiplier method. The key advantage of this method is reaching a mathematical problem solution by linearization assumption to be used as an initial approximation that promptly converges to an exact solution [19].

The following nonlinear differential equation elucidates the basic concept of the VIM, as follows:

$$L(y) + N(y) = g(x) \tag{1}$$

where L and N are the linear and nonlinear operators, respectively, while g(x) is an inhomogeneous term. VIM is used to formalize the following correction function:

$$\begin{aligned}
 y_{n+1}(x, t) &= y_n(x, t) \\
 &+ \int_0^t \lambda(\tau) [L(y_n(x, t)) + N(\bar{y}_n(x, t)) - g(x)] d\tau
 \end{aligned} \tag{2}$$

where $\lambda(\tau)$ is a general Lagrangian multiplier which can be determined through the variational theory and integration by parts, \bar{y}_n represents a restricted variation (i.e. $\delta \bar{y}_n = 0$), and the subscript n symbolizes the nth-order approximation.

The Lagrange multiplier $\lambda(\tau)$ is optimally determined through integration by parts. The consecutive approximations y_{n+1} , $n \geq 0$, of the solution $y_n(x, t)$ are determined using the Lagrange multiplier from the first step using any selective function y_0 . Thus, the solution will be as follows.

$$y(x, t) = \lim_{n \rightarrow \infty} y_n \tag{3}$$

3. Bernstein Polynomials

Polynomials are considered easily defined, calculated, differentiated, and integrated mathematical tools. The Bernstein based polynomials are tested to approximate the functions as they result in improved approximations to a function with a few terms. This feature has led to their widespread use in applied mathematics, physics, and computer based geometric

designs, combined with other methods like Galerkin and collocation to solve both differential and integral equations.

The nth degree Bernstein Polynomials form a complete basis over [0, 1], formalized as

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad 0 \leq i \leq n \tag{4}$$

in which $\binom{n}{i} = n!/i!(n-i)!$ represent the binomial coefficients.

Based on [13], we use the generalized Bernstein polynomials of nth degree which form a complete basis over [a, b] as follows.

$$B_{i,n}(x) = \frac{1}{(b-a)^n} \binom{n}{i} (x-a)^i (b-x)^{n-i}, \quad 0 \leq i \leq n \tag{5}$$

If the function f of two real variables over the interval square are

$$S : [a, b] \times [c, d] \tag{6}$$

then the two-variable Bernstein polynomial of (n, m) degree, equivalent to the function f, is determined in the following formula [20]:

$$(B_{n,m}f)(x, t) = \sum_{i=0}^n \sum_{k=0}^m f\left(\frac{i}{n}, \frac{k}{m}\right) P_{i,n}(x) P_{k,m}(t) \tag{7}$$

where

$$P_{j,s}(z) = \binom{s}{j} z^j (1-z)^{s-j} \tag{8}$$

We use (7) to convert the transcendental functions (trigonometric, exponential, and Logarithm functions) that appear in homogenous and nonhomogenous PDEs, such as the sine-Gordon equation, into the series approximation by means of Bernstein polynomial.

The following nonlinear partial differential equation contains the transcendental functions term as follows:

$$L(y) + N(y) + T(y) = g(x) \tag{9}$$

such that T(y) is the transcendental functions.

Using (2), we have the following.

$$\begin{aligned}
 y_{n+1}(x, t) &= y_n(x, t) + \int_0^t \lambda(\tau) [L(y_n(x, t)) \\
 &+ N(\bar{y}_n(x, t)) + T(\bar{y}_n(x, t)) - g(x)] d\tau
 \end{aligned} \tag{10}$$

Therefore, we approach the transcendent functions $T(\bar{y}_n(x, t))$ using Bernstein’s approximation, because it can only be integrated by numerical methods.

$$T(\bar{y}_n(x, t)) = B_{i,j} y_n(x, t) \tag{11}$$

4. Analysis of the Convergence

The variational iteration method converts the partial differential equations to a recurrence sequence of functions. The limit of that sequence leads to the solution of the PDE.

Now, let

$$F\left(y, T(\bar{y}_n), \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 y}{\partial x \partial t}\right) = g(x) \quad (12)$$

be the second- order partial differential equation and $T(\bar{y}_n(x, t))$ be the transcendent functions. Using VIM with Bernstein polynomial in (12), we have

$$y_{n+1}(x, t) = y_n(x, t) + \int_0^t \lambda \left[F\left(y_n, B_{n,m}(\bar{y}_n), \frac{\partial y_n}{\partial \tau}, \frac{\partial \bar{y}_n}{\partial x}, \frac{\partial^2 \bar{y}_n}{\partial x^2}, \frac{\partial^2 y_n}{\partial \tau^2}, \frac{\partial^2 \bar{y}_n}{\partial x \partial \tau}\right) - g(x) \right] d\tau \quad (13)$$

such that \bar{y}_n is He's monographs, i.e., $\delta(\bar{y}_n) = 0$ [21]. To find the value λ , (13) becomes as follows.

$$\delta y_{n+1}(x, t) = \delta y_n(x, t) + \delta \int_0^t \lambda \left[F\left(y_n, B_{n,m}(\bar{y}_n), \frac{\partial y_n}{\partial \tau}, \frac{\partial \bar{y}_n}{\partial x}, \frac{\partial^2 \bar{y}_n}{\partial x^2}, \frac{\partial^2 y_n}{\partial \tau^2}, \frac{\partial^2 \bar{y}_n}{\partial x \partial \tau}\right) - g(x) \right] d\tau \quad (14)$$

In fact the solution of (14) is considered as the fixed point with the initial condition $y_0(x, t)$ [21].

Theorem 1 (see [22]). *Let a sequence q_n satisfy $0 < q_n < 1$ and $q_n \rightarrow 1$ as $n \rightarrow \infty$. Then for any function $f \in C[0, 1]$,*

$$B_n(f, q_n; x) \rightarrow f(x), \quad [x \in [0, 1]; n \rightarrow \infty]. \quad (15)$$

In other words, $B_n(f, q_n; x)$ is Bernstein polynomials and uniformly convergent to $f(x)$.

Theorem 2 (see [21] (Banach's fixed point theorem)). *Assume that X is a Banach space and*

$$A : X \rightarrow X \quad (16)$$

is a nonlinear mapping, and suppose that

$$\|A[y] - A[\bar{y}]\| \leq \mu \|y - \bar{y}\|, \quad y, \bar{y} \in X, \quad (17)$$

for some constant $\mu < 1$. Then A has a unique fixed point. Furthermore, the sequence

$$y_{n+1} = A[y_n], \quad (18)$$

with an arbitrary choice of $y_0 \in X$, converges to the fixed point of A and

$$\|y_k - y_l\| \leq \|y_1 - y_0\| \sum_{j=l-1}^{k-2} \mu^j, \quad (19)$$

and, according to Theorem 2, for the nonlinear mapping

$$A[y] = y(x, t) + \int_0^t \lambda \left[F\left(y, \frac{\partial y}{\partial \tau}, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial \tau^2}, \frac{\partial^2 y}{\partial x \partial \tau}\right) \right] d\tau, \quad (20)$$

a sufficient condition for convergence of the variational iteration method is strictly contraction of A .

Corollary 3. *Using Theorems 1 and 2 we conclude that*

$$A[y] = y(x, t) + \int_0^t \lambda \left[F\left(y, B_{n,m}(y), \frac{\partial y}{\partial \tau}, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial \tau^2}, \frac{\partial^2 y}{\partial x \partial \tau}\right) - g(x) \right] d\tau \quad (21)$$

such that $B_{n,m}(y)$ is Bernstein approximation; then the sufficient condition for convergence of the variational iteration method is strictly contraction of A . Furthermore, the sequence (13) converges to the fixed point of A which is also the solution of the partial differential equation (9).

5. Numerical Examples

In this section, the following three examples of nonlinear PDEs are solved using the VIM with Bernstein polynomials. The results are generated using Maple 13. The accuracy of the results is determined when compared to the exact solutions.

Example 1. Let us take the sine-Gordon equation [17].

$$y_{tt} - y_{xx} + \sin(y) = 0 \quad t > 0; \\ y(x, 0) = 0; \quad (22)$$

$$y_t(x, 0) = \frac{8}{e^x + e^{-x}};$$

the exact solution of (9) is as follows.

$$y(x, t) = 4 \arctan\left(\frac{2t}{e^x + e^{-x}}\right) \quad (23)$$

Applying the Bernstein polynomials approximation for $\sin(y)$ in (7) with $n=m=2$ results in the following.

$$y_0 = \frac{8t}{e^x + e^{-x}} \\ \sin(y_0) = 1.818594854t + 0.2808027720x \\ - 0.1743777930tx^2 - 2.575397349t^2 \\ + 0.4435096958t^2x \\ + 0.7290227026t^2x^2 \quad (24)$$

Applying the VIM on (22), we get

$$y_{n+1}(x, t) = y_n + \int_0^t (s-t)(y_{ss} - y_{xx} + \sin(y_0)) ds, \tag{25}$$

$n \geq 1$

where the Lagrange multiplier is as follows.

$$\lambda(s) = s - t \tag{26}$$

Consider the first iterate numerical solution.

$$y_0 = \frac{8t}{e^x + e^{-x}}$$

$$y_1(x, t) = -\frac{1}{(e^{1.000000063x} + e^{-1.000000063x})^3} (1 \times 10^{-11}) 1.6$$

$$\times 10^{12}s + 8.000001 \times 10^{11}t^3 + 3.69591413$$

$$\times 10^9 t^4 x e^{3.000000189x} - 6.438493370$$

$$\times 10^{10} t^4 e^{1.000000063x} - 6.43849337$$

$$\times 10^{10} t^4 e^{-1.000000063x} + 9.09297427$$

$$\times 10^{10} t^3 e^{1.000000063x} + 9.09297427$$

$$\times 10^{10} t^3 e^{-1.000000063x} + \dots \tag{27}$$

$$y_2(x, t) = -\frac{1}{(e^{1.000000063x} + e^{-1.000000063x})^5} (2 \times 10^{-20}t)$$

$$\cdot 6.66666745 \times 10^{20}t^2 - 7.6666686 \times 10^{20}t^4 - 1$$

$$\times 10^{10}t^4 x e^{3.000000189x} - 1.6 \times 10^{21} e^{2.000000126x}$$

$$- 1.45314825 \times 10^{18}t^4 e^{1.000000063x} - 1.45314825$$

$$\times 10^{18}t^4 e^{-1.000000063x} - 8.33610388$$

$$\times 10^{19}t^3 e^{1.000000063x} - 8.33610388$$

$$\times 10^{19}t^3 e^{-1.000000063x}$$

The absolute error of VIM-Bernstein and the exact solution are presented in Tables 1-2 and Figures 1-2.

Tables 1-2 present the absolute error and mean square error of VIM with modified Bernstein polynomial when $n=m=2$ and $t = 0.1$ in Table 1 and $n=m=1$ and $x = 0.1$ in Table 2. The maximum errors generated using the modified Bernstein polynomial are of 10^{-4} .

Example 2. Consider the PDE [23]:

$$y_t = -e^y y_x + k(y_x)^2 + ky_{xx} + e^{-y}R(x, t); \tag{28}$$

$$y(x, 0) = g(x)$$

TABLE 1: The absolute and mean square errors using y_2 for Sine-Gordon equation when $n=m=2$ and $t = 0.1$.

X	VIM-Berns.	Exact	VIMB-Exact
0.1	0.397057362	0.396702532	3.548303×10^{-4}
0.2	0.391225182	0.390882208	3.429731×10^{-4}
0.3	0.381815965	0.381490269	3.256956×10^{-4}
0.4	0.369256745	0.368953065	3.036803×10^{-4}
0.5	0.354079776	0.353801996	2.777801×10^{-4}
0.6	0.336872278	0.336623332	2.489467×10^{-4}
0.7	0.318228605	0.318010464	2.1814014×10^{-4}
0.8	0.298710758	0.298524478	1.862801×10^{-4}
0.9	0.278820327	0.278666139	1.541880×10^{-4}
1.0	0.258982287	0.258859720	1.225671×10^{-4}
MSE			$7.020528502 \times 10^{-8}$

TABLE 2: The absolute and mean square errors using y_3 for Sine-Gordon equation when $n=m=1$ and $x = 0.1$.

t	VIM-Berns.	Exact	VIMB-Exact
0.01	0.03980022977	0.03979951651	7.132600×10^{-7}
0.02	0.07959685852	0.07959115424	5.704280×10^{-6}
0.03	0.1193862874	0.1193670439	1.924350×10^{-5}
0.04	0.1591649208	0.1591193346	4.558630×10^{-5}
0.05	0.1989291693	0.1988402037	8.896560×10^{-5}
0.06	0.2386754506	0.2385218656	1.535849×10^{-4}
0.07	0.2784001917	0.2781565807	2.436109×10^{-4}
0.08	0.3180998309	0.3177366650	3.631659×10^{-4}
0.09	0.3577708202	0.3572544980	5.163222×10^{-4}
0.1	0.3974096254	0.3967025322	7.070931×10^{-4}
MSE			$8.825264703 \times 10^{-7}$

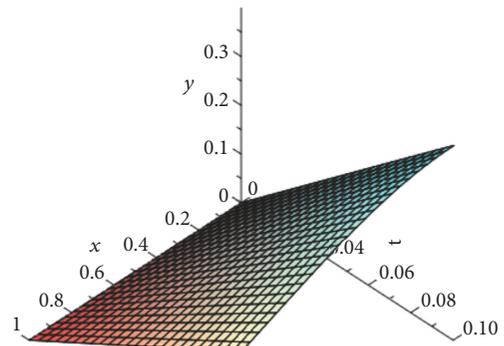


FIGURE 1: Numerical solution for (22) using VIMB when $n=m=2$ and $t=0.1$.

where k is constant and $R(x, t)$ is a given function. Thus, applying the Bernstein approximation when $n=m=8$:

$$y_0 = Ln(x), \quad x \neq 0$$

$$e^{y_0} = 0.000004448t^8 x^8 + 0.000645068t^8 x^2$$

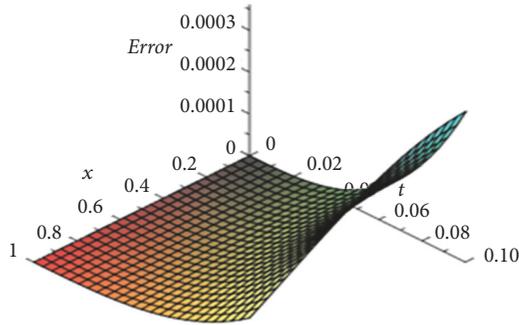


FIGURE 2: Absolute error for (22) using VIMB when n=m=2 and t=0.1.

$$\begin{aligned}
 & - 0.000047808t^8x^7 - 0.00023466t^8x \\
 & - 0.00100314t^8x^3 + 0.0009657t^8x^4 \\
 & - 0.000589584t^8x^5 + 0.0002232024t^8x^6 \\
 & + 0.00401256t^7x^3 + 0.000938864t^7x \\
 & - 0.002580272t^7x^2 - 0.0038628t^7x^4 + \dots \\
 e^{-y_0} = & 3.046076038 + 0.0000011442t^8x^8 \\
 & + 0.0005971496t^8x^2 - 0.0000153256t^8x^7 \\
 & - 0.0002611536t^8x - 0.0007634192t^8x^3 \\
 & + 0.000597954t^8x^4 - 0.0002945112t^8x^5 \\
 & + 0.0000893456t^8x^6 + 0.0030536768t^7x^3 \\
 & + 0.0010446144t^7x - 0.0023885984t^7x^2 \\
 & - 0.002391816t^7x^4 + \dots
 \end{aligned} \tag{29}$$

and the variational iteration method on (28) with $\lambda = -1$, $k = 1$, $R(x, t) = (1 + x + t)$, and $t \in [0, 1]$, $x \in [1, 2]$:

$$\begin{aligned}
 y_{n+1}(x, t) &= y_n(x, t) \\
 & - \int_0^t \left(-e^y y_x + (y_x)^2 + y_{xx} + e^{-y} (1 + x + t) \right) ds \tag{30} \\
 y_0 &= Ln(x) \\
 y_1 &= \frac{1}{x} \left(1 \times 10^{-16} \right) - 4.10577777810^{10}t^9 + 4.88162 \\
 & \times 10^{10}t^{10}x + 8.93456 \times 10^{10}t^{10}x^7 - 1.53256
 \end{aligned}$$

TABLE 3: The absolute and mean square errors using y_1 for (28) when n=m=8 and t = 0.01.

X	VIM-Berns.	Exact	VIMB-Exact
1.1	0.104613190	0.104360015	2.531755×10^{-4}
1.2	0.190945182	0.190620359	3.248227×10^{-4}
1.3	0.270371291	0.270027137	3.441544×10^{-4}
1.4	0.343920261	0.343589704	3.305573×10^{-4}
1.5	0.412406467	0.412109650	2.968162×10^{-4}
1.6	0.476485538	0.476234179	2.513593×10^{-4}
1.7	0.536693084	0.536493370	1.997142×10^{-4}
1.8	0.593472312	0.593326845	1.454676×10^{-4}
1.9	0.647194150	0.647103242	9.090796×10^{-5}
2.0	0.698172179	0.698134722	3.745730×10^{-5}
MSE			6.193132×10^{-8}

$$\begin{aligned}
 & \times 10^{10}t^{10}x^8 + 1.1442 \times 10^9t^{10}x^9 - 2.945112 \\
 & \times 10^{11}t^{10}x^6 - 2.611536 \times 10^{11}t^{10}x^2 5.971496 \\
 & \times 10^{11}t^{10}x^3 - 7.634192 \times 10^{11}t^{10}x^4 + 5.97954 \\
 & \times 10^{11}t^{10}x^5 + \dots
 \end{aligned} \tag{31}$$

the exact solution for (28) is as follows.

$$y(x, t) = Ln(x + t) \tag{32}$$

Tables 3-4 present the absolute error and mean square error of VIM with modified Bernstein polynomial when n=m=8 and t = 0.01 in Table 3 and n=m=2 and x = 2 in Table 4. The absolute errors generated using the modified Bernstein polynomial are of 10^{-4} .

Example 3. The inhomogeneous nonlinear equation (28) with initial condition [23]

$$\begin{aligned}
 y_t &= F(x, t) - e^y y_x + k(y_x)^2 + ky_{xx} \\
 & + e^{-y}R(x, t); \tag{33}
 \end{aligned}$$

$$y(x, 0) = 1$$

where $F(x, t) = -2t + x^2(1 - 4t^2) - e^{-(1+tx^2)} + 2txe^{1+tx^2}$ and $R(x, t) = 1$ is a given function, $k=1$.

Using the Bernstein approximation when n=m=2 we get the following.

$$\begin{aligned}
 F_0(x, t) &= 0.5x^2 + 0.5x^2 + 5.333341592xt \\
 & + 1.919535110tx^2 - 0.1899063900t^2x \\
 & + 3.947686046t^2x^2 - 2 \times 10^{-10}t^2 \\
 & - 0.3678794412 - 2.t
 \end{aligned}$$

TABLE 4: The absolute and mean square errors using y_3 for (28) when $n=m=2$ and $x = 2$.

t	VIM-Berns.	Exact	VIMB-Exact
0.01	0.6981473299	0.6981347221	1.260780×10^{-5}
0.02	0.7031437324	0.7030975114	4.622100×10^{-5}
0.03	0.7081301493	0.7080357931	9.435620×10^{-5}
0.04	0.7131001195	0.7129498079	1.503116×10^{-4}
0.05	0.7180469602	0.7178397932	2.071670×10^{-4}
0.06	0.7229637696	0.7227059828	2.577866×10^{-4}
0.07	0.7278434274	0.7275486073	2.948201×10^{-4}
0.08	0.7326785978	0.7323678937	3.107041×10^{-4}
0.09	0.7374617300	0.7371640660	2.976640×10^{-4}
0.1	0.7421850610	0.7419373447	2.477163×10^{-4}
MSE			2.330795×10^{-8}

TABLE 5: The absolute and mean square errors using y_2 for (33) when $n=m=2$ and $t = 0.01$.

X	VIM-Berns.	Exact	VIMB-Exact
0.1	1.00045061	1.00010000	3.506120×10^{-4}
0.2	1.00111203	1.00040000	7.120320×10^{-4}
0.3	1.00187539	1.00090000	9.753990×10^{-4}
0.4	1.00274069	1.00160000	1.1460690×10^{-3}
0.5	1.00370788	1.00250000	1.207884×10^{-3}
0.6	1.00477695	1.00360000	1.176957×10^{-3}
0.7	1.005947880	1.00490000	1.047886×10^{-3}
0.8	1.00722064	1.00640000	8.206490×10^{-4}
0.9	1.00859522	1.00810000	4.952220×10^{-4}
1.0	1.01007158	1.01000000	7.158300×10^{-5}
MSE			7.748606×10^{-7}

$$\begin{aligned}
 y_1(x, t) = & 1 - 0.06330212999t^3x + 1.315895349t^3x^2 \\
 & - 6.666666666 \times 10^{-11}t^3 \\
 & + 2.666670796t^2x + 0.9597675550t^2x^2 \\
 & - 1.t^2 - 2.32 \times 10^{-8}t + 0.5tx^2 + 0.5xt
 \end{aligned} \tag{34}$$

$$\begin{aligned}
 e^{y_1} = & 2.718282 + 7.241711382xt \\
 & + 8.737391384tx^2 + 9.650455035t^2x \\
 & + 330.8334627t^2x^2 - 0.5157182076t^2 \\
 & - 1.202563816t
 \end{aligned}$$

$$\begin{aligned}
 e^{-y_1} = & 1 \times 10^{-10}x^2 + 0.367879418 - 1 \times 10^{-10}x^2 \\
 & + 0.3678794180 - 1 \times 10^{-10}x \\
 & - 0.8709878762xt \\
 & + +0.1241462421tx^2 \\
 & - 0.9172842035t^2x \\
 & + 0.6669233058t^2x^2 + 0.4231463703t^2 \\
 & + 0.2089742357t
 \end{aligned}$$

$$\begin{aligned}
 F_1(x, t) = & 0.5x^2 + 0.5x + 5.333341592xt \\
 & + 1.919535110tx^2 - 0.18990639t^2x \\
 & + 3.947686046t^2x^2 - 2 \times 10^{-10}t^2 \\
 & - 0.3678794412 - 2.t
 \end{aligned}$$

$$\begin{aligned}
 y_2(x, t) = & 1 + 0.5000000001tx^2 + 0.4999999999xt \\
 & - 2.581106735t^3x - 1.998599240t^3x^2 \\
 & + 0.872035858t^2x - 1.351599946t^3
 \end{aligned}$$

TABLE 6: The absolute and mean square errors using y_2 for (33) when $n=m=3$ and $t = 0.01$.

X	VIM-Berns.	Exact	VIMB-Exact
0.1	1.000331808	1.000100000	2.318080×10^{-4}
0.2	1.000871722	1.000400000	4.717220×10^{-4}
0.3	1.001545844	1.000900000	6.458440×10^{-4}
0.4	1.002354355	1.001600000	7.5433550×10^{-4}
0.5	1.003297432	1.002500000	7.974320×10^{-4}
0.6	1.004375255	1.003600000	7.752550×10^{-4}
0.7	1.005587998	1.004900000	6.879980×10^{-4}
0.8	1.006935836	1.006400000	5.358360×10^{-4}
0.9	1.008418942	1.008100000	3.189420×10^{-4}
1.0	1.010037492	1.010000000	3.749200×10^{-4}
MSE			3.362928×10^{-7}

$$\begin{aligned}
 & - 0.047599240t^7x + 0.9894746115t^7x^2 \\
 & + 2.626895702t^6x + 0.9413523939t^6x^2 \\
 & - 145.1140716t^6x^3 - 1.675722501t^5x \\
 & - 182.0611395t^5x^2 - 131.6082865t^5x^3 \\
 & - 5.303369670t^4x - 52.10714546t^4x^2 \\
 & - 86.90129806t^4x^3 - 2.912463794t^3x^3 \\
 & + 0.0005724513804t^7 \\
 & - 0.06170965730t^6 + 1.669391367t^5 \\
 & + 2.233808862t^4 + 1.021840676t^2x^2 \\
 & - 1.075083382t^2 - 2.32 \times 10^{-8}t
 \end{aligned} \tag{35}$$

The exact solution is $y(x, t) = 1 + tx^2$.

Tables 5-6 present the absolute error and mean square error of VIM with modified Bernstein polynomial when $n=m=2$ and $t = 0.01$ in Table 5 and when $n=m=3$ and $t = 0.01$

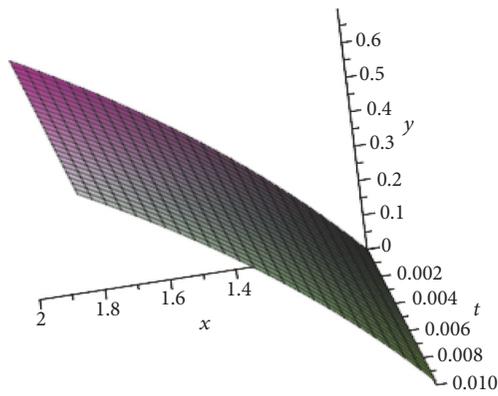


FIGURE 3: Numerical solution for (28) using VIMB when $n=m=8$ and $t = 0.01$.

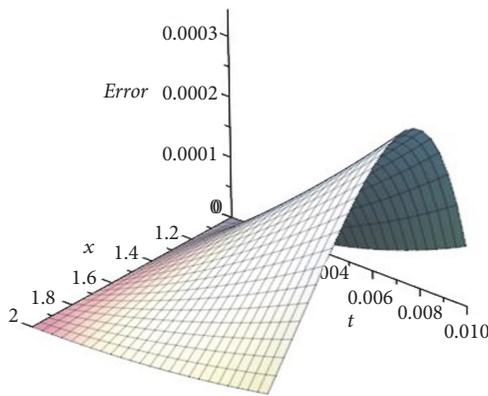


FIGURE 4: Absolute error for (28) using VIMB when $n=m=8$ and $t = 0.01$.

in Table 6. The absolute errors generated using the modified Bernstein polynomial are of 10^{-3} .

These examples are solved by the VIM with the Bernstein polynomials on the bound region $a \leq x \leq b$ and $c \leq t \leq d$. The obtained results are listed in Tables 1–6 and Figures 1–6 at $t = 0.1$ and $t = 0.01$. The results that were obtained showed that VIM with the Bernstein approximation solutions converges to the exact solution with less iterations. It is obvious that the approximate solutions found when using our proposed method are significantly accurate when increasing the number of iterations within the smallest value of time. We notice that the solution becomes faster by transforming the transcendent functions, which are difficult to integrate in some cases, into a series of polynomials where the solution becomes simpler and faster.

6. Conclusions

VIM is more reliable with Bernstein polynomial approximation when compared to dependent variable transcendental functions in differential equations. Comparing the above illustrated numerical results as shown in tables and figures,

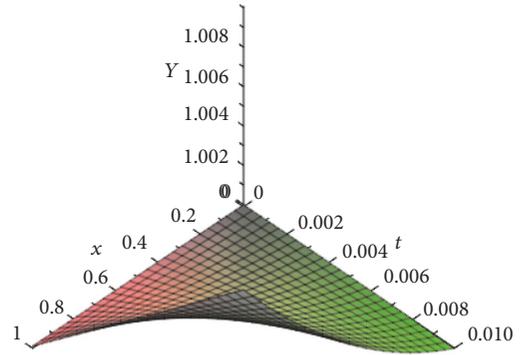


FIGURE 5: Numerical solution for (33) using VIMB when $n=m=2$ and $t = 0.01$.

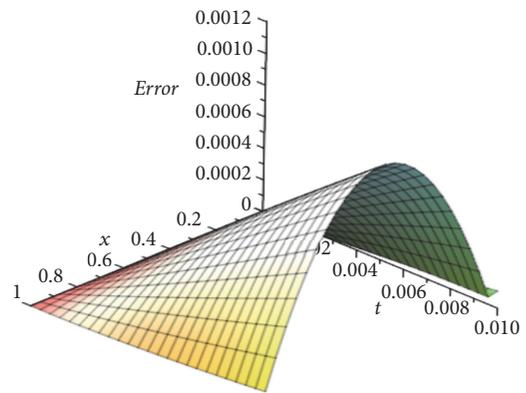


FIGURE 6: Absolute error for (33) using VIMB when $n=m=2$ and $t = 0.01$.

we conclude that VIMB provides reliably accurate results that are characterized by stability for all x at different values of t . The proposed method can be also applied to other differential equations that may include trigonometric functions of dependent variables. All the computations were carried out with the aid of the Maple 13 software. The VIM with the Bernstein polynomials used in this paper was applied directly without resorting to linearization or any kind of confining assumptions and was successful in determining approximate analytic solutions of nonlinear PDEs. Comparing numerical results of VIM with the Bernstein polynomial and the exact solutions proves power of VIM as a mathematical tool to solve nonlinear PDEs and with results that converge rapidly to the exact solution.

Data Availability

The data used to support the findings of this study are available from the corresponding authors upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The research is supported by College of Computer Sciences and Mathematics, University of Mosul, Republic of Iraq, under Project no. 21558430.

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