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Research Article

A Generalization of Arrow's Lemma on Extending a Binary Relation

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By examining whether the individualistic assumptions used in social choice could be used in the aggregation of individual preferences, Arrow proved a key lemma that generalizes the famous Szpilrajn's extension theorem and used it to demonstrate the impossibility theorem. In this paper, I provide a characterization of Arrow's result for the case in which the binary relations I extend are not necessarily transitive and are defined on abelian groups. I also give a characterization of the existence of a realizer of a binary relation defined on an abelian group. These results also generalize the well-known extension theorems of Szpilrajn, Dushnik-Miller, and Fuchs.

1. Introduction

One of the most fundamental results on extensions of binary relations is due to Szpilrajn [1], who shows that any transitive and asymmetric binary relation has a transitive, asymmetric, and complete extension. The Szpilrajn extension theorem has applications in many areas, including mathematical logic, order theory, mathematical social sciences, computer sciences, computability theory, fuzzy mathematics, and other fields in pure and applied mathematics. This result remains true if asymmetry is replaced with reflexivity; that is, any quasi-ordering has an ordering extension. Fuchs [2] obtained a more general theorem on extending a partially ordered group $(X, +, \succcurlyeq)$ to a linear one. In fact, Szpilrajn's theorem reduces to Fuchs's theorem, when $\{+\} = \emptyset$. Fuchs' result has been thoroughly investigated (see [3–6]). In fact, It has been proven that a partially ordered abelian group $(X, +, \succcurlyeq)$ has a linear extension if and only if X is torsion-free. Arrow [7, Page 64] states a generalization of Szpilrajn's theorem, which was the basis for his famous general impossibility theorem under the individualistic assumptions. This generalization may be stated as follows: *Suppose that $\succcurlyeq$ is a quasi-ordering defined on a set of alternatives X, and Y is the set of alternatives that any two elements in Y are incomparable according to $\succcurlyeq$ ($x, y \in Y$ implies $x \not\succcurlyeq y$). Then, given any ordering in Y, there exists an ordering in X which is compatible both with the given ordering in Y and with $\succcurlyeq$.* In other words, there exists an ordering of all the alternatives of X which is compatible with any ordering in Y. While the property of quasi-ordering satisfying the Arrow's assumptions is sufficient for the existence of an ordering extension, this is not necessary. As shown by Suzumura [8], S-consistency is a necessary and sufficient condition for a binary relation to have an extension satisfying Arrow's assumptions. The extension theorems of Szpilrajn type have played an important role in social choice and economic and game theories. For example, one way of assessing whether a preference relation is rational is to check whether it can be extended to a transitive and complete relation (see [9, 10]). Another example is the problem of the existence of maximal elements of binary relations. If $\succcurlyeq$ is a binary relation which has a linear extension $\geq$, then any maximal element of $\geq$ is also a maximal element of $\succcurlyeq$. In a very general sense, if $\succcurlyeq$ is defined in a topological space, then the existence of a linear extension of $\succcurlyeq$ satisfying some continuity conditions is equivalent to the existence of a continuous utility function representing $\succcurlyeq$ (a binary relation $\succcurlyeq$ defined on X is *represented* by a utility function $u : X \longrightarrow \mathbb{R}$, if for all $x, y \in X : x \succcurlyeq y \Longleftrightarrow u(x) \geq u(y)$) (see [11–14]). In the field of computer science, a topological sorting of a directed graph is a linear ordering of its vertices (it is well-known that directed graphs are useful in computer science because they serve as mathematical models of network structures) such that every edge between two vertices x and y, the vertex x comes before vertex y. That is, topological sorting can be used to convert a directed graph into a linear order such that if any event x requires that y be completed before x is initiated, then y will occur before y in the ordering. An elementary result in this direction says that a necessary and sufficient condition for a directed graph to have a topological sorting is acyclicity. The first implication is equivalent to the statement that every partial order has a linear order extension, which, as Knuth [15] noted, was proved by Szpilrajn [1930] for infinite as well as finite sets. Linearly ordered groups play an important role in many areas in pure and applied mathematics. For example, in economic and game theories, pairwise comparison matrices over a real linearly ordered group play a basic role in multicriteria decision-making methods such as the Analytic Hierarchy Process. (The Analytic Hierarchy Process provides a comprehensive and rational framework for structuring a decision problem, for representing and quantifying its elements, for relating those elements to overall goals, and for evaluating alternative solutions.) In utility theory, a major unsolved question is what can be said if a given preference relation cannot be represented by a utility function. In this case, one possibility is to seek utility functions with values in a suitable linearly ordered group (see [16]). In computer science, variants of duration calculus have been developed for discrete and abstract time (a system is said to be real-time if the total correctness of an operation depends not only on its logical correctness, but also on the time in which it is performed. In computer science, real-time computing describes hardware and software systems subject to a "real-time constraint". Duration Calculus is an interval-based logic for the specification of real-time systems), where an arbitrary (commutative) linearly ordered group can be the model of time (see [17]).

Building on the original result of Szpilrajn's Theorem, Dushnik and Miller [18] proved that any strict partial order is equal to the intersection of its linear ordering extensions and, based on this fact, defined the *dimension* of a partially ordered set as the smallest number of linear orderings the intersection of which is the partial order. Fuchs [2, Theorem 2] proved a more general Dushnik-Miller extension theorem on partially ordered groups. The extension theorems of Dushnik-Miller and Fuchs have many applications in pure and applied mathematics. Much of economic and social behaviour observed is either group behaviour or that of an individual acting for a group. Group preferences may be regarded as derived from individual preferences, by means of some process of aggregation. For example, if all voters agree that some alternative x is preferred to another alternative y, then the majority rule will return this ranking. In this case, there is one simple condition that is nearly always assumed called the *principle of unanimity* or *Pareto principle*. This declares that the preference relation for a group of individuals should include the intersection of their individual preferences. Another example of the use of intersections is in the description of simple games which can be represented as the intersection of weighted majority games [19].

In this paper, I prove extension theorems for binary relations in a general framework allowing abelian groups, which generalize the well-known extension theorems of Arrow, Suzumura, Szpilrajn, Dushnik-Miller, and Fuchs.

2. Notations and Definitions

Let X be a nonempty set, and let $\succcurlyeq \subseteq X \times X$ be a binary relation on X. If Y is any subset of X, the *restriction* of $\succcurlyeq$ to Y is the relation $\succcurlyeq / Y = \succcurlyeq \bigcap Y \times Y$. If $\succcurlyeq$ and $\trianglerighteq$ are two binary relations on X, then $\succcurlyeq \setminus \trianglerighteq$ denotes the set that results from removing the elements of $\trianglerighteq$ from $\succcurlyeq$. We denote $(x, y) \notin \succcurlyeq$ as $x \not\succcurlyeq y$. Let $\succcurlyeq$ be a binary relation on X and let $x, y \in X$. We say that x and y are $\succcurlyeq$*-incomparable* if $(x, y) \notin \succcurlyeq$ and $(y, x) \notin \succcurlyeq$. Let $\succ$ and $I(\succcurlyeq)$ denote, respectively, the *asymmetric part* of $\succcurlyeq$ and the *symmetric part* of $\succcurlyeq$, which are defined, respectively, by $\succ = P(\succcurlyeq) = \{(x, y) \in X \times X \mid x \succcurlyeq y \text{ and } y \not\succcurlyeq x\}$ and $I(\succcurlyeq) = \{(x, y) \in X \times X \mid x \succcurlyeq y \text{ and } y \succcurlyeq x\}$. We say that $\succcurlyeq$ on X is (i) *reflexive* if for each $x \in X$ $x \succcurlyeq x$; (ii) *transitive* if for all $x, y, z \in X$, $[x \succcurlyeq z \text{ and } z \succcurlyeq y] \Longrightarrow x \succcurlyeq y$; (iii) *antisymmetric* if for each $x, y \in X$, $[x \succcurlyeq y \text{ and } y \succcurlyeq x] \Longrightarrow x = y$; (iv) *complete* if, for each $x, y \in X$, we have $x \succcurlyeq y$ or $y \succcurlyeq x$. The *transitive closure* of a relation $\succcurlyeq$ is denoted by $\overline{\succcurlyeq}$, that is for all $x, y \in X$, $(x, y) \in \overline{\succcurlyeq}$ if there exist $m \in \mathbb{N}$ and $z_0, \ldots, z_m \in X$ such that $x = z_0, z_k \succcurlyeq z_{k+1}$ for all $k \in \{0, \ldots, m-1\}$ and $z_m = y$. Clearly, $\overline{\succcurlyeq}$ is transitive and because the case $m = 1$ is included, it follows that $\succcurlyeq \subseteq \overline{\succcurlyeq}$. *Acyclicity* says that m and $z_0, z_1, \ldots, z_m \in X$ do not exist such that $x = z_0, z_k \succcurlyeq z_{k+1}$ for all $k \in \{0, \ldots, m-1\}$ and $z_m = x$. The relation $\succcurlyeq$ is S-*consistent*, if, for all $x, y \in X$, $x \overline{\succcurlyeq} y$ implies $y \not\succ x$ (see [20]). The following combination of properties is considered in the next theorems. A binary relation $\succcurlyeq$ on X is (i) *partial order* if $\succcurlyeq$ is reflexive, transitive, and antisymmetric and (ii) *linear order* if $\succcurlyeq$ is a complete partial order. We say that $\succcurlyeq$ is *contradictory* if any two of $x \succ y$, $x = y$, and $x \prec y$ can not be satisfied simultaneously.

An abelian group is a set, G, together with an operation $+$ that combines any two elements a and b to form another element, denoted $a+b$ and satisfy the following requirements: (i) for all $a, b \in G$, $a + b \in G$; (ii) for all a, b and c in G, $(a+b)+c = a+(b+c)$; (iii) there exists an element 0 in G, such that, for every element $a \in G$, the equation $0 + a = a + 0 = a$ holds; (iv) for each $a \in G$, there exists an element $-a \in G$ such that $a + (-a) = (-a) + a = 0$; (v) for all $a, b \in G$, $a + b = b + a$. We write $a - b$ to mean $a + (-b)$. The sum $a + \ldots + a$ (n summands) is abbreviated as na (called a multiple of a), and $-a - \ldots - a$ (n summands) as $-na$ with $n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers). A binary relation $\succcurlyeq$ defined on an abelian group $(G, +)$ is *homogeneous* if it satisfies the following requirements: ($\mathfrak{a}$) $\succcurlyeq$ is contradictory; ($\mathfrak{b}$) if $a \succcurlyeq b$ and $c \succcurlyeq d$ hold for some $a, b, c, d \in G$, then $a+c \succcurlyeq b+d$ holds. An *ordered group* $(G, +, \succcurlyeq)$ is an abelian group $(G, +)$ equipped with a *homogeneous* binary relation $\succcurlyeq$. We say that $\succcurlyeq$ is *normal*, if $na = a + a + +a \succcurlyeq 0$ for some positive integer n implies $a \succcurlyeq 0$. An ordered group $(G, +, \succcurlyeq)$ is *cancellative* if for all $a, b, c \in G$, $a + c \geq b + c$ implies $a \geq b$. If an ordered group is cancellative, then $a+c = b+c$ means that $a+c \geq b+c$ and $b+c \geq a+c$, which yields $a \geq b$ and $b \geq a$, and therefore $a = b$. Every ordered group is automatically cancellative since $a + b \geq b + c$ implies $a + c + (-c) \geq b + c + (-c)$, and therefore $a \geq b$. An ordered group $(G, +, \succcurlyeq)$ is called: (1) *partially ordered group* if $\succcurlyeq$ is reflexive and transitive and (2) *linearly ordered group* if $\succcurlyeq$ is reflexive, transitive, antisymmetric, and complete. If $(G,+,\succcurlyeq)$ is an ordered group, then we say that $\succcurlyeq$ has a *linear ordering extension* $\trianglerighteq$ if and only if $(G,+,\trianglerighteq)$ is a linearly ordered group such that $\succcurlyeq\subseteq\trianglerighteq$ and $\succ\subseteq\triangleright$. In fact, $\trianglerighteq$ subsumes all the pairwise information provided by $\succcurlyeq$ and possibly further information.

3. The Main Results

In the context of examining if the individualistic assumptions used in economics can be used in the aggregation of individual preferences ([7, Definition 5, Theorem 2], Arrow proved a key lemma that extends the famous Szpilrajn's Theorem.

Arrow's Lemma ([7, pp. 64-68]). Let $\succcurlyeq$ be a quasi-ordering on X, Y a subset of X such that, if $x \neq y$ and $x, y \in Y$, then $x \not\succcurlyeq y$, and $\sqsupseteq$ an ordering on Y. Then, there exists an ordering extension $\trianglerighteq$ such that $\trianglerighteq/Y = \sqsupseteq$.

In fact, the lemma says that, if $\succcurlyeq$ is a binary relation defined on a set of alternatives X, then given any ordering $\sqsupseteq$ to any subset Y of $\succcurlyeq$-incomparable elements, there is a way of ordering all the alternatives which will be compatible both with $\succcurlyeq$ and with the given ordering $\sqsupseteq$ in Y. In this case, it is important that the linear extension of $\succcurlyeq$ inherits the relationship we put between the $\succcurlyeq$-incomparable elements of $\succcurlyeq$.

Definition 1. A binary relation $\succcurlyeq$ is called Δ-*consistent* if $I(\overline{\succcurlyeq}) \subseteq \Delta$.

Clearly, Δ-consistency implies S-consistency. The following proposition is evident.

Proposition 2. *A binary relation $\succcurlyeq$ is Δ-consistent if and only if $\succcurlyeq \setminus \Delta$ is acyclic.*

In the following, $\mathbb{N}$ denotes the set of all natural numbers.

Theorem 3. *Let $(G,+,\succcurlyeq)$ be an ordered group, Y be a subset of G such that, if $x \neq y$ and $x, y \in Y$ then $(x,y) \notin \overline{\succcurlyeq}$, and let $\sqsupseteq$ be a linear ordering on Y. Then, $\succcurlyeq$ has a linear ordering extension $\trianglerighteq$ such that $\trianglerighteq/Y = \sqsupseteq$ if and only if $\succcurlyeq$ is a normal Δ-consistent binary relation.*

Proof. To prove necessity, let $(G,+,\succcurlyeq)$ be an ordered group and let $\succcurlyeq$ be normal and Δ-consistent. Suppose that x and y be two $\overline{\succcurlyeq}$-incomparable elements of G.

We put

$$\begin{aligned}\succcurlyeq^* = \Delta \cup \succcurlyeq \cup \{(a,b),\ a,b \in G,\ a \neq b \mid \\ &\text{there are two non-negative integers } p,q, \\ &\text{not both zero, such that } p(a-b) \succcurlyeq q(x-y),\ x,y \\ &\in Y \text{ and } x \sqsupseteq y\}. \end{aligned} \quad (1)$$

We show that $(G,+,\succcurlyeq^*)$ is an ordered group with $\succcurlyeq^*$ being a normal, reflexive, and Δ-consistent extension of $\succcurlyeq$ satisfying $x \succ^* y$.

First of all, we note that p is never zero, because otherwise we should have $0 \succcurlyeq q(x-y)$ for some $q \in \mathbb{N}^*$, whence by normality of $\succcurlyeq$ we have $y \succcurlyeq x$ against hypothesis.

To show that $(G,+,\succcurlyeq^*)$ is an ordered group, we show that $\succcurlyeq^*$ is contradictory and homogenous.

To prove that $\succcurlyeq^*$ is contradictory, we show that any two of the three relations $a \succ^* b$, $a = b$, and $b \succ^* a$ cannot be satisfied simultaneously. We prove the case of $a \succ^* b$ and $b \succ^* a$; the other cases are obvious. We have four cases to consider.

Case 1. $a \succ b$ and $b \succ a$. This is impossible because of the homogeneity of the relation $\succcurlyeq$.

Case 2. $a \succ b$ as well as $p(b-a) \succcurlyeq q(x-y)$ for some $p,q \in \mathbb{N}\setminus\{0\}$. By adding p times the $a-b \succ 0$ in the second inequality, one obtains $p(a-b) + p(b-a) \succcurlyeq q(x-y)$; that is to say, $0 \succcurlyeq q(x-y)$. But then, by normality we are led to $y \succcurlyeq x$, a contradiction.

Case 3. $p(a-b) \succcurlyeq q(x-y)$ and $b \succ a$ for some $p,q \in \mathbb{N}\setminus\{0\}$. This is impossible as in the preceding case.

Case 4. In this case we have $p_1(a-b) \succcurlyeq q_1(x-y)$, $p_2(b-a) \succcurlyeq q_2(x-y)$, and $a \neq b$ for some $p_1, p_2, q_1, q_2 \in \mathbb{N}\setminus\{0\}$. By adding p_2 times the first, p_1 times the second inequality, we have $p_1p_2(a-b) + p_1p_2(b-a) \succcurlyeq (q_1p_2 + q_2p_1)(x-y)$; that is, $0 \succcurlyeq (q_1p_2 + q_2p_1)(x-y)$. If $q_1p_2 + q_2p_1 \neq 0$, by normality we have $y \succcurlyeq x$, which is impossible. On the other hand, if $q_1p_2 + q_2p_1 = 0$, then since $p_1, p_2 \neq o$ we have $q_1 = q_2 = 0$. It follows that $p_1(a-b) \succcurlyeq 0$ and $p_2(b-a) \succcurlyeq 0$. Therefore, by normality we have $a \succcurlyeq b$ and $b \succcurlyeq a$. Since $\succcurlyeq$ is contradictory, we conclude that $a = b$, an absurdity. Therefore, $\succcurlyeq^*$ is contradictory.

To prove homogeneity for $\succcurlyeq^*$, let $a,b,c,d \in G$ such that $a \succcurlyeq^* b$ and $c \succcurlyeq^* d$. We have four cases to consider: ($\mathfrak{a}$) $a \succcurlyeq b$ and $c \succcurlyeq d$; ($\mathfrak{b}$) $a \succcurlyeq b$ and $p(c-d) \succcurlyeq q(x-y)$ for some $p,q \in \mathbb{N}^*$; ($\mathfrak{c}$) $p(a-b) \succcurlyeq q(x-y)$ for some $p,q \in \mathbb{N}^*$ and $c \succcurlyeq d$; ($\mathfrak{d}$) $p_1(a-b) \succcurlyeq q_1(x-y)$ and $p_2(c-d) \succcurlyeq q_2(x-y)$ for some $p_1, p_2, , q_1, q_2 \in \mathbb{N}^*$. We only prove the fourth case, as the proof of the others is similar. In this case, we have that $p_1p_2[(a+c)-(b+d)] \succcurlyeq (p_2q_1 + p_1q_2)(x-y)$ and $p_1, p_1 \neq 0$. Thus, $a+c \succcurlyeq^* b+d$.

We proceed now to prove that $\succcurlyeq^*$ is Δ-consistent. Indeed, suppose to the contrary that there are $a,b \in G$ such that $a\overline{\succcurlyeq^*}b$ and $b \succ a$. Thus, there exists $z_0, z_1, \ldots, z_n \in G$ such that

$$a = z_0 \succcurlyeq^* z_1 \ldots z_{n-1} \succcurlyeq^* z_n = b \quad (2)$$

and $b \succ a$. Therefore, there exist nonnegative integers p_i, q_i, $i \in \{1, 2, \ldots, n\}$, $p_i \neq 0$, such that

$$\begin{aligned} p_1(a-z_1) &\succcurlyeq q_1(x-y), \\ p_2(z_1-z_2) &\succcurlyeq q_2(x-y), \ldots, \\ p_n(z_{n-1}-b) &\succcurlyeq q_n(x-y). \end{aligned} \quad (3)$$

By adding $p_2p_3 \ldots p_n$ times the first, $p_1p_3 \ldots p_n$, times the second, …, $p_1p_2 \ldots p_{n-1}$, times the n-th and by using the fact that $p_1p_2 \ldots p_n(b-a) \succ 0$, we have

$$0 = p_1 p_2 \dots p_n (a - a) \succcurlyeq (q_1 p_2 \dots p_m + p_1 q_2 p_3 \dots p_m + \dots + p_1 p_2 \dots p_{n-1} q_n)(x - y). \tag{4}$$

We have that $p_1 p_2 \dots p_n$ is nonzero. On the other hand, $q_1 p_2 \dots p_m + p_1 q_2 p_3 \dots p_m + \dots + p_1 p_2 \dots p_{m-1} q_m = 0$, because, otherwise, $0 \succcurlyeq (q_1 p_2 \dots p_m + p_1 q_2 p_3 \dots p_m + \dots + p_1 p_2 \dots p_{m-1} q_m)(x - y)$ implies that $y \succcurlyeq x$, an absurdity. It follows that $q_i = 0$, $i \in \{1, \dots, n\}$. But then,

$$a \succcurlyeq z_1 \succcurlyeq \dots \succcurlyeq z_{n-1} \succcurlyeq z_n = b \succ a, \tag{5}$$

a contradiction to Δ-consistency of $\succcurlyeq$. It follows that $\succcurlyeq^*$ is Δ-consistent.

To prove normality, let $na \succcurlyeq^* 0$ for some integer n and some $a \in G$. Then, $na \succcurlyeq 0$ or there exist nonnegative integers p, q such that $p(na - 0) \succcurlyeq q(x - y)$. If $na \succcurlyeq 0$, then by the normality of $\succcurlyeq$ we have $a \succcurlyeq 0$ which implies that $a \succcurlyeq^* 0$. Otherwise, $pn(a - 0) \succcurlyeq q(x - y)$ which implies $a \succcurlyeq^* 0$ as well.

It remains to prove that $\succcurlyeq^*$ is an extension of $\succcurlyeq$. Clearly, $\succcurlyeq \subseteq \succcurlyeq^*$. On the other hand, if $a \succ b$ for some $a, b \in G$, then $a - b \succ 0(x - y)$ which implies that $a \succ^* b$. It follows that $\succ \subseteq \succ^*$.

Finally, since $x - y \succcurlyeq x - y$ and $x \neq y$, we conclude that $x \succ^* y$.

Suppose that $\mathfrak{G} = \{(G, +, \succcurlyeq_i) \mid i \in I\}$ denotes the set of ordered groups such that, for each $i \in I$, $\succcurlyeq_i$ is a normal, reflexive, and Δ-consistent extension of $\succcurlyeq$ satisfying $x \succ_i y$. Since $(G, +, \succcurlyeq^*) \in \mathfrak{G}$ this set is nonempty. Let $\widetilde{\succcurlyeq} = (\succcurlyeq_j)_{j \in I}$ be a chain in $\{\succcurlyeq_i \mid i \in I\}$, and let $\widehat{\succcurlyeq} = \bigcup_{j \in I} \succcurlyeq_j$. It is easy to check that $(G, +, \widehat{\succcurlyeq})$ is an ordered group such that $\widehat{\succcurlyeq}$ is a normal, reflexive, and Δ-consistent extension of $\succcurlyeq$ satisfying $x \widehat{\succ} y$.

By Zorn's lemma, $(\succcurlyeq_i)_{i \in I}$ possesses an element, say $\trianglerighteq$, that is maximal with respect to set inclusion. It follows that $(G, +, \trianglerighteq)$ is an ordered group such that $\trianglerighteq$ is a normal, reflexive, and Δ-consistent extension of $\succcurlyeq$ satisfying $x \vartriangleright y$. Let $\overline{\trianglerighteq}$ be the transitive closure of $\trianglerighteq$. Then, $(G, +, \overline{\trianglerighteq})$ is an ordered group such that $\overline{\trianglerighteq}$ is a normal, reflexive, transitive, and Δ-consistent extension of $\succcurlyeq$ satisfying $x \overline{\vartriangleright} y$. To prove it, we show only the normality for $\overline{\trianglerighteq}$. All the other conditions are easily verified from the fact that they are also satisfied by $\trianglerighteq$. Indeed, let $na \overline{\trianglerighteq} 0$ for some integer n and some $a \in G$. Therefore, there exist $a_1, \dots, a_n \in G$ such that

$$na \trianglerighteq a_1 \trianglerighteq \dots \trianglerighteq a_{n-1} \trianglerighteq a_n \trianglerighteq 0. \tag{6}$$

Since $a_{n-1} \trianglerighteq a_n$ and $a_n \trianglerighteq 0$, by the homogeneity of $\trianglerighteq$ we conclude that $a_{n-1} + a_n \trianglerighteq a_n + 0$. But then, by the cancellativity of $\trianglerighteq$ we have that $a_{n-1} \trianglerighteq 0$ and, by an induction argument based on this logic, we obtain $na \trianglerighteq 0$ which implies that $a \trianglerighteq 0$. To prove that $(G, +, \overline{\trianglerighteq})$ is a linearly ordered group, it remains to show that $\overline{\trianglerighteq}$ is complete and antisymmetric. Since antisymmetry of $\overline{\trianglerighteq}$ is an immediate consequence of the Δ-consistency of $\trianglerighteq$, we prove the completeness of $\overline{\trianglerighteq}$. Suppose to the contrary that there exist $a, b \in X$ such that $(a, b) \notin \overline{\trianglerighteq}$ and $(b, a) \notin \overline{\trianglerighteq}$. We define

$$\trianglerighteq^* = \trianglerighteq \cup \{(a, b),\ a, b \in G,\ a \neq b \mid \text{there exists } p, q \in \mathbb{N}^*,\ \text{not both zero, such that } p(a - b) \trianglerighteq q(x - y),\ x, y \in Y \text{ and } x \sqsupseteq y\}. \tag{7}$$

Then, as in the case of $\succcurlyeq^*$ above, we can prove that $(G, +, \trianglerighteq^*)$ is an ordered group such that $\trianglerighteq^*$ is a normal, reflexive, and Δ-consistent extension of $\succcurlyeq$ satisfying $x \vartriangleright^* y$, a contradiction of maximality of $\trianglerighteq$. Therefore, $\overline{\trianglerighteq}$ is complete.

To complete the necessity part we show that $\overline{\trianglerighteq}/Y = \sqsupseteq$. Evidently, $\sqsupseteq \subseteq \overline{\trianglerighteq}/Y$. To prove the converse, let $a(\overline{\trianglerighteq}/Y)b$ for some $a, b \in G$. Suppose to the contrary that $(a, b) \notin \sqsupseteq$. Since $\sqsupseteq$ is complete, $b \sqsupseteq a$ holds which implies $b \succ^* a$ ($a \neq b$ since $a, b \in Y$). Since $\vartriangleright$ is a linear order extension of $\succ^*$, we have that $b \overline{\vartriangleright} a$. But then, $a \neq b$ and antisymmetry of $\trianglerighteq$ imply that $(a, b) \notin \overline{\trianglerighteq}/Y$, a contradiction. The last contradiction shows that $\overline{\trianglerighteq}/Y = \sqsupseteq$.

Conversely, suppose that $(G, +, \succcurlyeq)$ is an ordered group and $\succcurlyeq$ has a linear order extension $\trianglerighteq$. Suppose to the contrary that $\succcurlyeq$ is non-Δ-consistent. Then, $I(\overline{\succcurlyeq}) \nsubseteq \Delta$. Thus, there exists $a, b \in X$, such that $(a, b) \in I(\overline{\succcurlyeq}) \subseteq I(\overline{\trianglerighteq}) = I(\trianglerighteq)$ and $a \neq b$, which contradicts with the fact that $\trianglerighteq$ is antisymmetric. It remains to prove that $\succcurlyeq$ is normal. Suppose to the contrary that $na \succcurlyeq 0$ for some integer n and $a \in G$ and $a \not\succcurlyeq 0$. Then, as in the case of $\trianglerighteq$ above, the homogeneity and cancellativity of $\succcurlyeq$ implies that $(a, 0) \notin \overline{\succcurlyeq}$. On the other hands, $na \succcurlyeq 0$ implies that $na \trianglerighteq 0$. It follows that $(0, a) \notin \overline{\succcurlyeq}$, because otherwise $(0, a) \in \overline{\succcurlyeq} \subseteq \overline{\trianglerighteq}$ jointly to $na \trianglerighteq 0$ implies that $(a, o) \in I(\overline{\trianglerighteq}) = I(\trianglerighteq)$ which concludes that $a = 0$ ($\trianglerighteq$ is antisymmetric), a contradiction to $a \not\succcurlyeq 0$. Then, as in the case of $\succcurlyeq^*$ above, for $Y = \{0, a\}$ (0 and a are incomparable with respect to $\overline{\succcurlyeq}$) and $\sqsupseteq = \{(0, a)\}$ there exists a homogeneous extension $\succcurlyeq^{**}$ of $\succcurlyeq$ such that $0 \succ^{**} a$. It follows that $0 \succ^{**} na$. On the other hand, $na \succ 0$ ($na \succcurlyeq 0$ and $a \not\succcurlyeq 0$) implies $na \succ^{**} 0$, a contradiction to the contradictory of $\succ^{**}$. The last contradiction shows that $\succcurlyeq$ is normal. □

The following result of Arrow [7] and Suzumura (see [1, Main Theorem]) is an immediate consequence of Theorem 3.

Corollary 4. *Let $\succcurlyeq$ be a binary relation on X and let Y be a subset of X such that, if $x \neq y$ and $x, y \in Y$, then $(x, y) \notin \overline{\succcurlyeq}$, and let $\sqsupseteq$ be a linear ordering on Y. Then, $\succcurlyeq$ has a linear ordering extension $\trianglerighteq$ such that $\trianglerighteq/Y = \sqsupseteq$ if and only if $\succcurlyeq$ is an S-consistent binary relation.*

Corollary 5 (Szpilrajn's extension theorem [1]). *Every partial order $\succcurlyeq$ possesses a linear order extension $\trianglerighteq$. Moreover, if x and y are any two $\succcurlyeq$-incomparable elements of $\succcurlyeq$, then there exists a linear order extension $\trianglerighteq_1$ in which $x \trianglerighteq_1 y$ and a linear order extension $\trianglerighteq_2$ in which $y \trianglerighteq_2 x$.*

A consequence of Theorem 3 is also the result of Fuchs for extending partial orders of abelian groups to linear orders. The following corollary shows this fact.

Corollary 6 ([2, theorem 1]). *Let $(G, +, \succcurlyeq)$ be an ordered group and $x, y \in X$ be two $\succcurlyeq$-incomparable elements. Then, $\succcurlyeq$ has a linear ordering extension $\trianglerighteq$ such that $x \vartriangleright y$.*

Proof. The corollary is an immediate consequence of the necessity part of Theorem 3 for $\succcurlyeq$ being a partial order, $Y = \{x, y\}$ and $\sqsupseteq = \{(x, y)\}$. □

Moreover, if such an extension $\trianglerighteq$ exists, then $\succcurlyeq$ is necessarily normal [2, Lemma §3].

Definition 7. Let $(G,+,\succcurlyeq)$ be an ordered group and let $\mathscr{F}$ be a collection of linear order extensions of $\succcurlyeq$. Then $\mathscr{F}$ is a *realizer* of $\succcurlyeq$ if and only if the following conditions are satisfied: (α) the intersection of the members of $\mathscr{F}$ coincides with $\succcurlyeq$ and (β) for every pair of $\succcurlyeq$-incomparable elements $x, y \in X$, there exists a $\succcurlyeq^* \in \mathscr{F}$ with $x \succ^* y$.

The following result generalizes the classical Dushnik-Miller's type extension theorem for an ordered group (see [2, Theorem 2]).

Theorem 8. *Let $(G,+,\succcurlyeq)$ be an ordered group. Then, $\succcurlyeq$ has as realizer the set of its linear order extensions if and only if $\succcurlyeq$ is a normal Δ-consistent binary relation.*

Proof. To prove necessity, let $(G,+,\succcurlyeq)$ be an ordered group and let $\succcurlyeq$ be a normal Δ-consistent binary relation on G. Suppose that $\mathfrak{D} = \{\succcurlyeq_i \mid i \in I\}$ be the set of linear order extensions of $\succcurlyeq$. By Theorem 3, $\mathfrak{D}$ is nonempty. We show that $\succcurlyeq = \bigcap_{i\in I} \succcurlyeq_i$. Indeed, since $\succcurlyeq \subseteq \bigcap_{i\in I} \succcurlyeq_i$, we have to show that $\bigcap_{i\in I} \succcurlyeq_i \subseteq \succcurlyeq$. Assume by way of contradiction that there exists an $(x,y) \in \bigcap_{i\in I} \succcurlyeq_i$ with $x \not\succcurlyeq y$. It follows that $x \neq y$. On the other hand, $y \not\succcurlyeq x$ holds. Indeed, suppose to the contrary that $y \succcurlyeq x$. It follows that $y \succ_i x$ for some $i \in I$ which jointly to $x \succcurlyeq_i y$ contradicts with the fact that $\succcurlyeq_i$ is Δ-consistent. Therefore, $y \not\succcurlyeq x$. Then, as above (since $\succcurlyeq$ is homogeneous and cancellative) we conclude that $(x,y) \notin \overline{\succcurlyeq}$ and $(y,x) \notin \overline{\succcurlyeq}$. Define

$$\succcurlyeq^{\diamond} = \succcurlyeq \cup \{(a,b),\ a,b \in G,\ a \neq b \mid \text{there exists } p,q \in \mathbb{N}\setminus\{0\}, \text{such that } p(a-b) \succcurlyeq q(y-x)\}. \tag{8}$$

Clearly, $\succcurlyeq \subset \succcurlyeq^{\diamond}$. Then, as in the proof of Theorem 3, for $Y = \{x,y\}$ and $\sqsupseteq = \{(y,x)\}$, there exists a linear order extensions $\succcurlyeq_{i^*}$ of $\succcurlyeq$, $i^* \in I$, such that $y \succ_{i^*} x$. Since $\succcurlyeq_{i^*}$ is Δ-consistent and $x \neq y$ we have that $x \not\succcurlyeq_{i^*} y$, a contradiction to $(x,y) \bigcap_{i\in I} \succcurlyeq_i$. This contradiction confirms that $\bigcap_{i\in I} \succcurlyeq_i \subseteq \succcurlyeq$ and thus $\succcurlyeq = \bigcap_{i\in I} \succcurlyeq_i$. To finish the proof of necessity, it remains to show that $\mathfrak{D}$ is a realizer. But, this is an immediate consequence of Theorem 3 for $Y = \{x,y\}$ and $\sqsupseteq = \{(x,y)\}$.

To prove sufficiency, let $(G,+,\succcurlyeq)$ be an ordered group and let $\succcurlyeq$ have as realizer the set $\mathfrak{D} = \{\succcurlyeq_i \mid i \in I\}$ of all linear order extensions of $\succcurlyeq$. Then, $\succcurlyeq = \bigcap_{i\in I} \succcurlyeq_i$. Since for each $i \in I$, $\succcurlyeq_i$ is a linear order, we conclude that all the members of $\mathfrak{D}$ are normal and Δ-consistent binary relations. Since the intersection preserve the properties of normality and Δ-consistency we conclude that $\succcurlyeq$ is a normal and Δ-consistent binary relation. □

The following corollary is an immediate consequence of the necessity part of Theorem 8 for $\succcurlyeq$ being a partial order and $\{+\} = \emptyset$.

Corollary 9 (Dushnik-Miller's extension theorem [18, theorem 2.32]). *If $\succcurlyeq$ is any partial order on af set X, then there exists a collection $\mathscr{F}$ of linear orders on X which realize $\succcurlyeq$.*

The following corollary generalizes a result due to Fuchs [2, Theorem 2]. In fact, it is the Dushnik-Miller's type extension theorem for partially ordered groups.

Corollary 10 ([2, theorem 2]). *A partial order $\succcurlyeq$ defined on a group G has as realizer a certain set of linear orders if and only if $\succcurlyeq$ is normal.*

Proof. The sufficiency part is an immediate consequence of the fact that the intersection of linear orders is normal. The necessity part is obvious by Theorem 3, since a partial order is a Δ-consistent binary relation. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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