A Subclass of Bi-Univalent Functions Defined by Generalized Sâlãgean Operator Related to Shell-Like Curves Connected with Fibonacci Numbers

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The aim of this paper is to study certain subclasses of bi-univalent functions defined by generalized Sâlãgean differential operator related to shell-like curves connected with Fibonacci numbers. We find estimates of the initial coefficients and upper bounds for the Fekete-Szegő functional for the functions in this class. The results proved by various authors follow as particular cases.

1. Introduction and Preliminaries

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disc $U = \{ z : |z| < 1 \}$ with normalization $f(0) = f'(0) - 1 = 0$. By $S$, we denote the class of functions $f(z) \in A$ and univalent in $U$.

Let us denote by $B$ the class of bounded or Schwarz functions $w(z)$ satisfying $w(0) = 0$ and $|w(z)| \leq 1$ which are analytic in the open unit disc $U$ and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in U.$$ (2)

Consider two functions $f$ and $g$ analytic in $U$. We say that $f$ is subordinate to $g$ (symbolically $f \prec g$) if there exists a bounded function $u(z) \in B$ for which $f(z) = g(u(z))$. This result is known as principle of subordination.

By $S^*$, we denote the class of starlike functions $f \in S$ which satisfies the following condition:

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0$$

or

$$\frac{zf'(z)}{f(z)} < \frac{1+z}{1-z}, \quad (z \in U).$$ (3)

By $K$, we denote the class of convex functions $f \in S$ which satisfies the following condition:

$$\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > 0$$

or

$$\frac{zf''(z)}{f'(z)} < \frac{1+z}{1-z}, \quad (z \in U).$$ (4)
A function \( f \in S \) is said to be \( \alpha \)-convex if it satisfies the inequality
\[
\text{Re} \left( (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right) > 0 \quad (0 \leq \alpha \leq 1, \ z \in U).
\]

The class of \( \alpha \)-convex functions is denoted by \( M(\alpha) \) and was introduced by Mocanu [1]. In particular \( M(0) \equiv S^* \) and \( M(1) \equiv K \).

For \( \delta \geq 1 \) and \( f \in A \), Al-Oboudi [2] introduced the following differential operator:
\[
D^\delta_0 f(z) = f(z),
\]
\[
D^\delta_1 f(z) = (1-\delta) f(z) + \delta zf'(z),
\]
and in general,
\[
D^n\delta f(z) = D \left( D^{n-1}\delta f(z) \right)
\]
\[
= (1-\delta) D^{n-1}\delta f(z) + \delta z \left( D^{n-1}\delta f(z) \right)',
\]
\[
n \in \mathbb{N}
\]
or equivalent to
\[
D^n\delta f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + (k-1)\delta \right] a_k z^k,
\]
\[
n \in N_0 = \mathbb{N} \cup \{0\}
\]
with \( D^n\delta f(0) = 0 \). It is obvious that, for \( \delta = 1 \), the operator \( D^n\delta f(z) \) is equivalent to the Sălăgean operator introduced in [3]. So the operator \( D^n\delta f(z) \) is named as the Generalized Sălăgean operator.

The inverse functions of the functions in the class \( S \) may not be defined on the entire unit disc \( U \) although the functions in the class \( S \) are invertible. However using Koebe’s one quarter theorem [4] it is obvious that the image of \( S \) under every function \( f \in S \) contains a disc of radius \( 1/4 \). Hence every univalent function \( f \) has an inverse \( f^{-1} \), defined by
\[
f^{-1}(f(z)) = z (z \in U)
\]
and
\[
f(f^{-1}(w)) = w \left( |w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right)
\]
where
\[
g(w) = f^{-1}(w)
\]
\[
= w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots
\]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \).

By \( \Sigma \), we denote the class of bi-univalent functions in \( U \) defined by (1).

Lewin [5] discussed the class \( \Sigma \) of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [6] investigated certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients.

Sokol [7] introduced the class \( SL \) of shell-like functions \( f \in A \) defined as below.

**Definition 1.** A function \( f \in A \) given by (1) is said to be in the class \( SL \) of starlike shell-like functions if it satisfies the following condition:
\[
\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = 1 + \tau z^2
\]
where \( \tau = (1 - \sqrt{5})/2 = -0.618 \).

It should be observed that \( SL \) is a subclass of the class \( S^* \) of starlike functions.

Later Dziok et al. [8] introduced the class \( KSL \) of convex functions related to a shell-like curve as below.

**Definition 2.** A function \( f \in A \) given by (1) is said to be in the class \( KSL \) of convex shell-like functions if it satisfies the condition that
\[
1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = 1 + \tau^2 z^2
\]
where \( \tau = (1 - \sqrt{5})/2 = -0.618 \).

Again Dziok et al. [9] defined the following class of \( \alpha \)-convex shell-like functions.

**Definition 3.** A function \( f \in A \) given by (1) is said to be in the class \( SLM_\alpha \) of \( \alpha \)-convex shell-like functions if it satisfies the condition that
\[
(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec \tilde{p}(z)
\]
\[
= \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]
where \( \tau = (1 - \sqrt{5})/2 = -0.618 \).

Obviously \( SLM_0 \equiv SL \) and \( SLM_1 \equiv KSL \).

The function \( \tilde{p} \) is not univalent in \( U \), but it is univalent in the disc \( |z| < (3 - \sqrt{5})/2 \approx 0.38 \). For example, \( \tilde{p}(0) = \tilde{p}(-1/2 \tau) = 1 \) and \( \tilde{p}(e^{i\arccos(1/4)}) = \sqrt{5}/5 \), and it may also be noticed that
\[
\frac{1}{|r|} = \frac{|r|}{1 - |r|}
\]
which shows that the number \(|r|\) divides \([0, 1]\) such that it fulfills the golden section. The image of the unit circle \(|z| = 1\) under \(\bar{p}\) is a curve described by the equation given by

\[
(10x - \sqrt{5})^y = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,
\]

which is translated and revolved trisectrix of Maclaurin. The curve \(\bar{r}(\tau)\) is a closed curve without any loops for \(0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38\). For \(r_0 < r < 1\), it has a loop, and for \(r = 1\), it has a vertical asymptote. Since \(\tau\) satisfies the equation \(\tau^2 = 1 + \tau\), this expression can be used to obtain higher powers \(\tau^n\) as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of \(r\) and \(1\). The resulting recurrence relationships yield Fibonacci numbers \(u_i\):

\[
\tau^n = u_n,\tau + u_{n-1}.
\]

Also the subclasses of bi-univalent functions related to shell-like curves were studied by various authors [10–12].

The earlier work on bi-univalent functions related to shell-like curves connected with Fibonacci numbers motivated us to define the following subclass.

To avoid repetition, throughout the paper we assume that \(0 \leq \alpha \leq 1\), \(\tau = (1 - \sqrt{5})/2 \approx -0.618\) and \(z \in U\).

**Definition 4.** A function \(f \in \Sigma\) given by \((1)\) is said to be in the class \(S_L M_{\alpha, \Sigma}(n, \bar{p}(z))\) if it satisfies the following conditions:

\[
(1 - \alpha) \left( D_{\delta}^{n+1} f (z) \right) + \alpha \left( D_{\delta}^{n+1} f (z) \right)' < \bar{p}(z)
\]

\[
= \frac{1 + r^2 \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

\[
|a_2| \leq \frac{|r|}{\sqrt{\delta^2 (1 + \delta)^2 (1 + \alpha)^2 (1 - 3 \tau) + \tau \delta \left[ 2 (1 + 2 \alpha) (1 + 2 \delta)^n - (1 + 3 \alpha) (1 + \delta)^n \right]}}.
\]

\[
|a_3| \leq \frac{|r| (1 + \delta)^2 \left[ \delta^2 (1 + \alpha)^2 (1 - 3 \tau) - \tau \delta (1 + 3 \alpha) \right]}{2 (1 + 2 \alpha) \delta (1 + 2 \delta)^n \left[ \delta^2 (1 + \delta)^2 (1 + \alpha)^2 (1 - 3 \tau) + \tau \delta \left( 2 (1 + 2 \alpha) (1 + 2 \delta)^n - (1 + 3 \alpha) (1 + \delta)^n \right) \right]}
\]

**Proof.** As \(f \in S_L M_{\alpha, \Sigma}(n, \bar{p}(z))\), so by Definition 4 and using the principle of subordination, there exist Schwarz functions \(r(z)\) and \(s(z)\) such that

\[
(1 - \alpha) \left( D_{\delta}^{n+1} g (w) \right) + \alpha \left( D_{\delta}^{n+1} g (w) \right)' < \bar{p}(s(w))
\]

and

\[
(1 - \alpha) \left( D_{\delta}^{n+1} g (w) \right) + \alpha \left( D_{\delta}^{n+1} g (w) \right)' = \bar{p}(s(w))
\]

where \(r(z) = 1 + r_1 z + r_2 z^2 + \cdots\) and \(s(w) = 1 + s_1 w + s_2 w^2 + \cdots\).
On expanding, it yields
\[
(1 - \alpha) D^{n+1}_\delta f(z) + \alpha \left( D^{n+1}_\delta f(z) \right)' = 1 + (1 + \alpha) \delta \]
\[
\cdot \delta (1 + \delta)^n a_2 z + \delta \left[ 2 (1 + 2\alpha)(1 + 2\delta)^n a_3 
- (1 + 3\alpha)(1 + \delta)^{2n} a_5^2 \right] z^2 + \cdots
\]
(24)
and
\[
(1 - \alpha) D^{n+1}_\delta g(w) + \alpha \left( D^{n+1}_\delta g(w) \right)' = 1 - (1 + \alpha) \delta \]
\[
\cdot \delta (1 + \delta)^n a_2 w + \delta \left[ 4 (1 + 2\alpha)(1 + 2\delta)^n a_3 
+ (4 (1 + 2\alpha)(1 + 2\delta)^n - (1 + 3\alpha)(1 + \delta)^{2n} a_5^2 \right] a_2^2 \]
\[
\cdot w^2 + \cdots
\]
Again
\[
\tilde{p}(r(z)) = 1 + \frac{\tilde{p}_1 c_1 z}{2} + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{4} \tilde{p}_2 \right) z^2
\]
\[
+ \cdots
\]
(26)
and
\[
\tilde{p}(s(z)) = 1 + \frac{\tilde{p}_1 d_1 w}{2} + \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} + \frac{d_1^2}{4} \tilde{p}_2 \right) w^2
\]
\[
+ \cdots
\]
(27)
Using (24) and (26) in (22) and equating the coefficients of \( z \) and \( z^2 \), we get
\[
(1 + \alpha) \delta (1 + \delta)^n a_2 = \frac{c_1 \tau}{2}
\]
(28)
and
\[
\delta \left[ 2 (1 + 2\alpha)(1 + 2\delta)^n a_3 - (1 + 3\alpha)(1 + \delta)^{2n} a_5^2 \right]
= \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tau + \frac{c_1^3}{4} \tau^2.
\]
(29)
Again using (25) and (27) in (23) and equating the coefficients of \( w \) and \( w^2 \), we get
\[
-(1 + \alpha) \delta (1 + \delta)^n a_2 = \frac{d_1 \tau}{2}
\]
(30)
and
\[
\delta \left[ -2 (1 + 2\alpha)(1 + 2\delta)^n a_3 
+ (4 (1 + 2\alpha)(1 + 2\delta)^n - (1 + 3\alpha)(1 + \delta)^{2n}) a_5^2 \right]
= \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tau + \frac{d_1^3}{4} \tau^2.
\]
(31)
From (28) and (30), it is clear that
\[
c_1 = -d_1
\]
(32)
and
\[
2a_2^2 = \frac{(c_1^2 + d_1^2) \tau^2}{4\delta^2 (1 + \delta)^{2n} (1 + \alpha)^2}.
\]
(33)
Adding (29) and (31), it yields
\[
\delta \left[ 4 (1 + 2\alpha)(1 + 2\delta)^n - 2 (1 + 3\alpha)(1 + \delta)^{2n} \right] a_2^2
= \frac{1}{2} (c_2 + d_2) \tau - \frac{1}{4} (c_1^2 + d_1^2) \tau + \frac{3}{4} (c_1^2 + d_1^2) \tau^2.
\]
(34)
Putting (33) in (34), we get
\[
[\delta \tau \left( 4 (1 + 2\alpha)(1 + 2\delta)^n - 2 (1 + 3\alpha)(1 + \delta)^{2n} \right)
+ 2 (1 - 3\tau) \delta^2 (1 + \delta)^{2n} (1 + \alpha)^2] a_2^2
= \frac{1}{2} (c_2 + d_2) \tau.
\]
(35)
Using Lemma 5 and on applying triangle inequality in (35), (20) can be easily obtained.

Now subtracting (31) from (29), we obtain
\[
4\delta (1 + 2\alpha)(1 + 2\delta)^n a_3 - 4\delta (1 + 2\alpha)(1 + 2\delta)^n a_2^2
= \frac{1}{2} (c_2 - d_2) \tau.
\]
(36)
Applying triangle inequality and using Lemma 5 and (35) in (36), it yields
\[
4\delta (1 + 2\alpha)(1 + 2\delta)^n |a_3| 
\leq 2 |\tau| + 4\delta (1 + 2\alpha)(1 + 2\delta)^n |a_2|.
\]
(37)
From (37), result (21) is obvious.

For \( \delta = 1 \), Theorem 6 gives the following result.

**Corollary 7.** If \( f \in SLM_{\alpha, \pm}(n, \tilde{p}(z)) \), then
\[
|a_2| 
\leq \frac{|\tau|}{\sqrt{4^n (1 + \alpha)^2 + [2 (1 + 2\alpha) 3^n - (3\alpha^2 + 9\alpha + 4) 4^n] \tau}}.
\]
(38)
\[ |a_3| \leq \frac{|r| 4^n \left[ (1 + \alpha)^2 - (3\alpha^2 + 9\alpha + 4) r \right]}{2 (1 + 3\alpha) 3^n [4^n (1 + \alpha)^2 + (2 (1 + 2\alpha) 3^n - (3\alpha^2 + 9\alpha + 4) 4^n) r]} \]  

(39)

For \( \delta = 1, n = 0 \), Theorem 6 gives the following result due to Güney et al. [13].

Corollary 8. If \( f(z) \in SLM_{\alpha,\Sigma}(\bar{p}(z)) \), then

\[ |a_2| \leq \frac{|r|}{\sqrt{(1 + \alpha)^2 - (1 + \alpha)(2 + 3\alpha) r}} \]  

(40)

and

\[ |a_3| \leq \frac{|r| \left[ (1 + \alpha)^2 - (3\alpha^2 + 9\alpha + 4) (2 + 3\alpha) r \right]}{2 (1 + 2\alpha) (1 + \alpha) [(1 + \alpha) - (2 + 3\alpha) r]} \]  

(41)

For \( \delta = 1, n = 0, \alpha = 0 \), Theorem 6 agrees with the following result proved by Güney et al. [13] (Corollary 1).

Corollary 9. If \( f(z) \in SL_{\Sigma}(\bar{p}(z)) \), then

\[ |a_2| \leq \frac{|r|}{\sqrt{1 - 2r}} \]  

(42)

\[ |a_3| \leq \frac{|r| 4^n \left[ (1 + \alpha)^2 - (3\alpha^2 + 9\alpha + 4) r \right]}{2 \delta (1 + 2\alpha) (1 + \delta) [2 (1 + 2\alpha) 3^n - (1 + 3\alpha) (1 + \delta) 2^n (1 - 3r)] + \delta (1 + \alpha) \delta (1 + \delta) 2^n (1 - 3r)} \]  

(46)

Proof. From (35) and (36), it yields

\[ a_3 - \mu a_2^2 = (1 - \mu) \frac{\tau^2 (c_2 + d_2)}{4 \delta (1 + 2\alpha) (1 + 2\delta) 4^n} + \frac{\tau (c_2 - d_2)}{8 \delta (1 + 2\alpha) (1 + 2\delta)^n} \]  

(47)

Equation (47) can be expressed as

\[ a_3 - \mu a_2^2 = \left[ h(\mu) + \frac{\tau}{8 \delta (1 + 2\alpha) (1 + 2\delta)^n} \right] c_2 \]  

(48)
where
\[
h(\mu) = \frac{(1-\mu)\tau^2}{4\tau\delta\left[2(1+2\alpha)(1+2\delta)^n - (1+3\alpha)(1+\delta)^6n\right] + 4\delta^2(1+\delta)^2n(1+\alpha)^2(1-3\tau)}. \tag{49}
\]

Taking modulus, we obtain
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|r|}{2(1+2\alpha)(1+2\delta)^n} 
& 0 \leq |h(\mu)| \leq \frac{|r|}{8\delta(1+2\alpha)(1+2\delta)^n}, \\
\frac{|1-\mu|\tau^2}{(2(1+2\alpha)(1+\delta)^6n)(1+2\tau)^2n} 
& |h(\mu)| \geq \frac{|r|}{8\delta(1+2\alpha)(1+2\delta)^n}.
\end{cases} \tag{50}
\]

So (46) can be easily obtained from (50).

For \(\delta = 1, n = 0\), Theorem 11 gives the following result due to Güney et al. [13].

Corollary 12. Let \(f(z) \in SLM_{\alpha,\Sigma}(\tilde{p}(z))\), then
\[
|\mu - 1| \leq \frac{(2(1+2\alpha)^3n - (3\alpha^2 + 9\alpha + 4)^4n)\tau + (1+\alpha)^24^n}{2|r|(1+2\alpha)3^n}, \\
|\mu - 1| \geq \frac{(2(1+2\alpha)^3n - (3\alpha^2 + 9\alpha + 4)^4n)\tau + (1+\alpha)^24^n}{2|r|(1+2\alpha)3^n}. \tag{51}
\]

For \(\delta = 1, n = 0, \alpha = 0\), Theorem 11 agrees with the following result proved by Güney et al. [13] (Corollary 4).

Corollary 13. If \(f(z) \in SL_{\Sigma}(\tilde{\bar{p}}(z))\), then
\[
|\mu - 1| \leq \frac{(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]}{2|r|(1+2\alpha)}; \\
|\mu - 1| \geq \frac{(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]}{2|r|(1+2\alpha)}. \tag{52}
\]

For \(\delta = 1, n = 0, \alpha = 1\), Theorem 11 agrees with the following result proved by Güney et al. [13] (Corollary 5).

Corollary 14. If \(f(z) \in KSL_{\Sigma}(\tilde{\bar{p}}(z))\), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|r|}{2(1+2\alpha)}, 
& 0 \leq |h(\mu)| \leq \frac{|r|}{2|r|(1+2\alpha)}, \\
\frac{|1-\mu|\tau^2}{(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]} 
& |h(\mu)| \geq \frac{|r|}{2|r|(1+2\alpha)}.
\end{cases} \tag{53}
\]

For \(\delta = 1, n = 0, \alpha = 0\), Theorem 11 agrees with the following result proved by Güney et al. [13] (Corollary 5).

Corollary 15. If \(f(z) \in KSL_{\Sigma}(\tilde{\bar{p}}(z))\), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|r|}{6}, 
& 0 \leq |h(\mu)| \leq \frac{|r|}{3|r|}, \\
\frac{|1-\mu|\tau^2}{2(2-5\tau)} 
& |h(\mu)| \geq \frac{|r|}{3|r|}.
\end{cases} \tag{54}
\]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


