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Research Article

(p, q)-Growth of Meromorphic Functions and the Newton-Padé Approximant

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In this paper, we have considered the generalized growth ((p, q)-order and (p, q)-type) in terms of coefficient of the development p_{nn} given in the (n, n)-th Newton-Padé approximant of meromorphic function. We use these results to study the relationship between the degree of convergence in capacity of interpolating functions and information on the degree of convergence of best rational approximation on a compact of $\mathbb{C}$ (in the supremum norm). We will also show that the order of meromorphic functions puts an upper bound on the degree of convergence.

1. Introduction

Let $f(z) = \sum_{k=0}^{+\infty} a_n z^n$ be a nonconstant entire function and $M(f, r) = \max_{|z|=r} |f(z)|$.

It is well known that the function $r \longmapsto \log(M(f, r))$ is an indefinitely increasing convex function of $\log(r)$. To estimate the growth of f precisely, Boas (see [1]) has introduced the concept of order, defined by the number $\rho\,(0 \le \rho \le +\infty)$:

$$\rho = \limsup_{r \longrightarrow +\infty} \frac{\log\log(M(f, r))}{\log(r)}. \tag{1}$$

It is known that the order and type of an entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ are given, respectively, by

$$\rho_f = \limsup_{n \longrightarrow \infty} \frac{n \ln n}{\ln 1/|a_n|} \cdot \left(\sigma_f = \frac{1}{\rho e} \limsup_{n \longrightarrow \infty} n |a_n|^{\rho/n}\right). \tag{2}$$

The concept of type has been introduced to determine the relative growth of two functions of the same nonzero finite order. An entire function, of order ρ, $0 < \rho < +\infty$, is said to be of type σ, $0 \le \sigma \le +\infty$, if

$$\sigma = \limsup_{r \longrightarrow +\infty} \frac{\log(M(f, r))}{r^{\rho}}. \tag{3}$$

If f is an entire function of infinite or zero order, the definition of type is not valid and the growth of such function cannot be precisely measured by the above concept. Bajpai et al. (see [2]) have introduced the concept of index-pair of an entire function. Thus, for $p \ge q \ge 1$, they have defined the number

$$\rho(p, q) = \limsup_{r \longrightarrow +\infty} \frac{\log^{[p]}(M(f, r))}{\log^{[q]}(r)}, \tag{4}$$

$b \le \rho(p, q) \le +\infty$, where $b = 0$ if $p > q$ and $b = 1$ if $p = q$, where $\log^{[0]}(x) = x$, and $\log^{[p]}(x) = \log(\log^{[p-1]}(x))$, for $p \ge 1$.

The function f is said to be of index-pair (p, q) if $\rho(p - 1, q - 1)$ is a nonzero finite number. The number $\rho(p, q)$ is called the (p, q)-order of f.

Bajpai et al. have also defined the concept of the (p, q)-type $\sigma(p, q)$, for $b < \rho(p, q) < +\infty$, by

$$\sigma(p, q) = \limsup_{r \longrightarrow +\infty} \frac{\log^{[p-1]}((M(f, r)))}{\log^{[q-1]}(r)^{\rho(p,q)}}. \tag{5}$$

In their works, the authors established the relationship of (p, q)-growth of f with respect to the coefficients a_n in the Maclaurin series of f.

We also have many results in terms of polynomial approximation in the classical case. Let K be a compact subset of the complex plane $\mathbb{C}$ of positive logarithmic capacity, and f be a complex function defined and bounded on K. For $k \in \mathbb{N}$, put

$$E_n(K, f) = \|f - T_n\|_K, \tag{6}$$

where the norm $\|.\|_K$ is the maximum on K and T_n is the nth Chebyshev polynomial of the best approximation to f on K.

Bernstein showed (see [3], p. 14), for $K = [-1, 1]$, that there exists a constant $\rho > 0$ such that

$$\lim_{n \longrightarrow +\infty} n^{1/\rho} \sqrt[n]{E_n(K, f)}, \tag{7}$$

is finite, if and only if f is the restriction to K of an entire function of order ρ and some finite type.

This result has been generalized by Reddy (see [4, 5]) as follows:

$$\lim_{n \longrightarrow +\infty} \sqrt[n]{E_n(K, f)} = (\rho e \sigma) 2^{-\rho}, \tag{8}$$

if and only if f is the restriction to K of an entire function g of order ρ and type σ for $K = [-1, 1]$.

In the same way Winiarski (see [6]) generalized this result to a compact K of the complex plane $\mathbb{C}$ of positive logarithmic capacity, denoted by $c = \operatorname{cap}(K)$ as follows

If K is a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity, then

$$\lim_{n \longrightarrow +\infty} n (E_n(K, f))^{\rho/n} = c^{\rho} e \rho \sigma, \tag{9}$$

if and only if f is the restriction to K of an entire function of order ρ $(0 < \rho < +\infty)$ and type σ.

Recall that the capacity of $[-1, 1]$ is $\operatorname{cap}([-1, 1]) = 1/2$, and the capacity of a unit disc is $\operatorname{cap}(D(0, 1)) = 1$.

The authors considered, respectively, the Taylor development of f with respect to the sequence $(z_n)_n$ and the development of f with respect to the sequence $(\omega_n)_n$ defined by

$$\omega_n(z) = \prod_{j=1}^{j=n} (z - \eta_{nj}), \quad n = 1, 2, \ldots, \tag{10}$$

where $\eta^{(n)} = (\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn})$ is the nth extremal points system of K (see [6], p. 260).

We remark that the above results suggest that the rate at which the sequence $(\sqrt[k]{E_k(K, f)})_k$ tends to zero depends on the growth of the entire function (order and type).

Harfaoui (see [7–9]) obtained a result of generalized order and type in terms of approximation in L^p-norm for a compact of $\mathbb{C}^n$.

Recall that in the paper of Winiarski (see [6]), the author used the Cauchy inequality.

The aim of this paper is to generalize the growth $((p, q)$-order and (p, q)-type$)$, studied by K. Reczek (see [10]) in terms of coefficient of the development p_{nn} which will be defined later.

We use these results to study the relationship between the degree of convergence in capacity of interpolating functions and information on the degree of convergence of best rational approximation on a compact of $\mathbb{C}$ (in the supremum norm). We will also show that the order of meromorphic functions puts an upper bound on the degree of convergence.

A relation between the degree of convergence (in capacity) of Padé approximants and the degree of best rational is derived for functions in Goncar's class $\mathscr{R}_0$ (see [11]), where $\mathscr{R}_0$ is the class of functions f such that on some compact circular disk Δ_0 (depending on f) we have

$$\lim_{n \longrightarrow +\infty} \left\{ \inf_{r_n} \sup_{z \in \Delta_0} (f - r_n)(z) \right\}^{1/n} = 0, \tag{11}$$

where r_n ranges over the rational functions of type n with poles off Δ_0.

2. Auxiliary Results: The Newton-Padé Approximants

First, we recall some definitions and notations which will be used later.

Definition 1. If Δ is a compact subset of $\mathbb{C}$, we define its logarithmic capacity (transfinite diameter) by

$$cap(\Delta) = \lim_{n \longrightarrow +\infty} \left(\inf_{r_n} \|P_n\|_\Delta \right)^{1/n}, \tag{12}$$

where P_n ranges over all polynomials of degree n with leading coefficient 1 and $\|P_n\|_\Delta = \sup_\Delta |P_n(z)|$.

Let Δ be a compact subset of the complex plane $\mathbb{C}$ such that $\operatorname{cap}(\Delta) > 0$, and f is a complex function defined and bounded on Δ. For $n \in \mathbb{N}$, put (error of best rational approximation)

$$\epsilon_n(\Delta, f) = \inf_{r_n} \|f - r_n\|_\Delta. \tag{13}$$

We will denote by R, the class of functions f, such that on some compact circular disk Δ (depending on f) we have

$$\lim_{n \longrightarrow +\infty} \sqrt[n]{\epsilon_n(\Delta, f)} = 0, \tag{14}$$

where r_n ranges over the rational functions of type n $(r_n = P_n/Q_n)$ with poles off Δ.

Remark 1. If we let r_n range over the polynomials of degree n instead of over the rational functions, we get the class of entire functions.

We need the following notations and lemma which will be used in the sequel (see [2]):

(N) For $p \in \mathbb{Z}$, put

$$\begin{aligned} &\log^{[p]}(x) = \log\left(\log^{[p-1]}(x)\right); \\ &\log^{[0]}(x) = x; \Lambda_{[p]} = \prod_{k=1}^{p} \log^{[k]}(x), \\ &\exp^{[p]}(x) = \exp\left(\exp^{[p-1]}(x)\right); \\ &\exp^{[0]}(x) = x, \\ &E_{[p]}(x) = \prod_{k=0}^{p} \exp^{k}(x). \end{aligned} \tag{15}$$

Lemma 1 (see [2]). *With the above notations we have the following results:*

$$\begin{aligned}
E_{[-p]}(x) &= \frac{x}{\Lambda_{[p-1]}(x)},\\
\Lambda_{[-p]}(x) &= \frac{x}{E_{[p-1]}(x)},\\
\frac{\mathrm{d}}{\mathrm{d}x}\exp^{[p]}(x) &= \frac{E_{[p]}(x)}{x} = \frac{1}{\Lambda_{[-p-1]}(x)},\\
\frac{\mathrm{d}}{\mathrm{d}x}\log^{[p]}(x) &= \frac{E_{[-p]}(x)}{x} = \frac{1}{\Lambda_{[p-1]}(x)},\\
E_{[p]}^{-1}(x) &= \begin{cases} x, & \text{if } p = 0,\\ \log^{[p-1]}\left\{\log(x) - \log^{[2]}(x) + o\left(\log_{[3]}(x)\right)\right\}, & \text{if } p = 1, 2, \ldots, \end{cases}\\
\lim_{x\longrightarrow+\infty} \exp\left(E_{[p-2]}(x)\right) &= \begin{cases} e, & \text{if } p = 2,\\ 1, & \text{if } p \geq 3, \end{cases}\\
\lim_{x\longrightarrow+\infty} \left[\exp^{[p-1]}\left(E_{[p-2]}^{-1}(x)\right)\right]^{1/x} &= \begin{cases} e, & \text{if } p = 2,\\ 1, & \text{if } p \geq 3. \end{cases}
\end{aligned} \tag{16}$$

For more details of these results, see [2].

Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Suppose that f is a function holomorphic in a neighbourhood of the set $(z_n : 1 \leq n < \infty)$. Denote by $R_{n,m}$, the set of all rational functions, whose numerators and denominators are polynomials of degrees not greater than n and m, respectively. Let the function $f_{n,m}$ satisfy the following conditions:

(1) $f_{n,m} \in R_{n,m}$

(2) The function $f - f_{n,m}/\omega_n + m + 1$ is holomorphic at each point z_i for $1 \leq i \leq n + m + 1$

For each couple (n, m), there exists at most one function satisfying the above conditions. It is called the (n, m)-th Newton-Padé approximant of the function *f* with respect to the sequence $(z_n)_{n=1}^{\infty}$. In the sequel, we will consider the sequences of Newton-Padé approximants $(f_{n,m})$ with m fixed and with n tending to infinity. It will be useful to simplify the notations. Denote

$$f_n = f_{n,m} = \frac{p_n}{q_n}, \tag{17}$$

where

$$p_n(z) = \sum_{i=0}^{n} p_{ni} z^i, \tag{18}$$

$$q_n(z) = \left(z - z_{n,1}\right), \ldots, \left(z - z_{n,m_n}\right), \tag{19}$$

where $z_{n,1}, \ldots, z_{n,m_n}$ are the poles of the approximant f_n. Then, the polynomials p_n and q_n have no common divisors of degree higher than zero. Assume that

$$\left|z_{n,1}\right| \leq \cdots \leq \left|z_{n,m_n}\right|. \tag{20}$$

3. The (p, q)-Growth of Meromorphic Functions

In our work we assume that $p > q \geq 3$.

Let $M_m(\mathbb{C})$ be the class of meromorphic functions whose number of poles is not greater than *m*. The main result of this paper is as follows:

Lemma 2. *Let $(z_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers and let f be a function meromorphic in $\mathbb{C}$, holomorphic in a neighbourhood of the set $z_n : 1 \leq n < \infty$. Suppose that f has exactly m poles in C, counted with their multiplicities. Then,*

(1) For almost every n there exists the approximant f_n

(2) The poles of f_n tend to the poles of f when n tends to infinity

(3) $\limsup_{n\longrightarrow\infty} = f(z)$ in $\mathbb{C}$, except for the poles of f

(4) f can be extended to a function of the class $M_m(\mathbb{C})$

This lemma is a slight modification of the staff theorem, so we omit the proof.

Theorem 1. *Let $(z_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers. Let ω be a domain containing the set $z_n : 1 \le n < \infty$. Assume that there exists a limit point of the sequence (z_n) in ω. Let f be a function meromorphic in ω and holomorphic at each point of z_n for $1 \le n < \infty$. Assume that for almost every n, there exists the (n, m)-th Newton-Padé approximant f_n with respect to the sequence $(z_n)_{n=1}^{\infty}$ and that for some positive numbers μ and ν*

$$\limsup_{n \longrightarrow +\infty} \frac{\log^{[p-2]}(n)}{\left(\log^{[q-2]}\left(-(1/n)\log\left(\left|p_{nn}\right|\right)\right)\right)^{\mu}} \le \nu, \tag{21}$$

Then,

(1) The order of f is not greater than ν.

(2) If $\rho(f) = \nu$ then the type of f is not greater than μ.

(3) If

$$\limsup_{n \longrightarrow +\infty} \frac{\log^{[p-2]}(n)}{\left(\log^{[q-2]}\left(-(1/n)\log\left(\left|p_{nn}\right|\right)\right)\right)^{\mu}} = \nu, \tag{22}$$

and if

$$\limsup_{n \longrightarrow \infty} \left|z_{n,m}\right|^{1/n} \le 1, \tag{23}$$

then $\rho(f) = \mu$ and $\sigma(f) = \nu$.

Proof. Let $z \in C/D_\theta$, suppose that there exists a sequence (n_l) and a neighbourhood U of the point z such that for every l the function f_{n_l} has no poles in U. Then, it can be shown in the previous way that $\lim_{n\longrightarrow\infty} f_n(z) = \lim_{n\longrightarrow\infty} f_{n_l}(z) = f(z)$. So, we have shown that $\lim_{n\longrightarrow\infty} f_n(z) = f(z)$ in C/D_θ except for at most m points. We can choose a number R_0 such that for every point

$$z \in \left(\frac{C}{D_\theta}\right) B\left(0, R_0\right),$$
$$\lim_{n \longrightarrow \infty} f_n(z) = f(z). \tag{24}$$

Assume that R_0 is so great that $M_f(R)$ is an increasing function for R larger than R_0. Let R be greater than R_0. Then,

$$M_f(R) \le M_f(R_\theta) \le \left\|f_{n_0}\right\| C(0, R_\theta) + \sum_{n=n_0+1}^{\infty} \left\|f_n - f_{n-1}\right\| C(0, R_\theta). \tag{25}$$

According to (25), we have

$$M_f(R) \le A_1 (R_\theta)^n + \sum_{n=n_0+1}^{\infty} \left|p_{nn}\right| \theta^{-2mn} (R_\theta + s)^{m+n}. \tag{26}$$

Let K be an arbitrary number greater than μ. Then, it follows from (21) that there exists a number $n_1 \ge n_0$ such that

$$\left|p_{nn}\right| \le \exp\left(-n\left(\frac{\exp^{[q-2]}\log^{[p-2]}(n)}{\nu}\right)^{1/\mu}\right), \quad \text{for } n \ge n_1, \tag{27}$$

if R is large enough, then $(R_\theta + s) \le \theta^{-m} \cdot R$. It follows from (25) and (26) that

$$M_f(R) \le A_2 R^{n_1} + \left(\theta^{-m} \cdot R\right)^m \sum_{n=n_1+1}^{\infty} \exp\left(-n\left(\frac{\exp^{[q-2]}\log^{[p-2]}(n)}{\nu}\right)^{1/\mu}\right)\left(\theta^{-3m} \cdot R\right)^n, \tag{28}$$

where A_2 depends only on θ.

Let n_R be the smallest integer greater than $\exp^{[p-2]}(K(\log^{[q-1]}(2\theta^{-3m} \cdot R))^{\mu})$. Then, n_R is greater than n_1, if R is large enough, and the sum $\sum_{n \ge n_R} |p_{nn}| (\theta^{-3m} \cdot R)^n$ is smaller than 1. Consequently,

$$M_f(R) \le A_2 R^{n_1} + \left(\theta^{-m} \cdot R\right)^m \cdot \left(n_R \cdot \max\left\{\exp\left(-n\left(\frac{\exp^{[q-2]}\log^{[p-2]}(n)}{\mu}\right)^{1/\nu}\right) \cdot \left(\theta^{-3m} \cdot R\right)^n\right\} + 1\right), \tag{29}$$

when R is large enough. From (29), we get

$$M_f(R) \le A_2 R^{n_1} + \left(\theta^{-m} \cdot R\right)^m \cdot \left(n_R \cdot \left(R\exp\left(-\exp^{[q-2]}(\log(R) - \epsilon)\right)\right)^n\right), \tag{30}$$

where A_2 depends only on θ, μ, and K. Therefore, we can show that using the general formula of a (p, q) type

$$\left(R\exp\left(-\exp^{[q-2]}(\log(R) - \epsilon)\right)\right)^n \le \exp^{[p-1]}\left(K\log^{[q-1]}(R)^{\mu} - \epsilon\right), \tag{31}$$

which implies that the order of f is not greater than μ and if $\rho(f) = \mu$, then the type of f does not exceed K, consequently not greater than ν. This proves 1 and 2.

Now, assume that the conditions (22) and (23) are satisfied. Then, of course, f can be extended to a function of the class $M_m(C)$. Then, we can write $f = \varphi/Q$, where φ is an entire function and Q is a polynomial of the form

$$Q(z) = (t - \xi_1), \ldots, (t - \xi_k), \tag{32}$$

where k is the number of poles of f. Then, of course, the order of φ is equal to the order of f and the type of φ is equal to the type of f.

Assume that either the order of f is smaller than μ or the type of f is smaller than ν. Then, there exist a number $K < \nu$, such that

$$|\varphi(z)| \le \exp^{[p-1]}\left(K \log^{[q-1]}(z)\right)^{\mu}, \quad (33)$$

when $|z|$ is large enough. Using the Cauchy formula we get from (17) and (32),

$$p_{nn} = \frac{1}{2\pi i}\int_{C(0,r)} \frac{\varphi(z) q_n(z)}{w_{m+n+1}(z)} dz, \quad (34)$$

for $r > s$. Using (17), (18), and (33), we obtain the estimation

$$\left|p_{nn}\right| \le \frac{r \cdot 2^m\left(r^m + \left|z_{n,m}\right|^m\right)\exp^{[p-1]}\left(K\log^{[q-1]}(r)\right)^{\mu}}{\min\left|w_{m+n+1(z)}\right| : |z| = r}, \quad (35)$$

when r is large enough. Put $r = \exp^{[q-1]}(\log^{[p-2]}(n)/K)^{1/\rho}$. Then, for almost every n, the estimation (35) is true. Hence, we derive

$$\begin{aligned}\left|p_{nn}\right| &\le \exp\left(-n\exp^{[q-2]}\left(\frac{\log^{[p-2]}}{K}\right)^{1/\mu}\right)\\ &< \exp\left(-n\exp^{[q-2]}\left(\frac{\log^{[p-2]}}{K}\right)^{1/\nu}\right),\end{aligned} \quad (36)$$

and this contradicts the assumed equality (22). We have proved 3Â°. □

4. Best Rational Approximation in Terms of (p,q)-Growth

The aim of this section is to give a generalisation of the following theorems (see [11]).

Theorem 2. *Let f be a meromorphic function of order at most ρ, $(0 < \rho < \infty)$. Then,*

$$\sqrt[n]{\epsilon_n(\Delta, f)} < \frac{1}{n^{\alpha}}, \forall\alpha \quad (37)$$

Remark 2. A function f is entire of order at most ρ, $(0 < \rho < \infty)$, if and only if

$$\sqrt[n]{\epsilon_n(\Delta, f)} < \frac{1}{n^{\alpha}}, \forall\alpha \quad (38)$$

where r_n are replaced by polynomials.

Theorem 3. *Let f be a meromorphic function of order $\le \rho(p,q)$, $0 \le \rho(p,q) \le \infty$. Then,*

$$\limsup_{n\longrightarrow+\infty} \frac{\log\left(\log^{[p-2]}(n)^{\alpha}\right)}{\log^{[q-1]}\left(-(1/n)\log\left(\epsilon_n(\Delta, f)\right)\right)} \le 1. \quad (39)$$

Proof. By the Poisson–Jensen formula, we have

$$\begin{aligned}\log\left|\frac{(fQ_n - P_n)(z)}{z^{2n+1}}\right| &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{(fQ_n - P_n)(Re^{i\theta})}{(Re^{i\theta})^2 n + 1}\right|\\ &\quad\cdot \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta - \phi)}d\theta\\ &\quad + \sum_{|b_v|<R}\log\left|\frac{R^2 - \overline{b_v}z}{R(z - b_v)}\right|\\ &\quad - \sum_{|a_v|<R}\log\left|\frac{R^2 - \overline{a_v}z}{R(z - a_v)}\right|,\end{aligned} \quad (40)$$

where $z = re^{i\phi}$ and a_v and b_v are the zeros and poles, respectively, of $f\,Q_n - P_n$. Let $Q_n(z) = \prod\ (z - z_v)$. Since P_n is the nth Taylor polynomial to fQ_n, and hence majorized by a constant times $|Q_n|$ near the origin, we have on $|\omega| = R$: $|P_n(\omega) \le \prod\ (1 + |z_v|)|(R\,\text{const})^n$ by the Walsh–Bernstein lemma. With the usual notation of the Nevanlinna theory,

$$\begin{aligned} m(R) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(Re^{i\theta}\right)\right|d\theta,\\ N(R) &= \log\prod_{|b_v|<R}\frac{R}{b_v}, T(R) = N(R) + m(R),\end{aligned} \quad (41)$$

by replacing the integrand with $\text{const} + \log^+|f| + \log(\text{const}^n R^n \prod\ (1 + |z_v|)) - \log R^{2n+1}$ and integrating, using the fact that the Poisson kernel had integral 1 and is bounded for $r < R$,we get

$$\begin{aligned}\log\left|(fQ_n - P_n)(z)\right| &\le \log r^{2n+1} - \log R^{2n+1} + \text{const} + \text{const}\, m(R)\\ &\quad + \log\left(\text{const}^n R^n \prod\left(1 + \left|z_v\right|\right)\right)\\ &\quad + \sum_{2r<|b_v|<R}\log\left(\frac{2R}{|b_v|} + 2\right)\\ &\quad + \sum_{|b_v|\le 2r}\log\left(\frac{2R}{\epsilon}\right),\end{aligned} \quad (42)$$

if $|z - b_v| > \epsilon$. Now, if f is of order $\le \rho$, we have by definition that $T(R) \le R^{1/\alpha}$ for any $\alpha < \rho^{-1}$ for sufficiently large R, and we get

$$\begin{aligned}&\log\left|(f\,Q_n - P_n)(z)\right|\\ &\le \log\left(\frac{r}{R}\right)^{2n+1} + n\log\text{const} + \log R^n\prod\left(1 + \left|z_v\right|\right) + \text{const}\,R^{1/\alpha}\\ &\quad + \sum_{|b_v|\le 2r}\log 2R - \sum_{|b_v|\le 2r}\log\epsilon.\end{aligned} \quad (43)$$

We take $R = \exp^{q-1}(\log^{p-2}(n)^{\alpha})$ for so small r, and the two sums will disappear, and then subtract $\log|Q_n|$ to get

$\log|(f-(P_n/Q_n))(z)| \le \log(r^{2n+1}2^n/\exp^{q-1}(\log^{p-2}(n)\ ^{\alpha})^{n+1}) + n\log\text{const} - \log\delta^n$, except when $|Q_n(z)| \le \delta^n$.

Exponentiating, we get

$$\epsilon_n(\Delta, f) = \left|\left(f - \frac{P_n}{Q_n}\right)(z)\right|. \quad (44)$$

Then,

$$\begin{aligned}\frac{-1}{n}\log(\epsilon_n(\Delta, f)) &\ge \log\left(r^{(1/n)+2}2\,\text{const}\right)\\ &\quad + \exp^{q-2}\left(\log^{p-2}(n)^{\alpha}\right)^{1+(1/n)},\end{aligned} \quad (45)$$

for n assez grand

$$\begin{gathered}\frac{-1}{n}\log(\epsilon_n(\Delta, f)) \ge \exp^{q-2}\left(\log^{p-2}(n)^{\alpha}\right),\\ \frac{\log\left(\log^{p-2}(n)^{\alpha}\right)}{\log^{q-1}((-1/n)\log(\epsilon_n(\Delta, f)))} \le 1.\end{gathered} \quad (46)$$

Hence, the theorem is proved. □

Theorem 4. *Let f be a meromorphic function of finite order $\rho(p,q)$, $0 < \rho(p,q) < \infty$, and finite type. Then,*

$$\limsup_{n \longrightarrow +\infty} \frac{\left(\log^{[p-2]}\right)^{\alpha}(n)}{\log^{[q-2]}(-(1/n)\log(\epsilon_n(\Delta, f)))} \le c, \quad (47)$$

where c is a constant.

Proof. For the proof we use exactly the same step of Theorem 3 □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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