Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2020, Article ID 1853467, 10 pages
https://doi.org/10.1155/2020/1853467

Research Article

Fixed Points, Symmetries, and Bounds for Basins of Attraction of Complex Trigonometric Functions

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Received 17 December 2019; Accepted 3 February 2020; Published 23 April 2020

Academic Editor: Bao Q. Li

The dynamical systems of trigonometric functions are explored, with a focus on $t(z) = \tan(z)$ and the fractal image created by iterating the Newton map, $F_t(z)$, of $t(z)$. The basins of attraction created from iterating $F_t(z)$ are analyzed, and some bounds are determined for the primary basins of attraction. We further prove x- and y-axis symmetry of the Newton map and explore the nature of the fractal images.

1. Introduction

Newton's method is a very well-known iterative procedure for finding roots. Rather than applying Newton's method to one point at a time, we can instead consider the Newton map:

$$F_f(z) = z - \frac{f(z)}{f'(z)}, \tag{1}$$

of the function $f(z)$. Iterating $F_f(z)$ from a given starting value yields the familiar sequence produced by Newton's method. Iterating the function $F_f(z)$ yields a sequence of maps with fascinating dynamics.

The dynamics of the Newton map have been studied for the cases when the original function f is a polynomial or rational function (see Dwyer et al. [1]; Dwyer et al. [2]). In this paper, we consider the case $f(z) = \tan(z)$.

We first present an outline of the primary definitions and theorems, with an emphasis on the concepts used in this paper. Next, we describe the Newton maps of $\sin(z)$ and $\cos(z)$ before moving to the much more complex case of $\tan(z)$. We illustrate the symmetries of the Newton map for $\tan(z)$ and provide bounds for the basins of attraction.

1.1. Preliminaries in Complex Analysis. An expression of the form $z = x + iy$, where x and y are real numbers, is a complex number and $\mathbb{C}$ is the set of all complex numbers. We call x the real part, denoted as $\mathrm{Re}(z)$, and y the imaginary part, denoted as $\mathrm{Im}(z)$. The *modulus* (or absolute value) of z, $|z|^2 = x^2 + y^2$, is a real number which measures the distance from the origin, and $\overline{z} = x - iy$ is the *complex conjugate* of z.

Each complex number z in $\mathbb{C}$ can be identified with the unique point $(\mathrm{Re}(z), \mathrm{Im}(z))$ in the plane $\mathbb{R}^2$. We can establish polar coordinates, r and θ, for $z = x + iy$by noting $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{(x^2 + y^2)}$, and θ is the angle between the positive real axis and the line segment from 0 to z in the counterclockwise direction. Hence, the complex number $z = x + iy$ can be written in the polar form $z = re^{i\theta}$, and using Euler's equation we obtain $z = r(\cos\theta + i\sin\theta)$.

A complex-valued function $f(z) = f(x + iy)$ assigns to each z in the domain exactly one complex number $w = f(z)$. Just as z decomposes into real and imaginary parts, each complex-valued function can be written as $f(z) = u(x, y) + iv(x, y)$, where u and v are each real-valued functions. In essence, $f(z)$ is a pair of real functions of two real variables that maps regions from its domain in the complex plane onto its range in another copy of the complex plane.

The derivative of a complex function is defined by an extension of the definition of the real case. If G is an open set in the complex plane and $f: G \longrightarrow \mathbb{C}$, then f is *differentiable* at a point $z_0 \in \mathbb{C}$ if $\lim_{z \longrightarrow z_0} (f(z) - f(z_0))/(z - z_0)$ exists. The value of this limit is denoted as $f'(z)$ and is called the *derivative* of f at z_0.

It is possible for a complex function to be differentiable solely at isolated points, thus analyticity, a property defined over open sets, is a stronger condition. A complex-valued function $f(z)$ is said to be *analytic* on an open set G if it has a derivative at every point of G. If $f(z)$ is analytic on $\mathbb{C}$, then it is said to be *entire*.

The complex number z is a *zero* (or root) of the function $f(z)$ if it is a solution to the equation $f(z) = 0$.

Singularities of functions can result in particularly interesting fractal images. Let $B_r(a)$ denote the ball of radius r about a. Then, we have the following: a function f has an *isolated singularity* at $z = a$ if there exists $r > 0$ such that f is defined and analytic in $B_r(a) \backslash \{a\}$ but not in $B_r(a)$. That is, f is an analytic in some neighborhood of a but not at a itself. In addition, if $\lim_{z \longrightarrow a} |f(z)| = \infty$, then a is called a *pole* of f. The point a is called a *removable singularity* if there is an analytic function g: $B_r(a) \longrightarrow \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < r$. If an isolated singularity is neither a pole nor a removable singularity, it is called an *essential singularity*.

A function f is said to be *meromorphic* in a domain G if at every point of G it is either analytic or has a pole. In particular, we regard analytic functions on G as being special cases of meromorphic functions, and in this paper, we consider only analytic and meromorphic functions.

1.2. Preliminaries in Complex Dynamical Systems. For an analytic function f and given z_0 in $\mathbb{C}$, the *orbit of* z_0 is the sequence of iterates $\{z_0, f(z_0), f^2(z_0), \ldots, f^n(z_0), \ldots\}$ where $f^2(z_0)$ means $f(f(z_0)$ and $f^n(z_0)$ is the n^{th} application of the function f to the value z_0. The initial value z_0 is called the *seed value*, and dynamics is interested in the fate of orbits, that is, the behavior of $\{f^n(z_0)\}$ as $n \longrightarrow \infty$. Do they converge, diverge, cycle, or behave chaotically? A *fixed point* occurs when $f(z) = z$. If $f^n(z) = z$ for some $n \in \mathbb{Z}$, and z, $f(z), \ldots, f^{n-1}(z)$ are distinct points, then z is a *periodic point* with period *n*, defined as the smallest n for which $f^n(z) = z$. The set $\{z, f(z), \ldots, f^{n-1}(z)\}$ is called a *n-cycle for* f. If the orbit of z contains preliminary values before settling at either a fixed point ($f^{n+1}(z) = f(z)$ for some $n > 1$) or a periodic orbit ($f^{n+p}(z) = f^n(z)$ for some $n > 1$, where p is the period of the periodic orbit), then z is called an *eventually fixed point* or *eventually periodic*, respectively.

For example, consider the complex function $f(z) = z^2$. Fixed points of f occur when $z^2 = z$, and thus the fixed points are 0 and 1. Seed values $-1, i$, and $-i$ have orbits $\{-1, 1\}$, $\{i, -1, 1\}$, and $\{-i, -1, 1\}$, respectively. Since each orbit lands on the fixed point 1 after several iterations, these are all eventually fixed points. Similarly $z_0 = e^{2\pi i/3}$ and $z_0 = e^{4\pi i/3}$ are periodic fixed points of period two. If $|z_0| > 1$, then $\{f^n(z_0)\} \longrightarrow \infty$ as $n \longrightarrow \infty$, and if $|z_0| < 1$, then $\{f^n(z_0)\} \longrightarrow 0$ as $n \longrightarrow \infty$.

Theorem 1. *Let f be a (real or complex) continuous function from its domain set to itself. Suppose a and z_0 are both in the domain set and $\{f^n(z_0)\} \longrightarrow a$, then $f(a) = a$.*

Proof. Since the sequence $\{f^n(z_0)\} \longrightarrow a$ and f is continuous, we must have

$$f(a) = f\left(\lim_{n \longrightarrow \infty} f^n(z_0)\right) = \lim_{n \longrightarrow \infty} f^{n+1}(z_0) = a. \tag{2}$$

□

1.3. Newton's Method. For a real-valued function $f(x)$, $x \in \mathbb{R}$, with root a, we define the *Newton map of* f as

$$F_f(x) = x - \frac{\mathrm{f}(\mathrm{x})}{\mathrm{f}'(\mathrm{x})}. \tag{3}$$

The sequence of iterates $\left\{F_f^n(x_0)\right\}$ converges to the sought-after root, a, for an initial guess x_0 "close enough" to a. Note that when $f(x) = 0$, $F(x) = x$; therefore, the convergence of $\left\{F_f^n\right\}$ to a root of $f(x)$ can be thought of as a convergence to a fixed point of $F(x)$. Henceforth, we will be concerned with the orbits $\left\{F_f^n(z_0)\right\}$ for various analytic or meromorphic functions f and seed values z_0 and explore the fractal nature of the images created by these iterations.

Example 1. Consider the function $f(z) = (z-2)(z-3)$, which has zeros at $z = 2$ and $z = 3$, and Newton map $F_f(z) = z - ((z^2 - 5z + 6)/(2z - 5)) = (z^2 - 6)/(2z - 5)$. For seed value $z_0 = 1/2$, the iterates are $z_0 = 0.5, z_1 = 1.4375, z_2 = 1.8511, z_3 = 1.9829, z_4 = 1.9997, z_5 = 1.9999, \ldots$ which are clearly converging to root $z = 2$. Experiments with different values of z_0 will reveal that seed values which are closer to the root 2 will produce orbits that converge to 2, and likewise, points closer to 3 will iterate to 3 under F_f. Note that if we choose $z_0 = 5/2$, Newton's method will fail. Analytically, we can see this because 5/2 leads to a zero in the denominator of F and informally we see that the point 5/2 lies directly halfway between roots 2 and 3 and is thus pulled equally in both directions by each root. That is, 5/2 separates those points on the real line which will iterate to 2 and those which will iterate to 3, so it seems our method should fail at this point. Figure 1 shows the dynamics for F_f. Iterates of seed values from the green region will converge to 2, whereas seed values from the blue region will iterate to 3.

For any real or complex function f and any real or complex number w, we define $A_f(w)$, the *attracting basin of* w, under the function f, to be the set of all starting points whose iterates limit to the point w. That is,

$$A_f(w) = \{z \in G\colon f^n(z) \longrightarrow w\}. \tag{4}$$

For any real or complex function f and any real or complex number w, we define $A_f^*(w)$, the *primary* (or immediate) *basin of attraction of* w under f, to be the largest

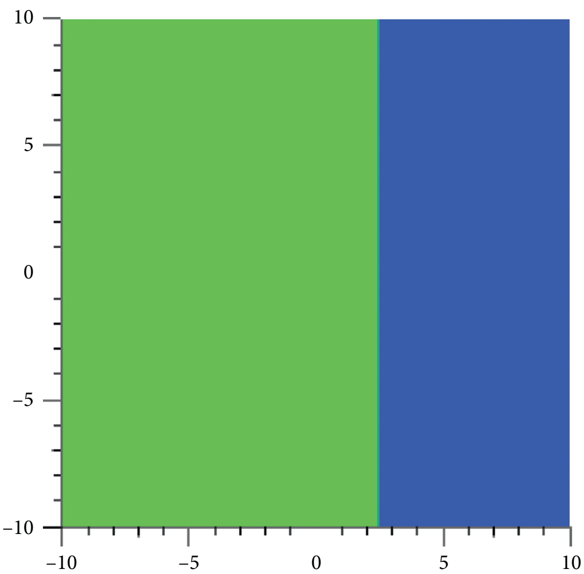


FIGURE 1: The dynamics of the Newton map of $f(z) = (z-2)(z-3)$, as discussed in Example 1.

FIGURE 2: The global dynamics of $F(z) = (-z^3 + 3z^2 + 1)/3z$, the Newton map for $f(z) = z^3 - 1$. The blue region represents $A_F(1)$, green represents $A_F(e^{2\pi i/3})$, and yellow represents $A_F(e^{-2\pi i/3})$.

set containing w that lies in the basin of attraction of w. Equivalently, the primary basin of attraction is the connected component of the basin of attraction containing w.

Theorem 2 (attracting property of Newton's method). *Given any real or complex analytic function f with root a, there exists $r > 0$ such that all points within a distance r of a are necessarily in the set $A_F(a)$, where F is the Newton map of f. That is, for all initial values z_0 that are "close enough" to a, the orbit $F^n(z_0)$ converges to a. An outline of the proof is discussed by Brilleslyper et al. [3].*

Note that $A_F^*(w) \subseteq A_F(w)$ and there is no universal value for r, as it depends heavily on the function f. The value r gives a lower bound on how close a seed value must be in order for guaranteed convergence under Newton's method. Furthermore, as f becomes more complicated, so do the dynamics of F, and it may not simply be the case that all seed values close to a certain root converge to that root. Consider Figure 2, where the picture of the dynamics for $f(z) = z^3 - 1$ is shown.

1.4. Classification of Fixed Points. The behavior of a function near its fixed points can vary, and we use this to classify fixed points. In general, if the action of f is to move points closer to the fixed point, we call it attracting. Conversely, a fixed point may be repelling, in which case, no matter how close the seed is to the point, iterates of the seed will diverge from the fixed point. Formally, we have the following definition.

Definition 1. Let f be a map from its domain set G (a subset of either $\mathbb{R}$ or $\mathbb{C}$) into itself.

(a) A finite fixed point a in $\mathbb{C}$ is an *attracting fixed point* (of f) if there exists a neighborhood U of a such that for any point $z \in G \cap U \setminus \{a\}$, we have $|f(z) - a| < |z - a|$.

(b) We call ∞ an *attracting fixed point* of f if there exists a neighborhood U of ∞ such that for any point $z \in G \cap U \setminus \{\infty\}$, we have $|f(z)| > |z|$. That is, the action of f is to move each point in $G \cap U \setminus \{\infty\}$ closer to ∞ (as measured by the spherical metric).

(c) A finite fixed point a in $\mathbb{C}$ is called a *repelling fixed point* (of f) if there exists a neighborhood U of a such that for any point $z \in G \cap U \setminus \{a\}$, we have $|f(z) - a| > |z - a|$.

(d) We call ∞ a *repelling fixed point* of f if there exists a neighborhood U of ∞ such that for any point $z \in G \cap U \setminus \{\infty\}$, we have $|f(z)| < |z|$.

Observe that in the case of (a) and (b) in Definition 1, $UA_F^*(a) \subseteq A_F(a)$. Thus, if a fixed point is attracting, the iterates of any seed value in the neighborhood U converge *monotonically* to the fixed point, while points in $A_F(a) \backslash U$ will *eventually* converge to a. The following theorem is helpful in determining the classification of a given fixed point.

Theorem 3. *Let $f(z)$ be an analytic map on a domain $G \subset \mathbb{C}$ such that $f(a) = a$ for some a in G. Then, we have the following:*

(a)a is an attracting fixed point if and only if $|f'(a)| < 1$,

(b) a is a repelling fixed point if and only if $|f'(a)| > 1$.

We note that if $|f'(a)| = 0$, a is called a *superattracting fixed point*, and if $|f'(a)| = 1$, a is a *neutral fixed point*. The behavior of neutral fixed points is complicated: sometimes they can exhibit both a partial attracting nature and a partial repelling nature. For example, for the complex function $f(z) = \sin(z)$, we have a neutral fixed point at $a = 0$. For real-valued seeds, a has attracting properties, but for purely imaginary-valued seeds, $a = 0$ has a repelling nature. Indeed, for $z = bi$, $b \in \mathbb{R}$, $\sin(bi) = i\sinh(b)$. Now, since $\sinh(b)$ is an increasing function, for all $b \in \mathbb{R}^+$, $\{\sin^n(bi)\} \longrightarrow \infty$ as $n \longrightarrow \infty$, and for all $b \in \mathbb{R}^-$, $\{\sin^n(bi)\} \longrightarrow -\infty$ as $n \longrightarrow \infty$. Similarly, neutral fixed points can act in neither an attracting nor repelling fashion.

We can see that periodic points correspond exactly to fixed points of higher iterates f^n of f, so the subsequent classifications follow from Theorem 3.

Definition 2. Suppose $\{z_0, f(z_0), \ldots, f^{n-1}(z_0)\}$ forms a p-cycle for the map f. That is, $f^n(z_0) = z_0$. Let $\lambda = \mathrm{d}/\mathrm{d}z f^n(z)|_{z=z_0}$, then the p-cycle of f is called

(a) Superattracting if $\lambda = 0$,

(b) Attracting if $0 < |\lambda| < 1$,

(c) Repelling if $|\lambda| > 1$,

(d) Neutral if $|\lambda| = 1$.

2. Newton's Method and Trigonometric Functions

As discussed by Alexander et al. [4], the beginning of the study of complex dynamics dates back to 1870 where Ernst Schröder studied iterative equation solving algorithms in the complex plane and hence the nature of the infinite sequence $\{z, f(z), f^2(z), \ldots, f^n(z), \ldots\}$ from a theoretical viewpoint. Schröder's interest in iterations led to the cornerstones of complex dynamics: Schröder's fixed-point theorem and fixed point classifications. Schröder used a heuristic approach in his proof of the fixed-point theorem that relied on the Taylor series expansion of f about an attracting fixed point and led to an explanation as to why Newton's method works. He also studied convergence rates. Namely, for simple roots, a, of a polynomial, Newton's method converges quadratically in a neighborhood U of a. For roots of multiplicity greater than 1, Schröder modified Newton's method to maintain this desirable convergence, and later, developed a family of similar root-solving algorithms which would either increase the rate of convergence, or have convergence of an arbitrary order.

Schröder was successful in developing fundamental tools of complex dynamics and applying them to the Newton map of $q(z) = z^2 - 1$: "Schröder observed, first, that there were periodic points of $F_q(z)$ of every order on the imaginary axis.His second observation was that if z was on the imaginary axis, but not eventually periodic, then $F_q^k(z)$ takes on infinitely many values" [4]. In other words, the forward orbit of z consists of infinitely many distinct points. Yet, his work raised an unanswered fundamental question: how far away can an arbitrary point z be from an attracting fixed point a of a function f, such that $f^n(z) \longrightarrow a$? It was not until the emergence of independent work by Pierre Fatou and Gaston Julia near the end of WWI that solutions to this query were explored. Furthermore, Schröder was unable to extend his results to higher-degree polynomials, in particular, he failed in an attempt to understand Newton's method for the cubic $c(z) = z^3 - 1$ and it took more than 45 years before the dynamics of $F_c(z)$ were well understood [4].

The 1918 *Grand Prix des Sciences Mathematiques* competition was devoted to the study of the iteration of complex functions and hence sparked a flurry of papers published by Fatou and Julia on the global behavior of iterates of complex rational functions. During this time, they were able to bring about a substantial global theory of rational dynamics, but little was known about the global iterative behavior of other kinds of complex functions [4]. Unfortunately, it was not until the onset of the personal computer that complex dynamics received considerable public attention again, and in the early 1980s the field exploded as computer-generated images of the Mandelbrot set and Newton's method on cubic polynomials circulated widely.

As shown later, the main focus of this paper will be the iteration of the function $z - \sin(z)\cos(z)$, so a look at literature concerning the dynamics of trigonometric functions will be of use. Unfortunately, this particular set of research is more limited than that of polynomials and rational functions. Schubert [5] proves that the area of the Fatou (or stable) set of the sine function in a vertical strip of width 2π is finite. He also references the dynamical work, both historical and recent, carried out with functions such as $\sin(z)$, $\cos(z)$, $h\sin(z) + a$ for $0 < h < 1$ real, $\sin(az + b)$ for $a \neq 0$, and $\lambda \sin(z)$.

Devaney [6] discusses the special class of meromorphic functions $f(z)$ whose Schwarzian derivative:

$$S(f(z)) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{5}$$

is a polynomial including the family of functions $\lambda\tan(z)$. As we know, the fate of asymptotic and critical values under iteration plays a crucial role in determining dynamics, and the main property of maps with polynomial Schwarzian derivatives is that they have a finite number of asymptotic values (all of which are isolated) and no critical values. In particular, for $T_\lambda(z) = \lambda\tan(z) = (\lambda/i)(e^{iz} - e^{-iz})/(e^{iz} + e^{-iz})$, where $\lambda > 0$, we have $S(T_\lambda(z)) = 2$, T_λ has asymptotic values at λi and $-\lambda i$, and T_λ preserves the real axis.

The Julia set $J(T_\lambda)$ is the closure of the set of repelling periodic points, or equivalently, $J(T_\lambda)$ is the closure of the set which consists of the union of all of the preimages of the poles of T_λ. Furthermore, all the poles and their preimages are dense in the Julia set. Devaney shows that $J(T_\lambda)$ is not a fractal set, indeed, it is a smooth manifold of $\mathbb{C}$. If $\lambda = 1$, then $J(T_\lambda) = \mathbb{R}$, and all points $z = x + iy$ such that $y \neq 0$ tend asymptotically to the neutral fixed point 0. When $\lambda < -1$, T_λ has an attracting periodic cycle of period two, and points in the upper and lower half planes hop back and forth as they are attracted to the cycle. And, since $|T_\lambda'(x)| > 1$ for all $x \in \mathbb{R}$, $J(T_\lambda) = \mathbb{R}$ for $\lambda < -1$ [6].

Devaney's results show that for $0 < |\lambda| < 1$, zero is an attracting fixed point for T_λ and $J(T_\lambda)$ breaks up into a Cantor set. In fact, the basin of zero is infinitely connected, contrasting the situation for polynomial or entire maps (such as $z - \sin(z)\cos(z)$) in which finite attracting fixed points always have a simply connected immediate basin of attraction (Devaney [6]). A full picture of the parameter plane for the tangent family is presented by Keen and Kotus [7]; however, an extensive literature search has failed to find work on the dynamics of Newton's method applied to trigonometric functions.

2.1. Introduction to the Dynamics of Trig Functions. This paper focuses on the dynamics of the Newton map of $t(z) = \tan(z)$, so as a means of comparison, we first look at some of the basic properties and dynamics of the entire functions $s(z) = \sin(z)$ and $c(z) = \cos(z)$.

First, $s(z)$ has a Newton map:

$$F_s(z) = z - \frac{\sin(z)}{\cos(z)} = \frac{z\cos(z) - \sin(z)}{\cos(z)}. \tag{6}$$

For all $n \in \mathbb{Z}$, the fixed points of F_s are $z_n := n\pi$ and $|F_s'(z_n)| = |\tan^2(n\pi)| = 0$ thus, z_n is a superattracting fixed point of F_s.

Similarly, $c(z)$ has a Newton map:

$$F_c(z) = z + \frac{\cos(z)}{\sin(z)} = \frac{z\sin(z) + \cos(z)}{\sin(z)}, \tag{7}$$

which has fixed points at $\tilde{z}_n := (\pi/2) + n\pi$ and $|F_c'(\tilde{z}_n)| = |\cot^2((\pi/2) + n\pi)| = 0$. Thus, $\tilde{z}_n$ is a superattracting fixed point of F_c.

The images produced from the iteration of F_s and F_c look as one might expect: with strips of width π about each fixed point, where points within this strip will converge (under iterations of the Newton map) to each respective root (see Figure 3(a)). However, we can see fractal properties about the boundaries and the dynamics are far from trivial. Notice that the Newton maps of $\sin(z)$ and $\cos(z)$ have singularities at $(\pi/2) + n\pi$ and $n\pi$, respectively, which leads to the boundary behavior shown in Figure 3(b).

Now, consider the meromorphic function $t(z) = \tan(z)$ which yields the corresponding Newton map:

$$F_t(z) = z - \frac{\tan(z)}{\tan'(z)}. \tag{8}$$

Note that $\tan'(z) = 1/\cos^2(z)$ is never zero, so $F_t(z) = z - \sin(z)\cos(z)$ for all $z \in \mathbb{C}$ and F_t is an entire function. Furthermore, the roots of $t(z)$ are the roots of $\sin(z)$, $z_n = n\pi$ for all $n \in \mathbb{N}$; however, the fixed points of F_t occur at both $z_n = n\pi$ and $\tilde{z}_n = \pi/2 + n\pi$. Taking the derivative of the Newton map:

$$F_t'(z) = 1 - \left(\cos^2(z) - \sin^2(z)\right) = 2\sin^2(z), \tag{9}$$

we see that $F_t'(z_n) = 0$ and $F_t'(\tilde{z}_n) = 2 > 1$; thus, by Theorem 3, for all $n \in \mathbb{Z}$, z_n is a superattracting fixed point of F_t and $\tilde{z}_n$ is a repelling fixed point of F_t. Figure 4 shows the computer-generated global dynamics of $F_t(z)$. Each root z_n lies at the center of one of the main colored bulbs. Seed values found outside of this strip of bulbs fail to iterate to any root under F_t.

Recall that for $F_t(z) = z - \sin(z)\cos(z)$, we have

$$A_F(z_n) = \left\{ z \in \mathbb{C} \mid F_t^k(z) \longrightarrow z_n, \text{for} k = 1, 2, \ldots \right\}, \tag{10}$$

and $A_F^*(z_n)$ is the largest connected component of $A_F(z_n)$ containing the root z_n.

A closer inspection of any one primary basin of attraction gives greater insight into what is happening. Continuously zooming in on the boundary of each $A_F^*(z_n)$ reveals the seemingly infinite nature of the fractal image. Each bulb consists of a boundary of bulbs, and it appears that there are an infinite number of them, each consisting of seed values which converge to a different root under iterations of F_t. Figure 5(a) shows a closer view of the dynamics about $z_0 = 0$. Every seed value from $A_F^*(0)$ will iterate to 0. Furthermore, notice that from this distance, we see no more subsets of $A_F(z_0)$ about the boundary of $A_F^*(z_0)$. Indeed, it seems that none of the bulbs that touch $A_F^*(z_n)$ contain seed values that converge to z_n.

Figure 5(b) displays a close-up of the largest bulbs stemming from $A_F^*(z_0)$ in the first quadrant. Notice that this close-up imitates the broader picture of Figure 5(a) in shape and bulb placement. Namely, the hexagonal-like shape of this bulb, while slightly distorted, resembles that of $A_F^*(z_0)$ with the positioning of boundary bulbs in the same general area. Moreover, these loose properties can be observed in any bulb one chooses to zoom in on.

2.2. Symmetry of $F_t(z)$. Our first exploration into the dynamics seen in Figure 5(a) is in the symmetry of the Newton map. We employ the following standard properties for all complex numbers z:

(i) $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$,
(ii) $\sin(\overline{z}) = \overline{\sin(z)}$ and $\cos(\overline{z}) = \overline{\cos(z)}$.

We first show that $F_t(z)$ is symmetric about the x–axis for all $z \in \mathbb{C}$. Let $F_t(z) = z - \sin(z)\cos(z) = z_1$. Then,

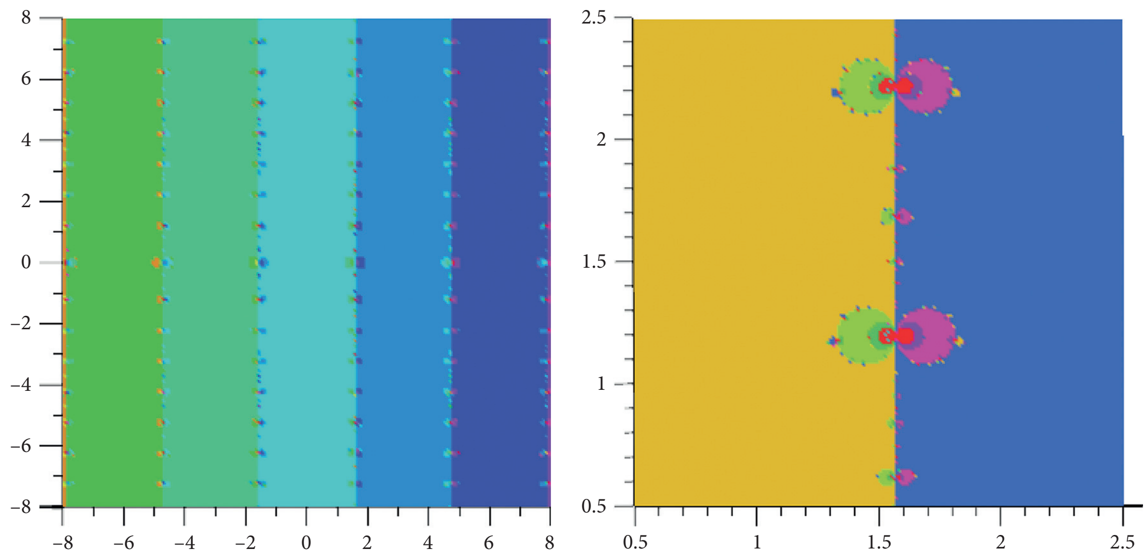
$$\begin{aligned} F_t(\overline{z}) &= \overline{z} - \sin(\overline{z})\cos(\overline{z}) = \overline{z} - \overline{\sin(z)\cos(z)} \\ &= \overline{z - \sin(z)\cos(z)} = \overline{z_1} = \overline{F_t(z)}, \end{aligned} \tag{11}$$

that is, for any $z = x + iy$, if F_t takes z to $z_1 = x_1 + iy_1$, then it takes $\overline{z} = x - iy$ to $\overline{z_1} = x_1 - iy_1$.

In a similar manner, it can be shown that if F_t takes $z = x + iy$ to $z_1 = x_1 + iy_1$, then it takes $-\overline{z} = -x + iy$ to $-\overline{z_1} = -x_1 + iy_1$. Hence, F_t is symmetric about the y–axis.

The symmetry of F_t means that an exploration of the dynamics which occur in the first quadrant is sufficient to understand the global dynamics of $\tan(z)$ under Newton's method (see Figure 6).

2.3. Bounding the Primary Basins. What can we say about the basins of attraction these images present us with? We would like to, in some way, bound the different sets of seed values converging to distinct roots, and in this section, we will focus on the primary basins about roots z_n. Recall that the function being iterated is $F_t(z) = z - \sin(z)\cos(z)$, $\sin(z) = (e^{iz} - e^{-iz})/2i$, and $\cos(z) = (e^{iz} + e^{-iz})/2$. So, for purely imaginary seed values $z = bi$, $b \in \mathbb{R}$, we have



(a) (b)

Figure 3: (a) Global dynamics of $F_s(z)$ and (b) the fractal boundary of $F_s(z)$ at $z = \pi/2 + iy$.

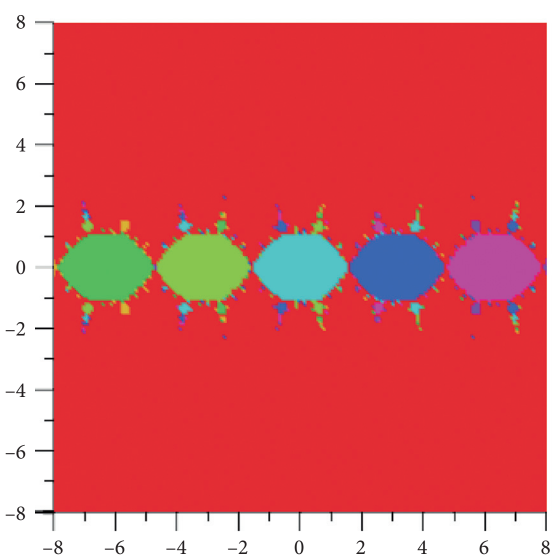


Figure 4: Global dynamics of $F_t(z)$.

$$
\begin{aligned}
F_t(bi) &= bi - \sin(\mathrm{bi})\cos(\mathrm{bi}) \\
&= bi - \frac{1}{4i}\left(e^{-b} - e^{b}\right)\left(e^{-b} + e^{-b}\right) = bi \\
&\quad + \frac{i}{4}\left(e^{-2b} - e^{2b}\right) = i\left(b + \frac{1}{4e^{2b}} - e^{2b}\right),
\end{aligned}
\tag{12}
$$

that is, points on the imaginary axis remain on the imaginary axis under iteration. Similarly, points on the real axis remain on the real axis since for $z = x \in \mathbb{R}$, $F_t(x) = x - \sin(x)\cos(x)$ is in $\mathbb{R}$.

Now, according to Definition 1, for each attracting fixed point z_n, there exists a neighborhood U about z_n such that all points in U converge monotonically to z_n. We will first examine sets of real points and points of the form $z = z_n + iy$.

Proposition 1. *Along the real axis, $U = (-(\pi/2) + z_n, (\pi/2) + z_n)$. That is, for all $z \in \mathbb{R} \cap (-(\pi/2) + z_n, (\pi/2) + z_n) \backslash \{z_n\}$, $n \in \mathbb{Z}$, we have*

$$
\left|F_t(z) - z_n\right| < \left|z - z_n\right|. \tag{13}
$$

Proof. Before proving this result, note the following:

(i) For $a, z \in \mathbb{R}$, if $0 < a < z$, then $z - a < z$ hence $|z - a| < |z|$,

(ii) For $a, z \in \mathbb{R}$, if $z < a < 0$, then $z - a > z$ hence $|z - a| < |z|$,

(iii) For $z \in \mathbb{R}$, if $0 < z < \pi$, then $z > \sin(z)\cos(z)$, and if $-\pi < z < 0$, then $z < \sin(z)\cos(z)$,

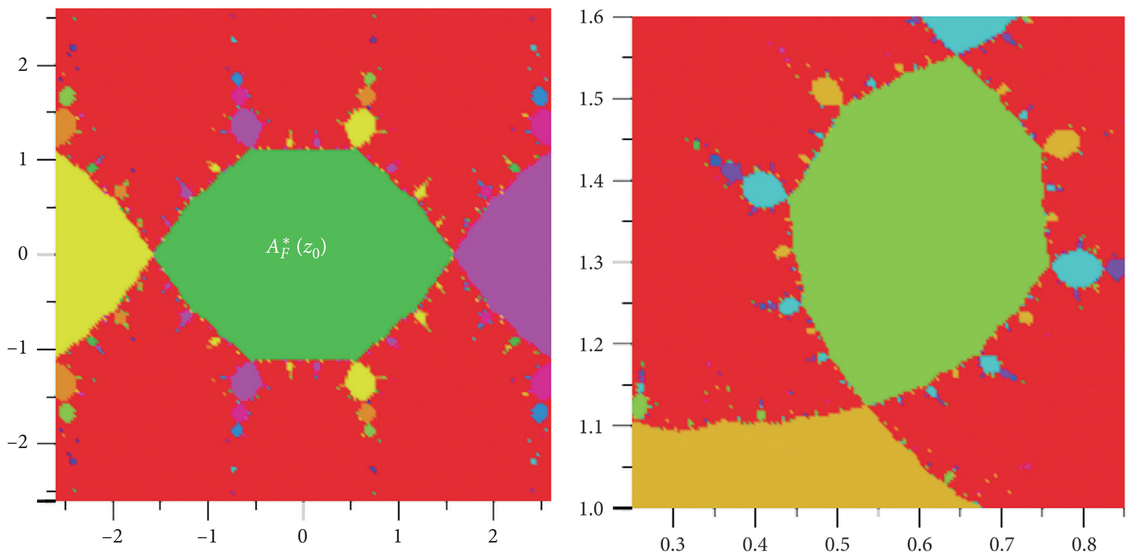
(iv) For $z \in \mathbb{R}$, $\sin(z)\cos(z)$ has period π.

Now, we first prove the result for $n = 0$. That is, we will show that for all

$$
\begin{gathered}
z \in \mathbb{R} \cap \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \backslash \{0\}, \\
\left|F_t(z)\right| < |z|.
\end{gathered}
\tag{14}
$$

Suppose that $z \in (0, \pi/2)$. Then, $z > 0$, $\sin(z)\cos(z) > 0$, and $z > \sin(z)\cos(z)$, so by (28a), $|z - \sin(z)\cos(z)| < |z|$. Suppose $z \in (-\pi/2, 0)$. Then, $z < 0$, $\sin(z)\cos(z) < 0$, and $z < \sin(z)\cos(z)$. Hence, by (28b), $|z - \sin(z)\cos(z)| < |z|$. Thus, for all $z \in U = (-\pi/2, \pi/2)$, we have $|z - \sin(z)\cos(z)| < |z|$, that is, $|F_t(z)| < |z|$.

Let us now consider the general case. For $z \in (z_n, (\pi/2) + z_n)$, let $w = z - z_n = z - \mathrm{n}\pi$. Then, $0 < w < \pi/2$ implies $w > \sin(w)\cos(w)$ and $\sin(w)\cos(w) > 0$. Thus, by the above remarks, we have



(a) (b)

Figure 5: (a) Primary basin of attraction of F_t about $z_0 = 0$. (b) The largest bulb attached to $A_F^*(z_0)$ in quadrant I.

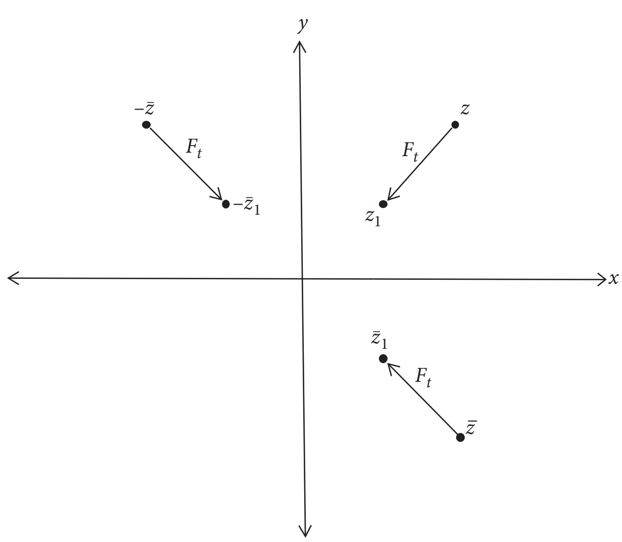


Figure 6: x- and y-axis symmetry of $F_t(z)$.

$$\begin{aligned}&|w - \sin(w)\cos(w)| < |w| |w - (-\sin(w))(-\cos(w))| < |w| \\ &\Longleftrightarrow |w - \sin(w + n\pi)\cos(w + n\pi)| < |w| |z - z_n \\ &\quad - \sin(z)\cos(z)| < |z - z_n| \\ &\Longleftrightarrow |F_t(z) - z_n| < |z - z_n|.\end{aligned} \tag{15}$$

Similarly, for $z \in (-\pi/2 + z_n, z_n)$, let $w = z - z_n$. Then, $-\pi/2 < w < 0$ and by the same reasoning, we have

$$|w - \sin(w)\cos(w)| < |w| \Longleftrightarrow |F_t(z) - z_n| < |z - z_n|. \tag{16}$$

Therefore, for all $z \in \mathbb{R} \cap (-(\pi/2) + z_n, (\pi/2) + z_n)\backslash\{z_n\}$, $n \in \mathbb{Z}$, we have

$$|F_t(z) - z_n| < |z - z_n|. \tag{17}$$

To place similar bounds along the imaginary axis, we define the set:

$$\beta_n : \{z_n + bi : b \in \mathbb{R}, b \neq 0, \text{and} |\sinh(2b)| < |4b|\}, \tag{18}$$

and we have the following result. ☐

Proposition 2. *For all $z \in \mathbb{C} \cap \beta_n \backslash \{z_n\}$, we have $|F_t(z) - z_n| < |z - z_n|$.*

Proof. We will first prove the result for the attracting fixed point $z_0 = 0$, and then extend it to the general case $z_n = n\pi$. For all $z \in \mathbb{C} \cap \beta_0 \backslash \{z_0\}$, we have

$$\begin{aligned}\sin(z)\cos(z) = \sin(bi)\cos(bi) &= \frac{1}{4i}\left(e^{-b} - e^{b}\right)\left(e^{-b} + e^{-b}\right) \\ &= \frac{-i}{4}\left(e^{-2b} - e^{2b}\right).\end{aligned} \tag{19}$$

Thus,

$$\begin{aligned}&|z - \sin(z)\cos(z) - z_0| < |z - z_0| \Longleftrightarrow |bi - \sin(bi)\cos(bi)| < |bi| \\ &\Longleftrightarrow \left|bi + \frac{i}{4}\left(e^{-2b} - e^{2b}\right)\right| < |bi| \left|b + \frac{1}{4}\left(e^{-2b} - e^{2b}\right)\right| < |b| \\ &\Longleftrightarrow b + \frac{1}{4}\left(e^{-2b} - e^{2b}\right) < |b| \text{and} b + \frac{1}{4}\left(e^{-2b} - e^{2b}\right) > -|b|.\end{aligned} \tag{20}$$

We first consider the case where $b > 0$:

$$b+\frac{1}{4}\left(e^{-2b}-e^{2b}\right)<b\Longleftrightarrow\frac{1}{4}\left(e^{-2b}-e^{2b}\right)<0e^{-2b}<e^{2b},\quad b\in(0,\infty),$$
$$b+\frac{1}{4}\left(e^{-2b}-e^{2b}\right)>-b-\frac{1}{4}\left(e^{-2b}-e^{2b}\right)<2b\frac{1}{4}\left(e^{2b}-e^{-2b}\right)<2b,$$
$$\Longleftrightarrow\sinh(2b)<4b. \tag{21}$$

Hence, the set $\{b:\sinh(2b)<4b\}$ satisfies the desired inequality.

Now, suppose that $b<0$ and let $b=-a$, where $a>0$. Then, we have

$$b+\frac{1}{4}\left(e^{-2b}-e^{2b}\right)<|b|-a+\frac{1}{4}\left(e^{2a}-e^{-2a}\right)<|-a|=a$$
$$\Longleftrightarrow\sinh(2a)<4a-\sinh(2b)<-4b\sinh(2b)>4b,$$
$$b+\frac{1}{4}\left(e^{-2b}-e^{2b}\right)>-|b|-a+\frac{1}{4}\left(e^{2a}-e^{-2a}\right)>-|-a|=-a$$
$$\Longleftrightarrow\frac{1}{4}\left(e^{2a}-e^{-2a}\right)>0e^{2a}>e^{-2a}e^{-2b}>e^{2b},\quad b\in(-\infty,0). \tag{22}$$

Therefore, for the attracting fixed point z_0, for all $z\in\mathbb{C}\cap\beta_0\setminus\{z_0\}$, we have

$$\left|F_t(z)-z_0\right|<\left|z-z_0\right|. \tag{23}$$

We now extend this result to the general case. Let $z\in\mathbb{C}\cap\beta_n\{z_n\}$ and let $w=z-z_n$. Since $z=z_n+bi$, we have $w=bi$ for all b such that $|\sinh(2b)|<|4b|$, hence by the first part we have

$$|w-\sin(w)\cos(w)|<|w||w-\sin(w+n\pi)\cos(w+n\pi)|<|w|$$
$$\Longleftrightarrow\left|z-\sin(z)\cos(z)-z_n\right|<\left|z-z_n\right|. \tag{24}$$

Thus, for the attracting fixed point z_n, for all $z\in\mathbb{C}\cap\beta_n\setminus\{z_n\}$, we have

$$\left|F_t(z)-z_n\right|<\left|z-z_n\right|. \tag{25}$$

The proof of the last result of this section makes use of both Schwarz's lemma and the maximum modulus principle. □

Lemma 1 (Schwarz). *Let $D=\{z:|z|<1\}$, and suppose f is an analytic on D with*

(a) $|f(z)|\le 1$ for $z\in D$,

(b) $f(0)=0$.

Then, $|f'(0)|\le 1$ and $|f(z)|\le|z|$ for all z in the open disk D. Moreover, if $|f'(0)|=1$ or if $|f(z)|=|z|$ for some $z\neq 0$, then there is a constant c, $|c|=1$, such that $f(w)=cw$ $\forall w\in D$.

Theorem 4 (Maximum modulus). *Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on the closure of G, $\overline{G}$, and analytic in G. Then,*

$$\max\{|f(z)|: z\in\overline{G}\}=\max\{|f(z)|: z\in\partial G\}, \tag{26}$$

where ∂G is the boundary of G.

The maximum modulus principle says that in a bounded domain, an analytic function that has a continuous extension to the boundary attains its maximum modulus on the boundary. Equivalently, if the modulus of an analytic function achieves its maximum value at some point *inside* the domain G, then the function is constant in G.

We can now show that the unit disk is contained in the primary basin of attraction.

Theorem 5. *For all $z\in\mathbb{C}\cap D_n\setminus\{z_n\}$, where $D_n:=\{z:|z-z_n|<1\}$, we have*

$$\left|F_t(z)-z_n\right|<\left|z-z_n\right|. \tag{27}$$

Proof. This proof will make use of the following equivalencies for all $z\in\mathbb{C}$:

$$\sin(z)\cos(z)=\frac{1}{2}\sin(2z), \tag{28a}$$

$$\begin{aligned}\sin(2\overline{z})&=\sin(2x)\cos(2iy)-\cos(2x)\sin(2iy)\\&=\sin(2x)\cosh(2y)-i\cos(2x)\sinh(2y),\end{aligned} \tag{28b}$$

$$\operatorname{Re}(z\sin(\overline{z}))=x\sin(2x)\cosh(2y)+y\cos(2x)\sinh(2y), \tag{28c}$$

$$\begin{aligned}|\sin(2z)|^2&=\frac{1}{4}\Big[\sin^2(2x)\left(e^{2y}+e^{-2y}\right)^2\\&\quad+\cos^2(2x)\left(e^{2y}-e^{-2y}\right)^2\Big]\\&=\frac{1}{4}\left[e^{4y}+e^{-4y}+2-4\cos^2(2x)\right]\\&=\frac{1}{2}\left[\cosh(4y)+\left(1-2\cos^2(2x)\right)\right]\\&=\frac{1}{2}(\cosh(4y)-\cos(4x)).\end{aligned} \tag{28d}$$

We will first prove this result for the $n=0$ case, i.e., we consider $z_0=0$ and the open unit disk D_0.

$F_t(z)=z-\sin(z)\cos(z)$ is analytic and continuous on all of $\mathbb{C}$, in particular, F_t is continuous on $\overline{D_0}$ and analytic on D_0. Thus, by the maximum modulus principle, $\forall z\in\overline{D_0}$, $|F_t(z)|=|z-\sin(z)\cos(z)|$ attains its maximum value when $|z|=1\Longleftrightarrow y=\pm\sqrt{1-x^2}$. Thus, we have

$$\begin{aligned}|z-\sin(z)\cos(z)|^2 &= |z|^2 - 2\mathrm{Re}(z\sin(\bar{z})\cos(\bar{z})) + |\sin(z)\cos(z)|^2\\
&= |z|^2 - \mathrm{Re}(z\sin(2\bar{z})) + \frac{1}{4}|\sin(2z)|^2\\
&= |z|^2 - x\sin(2x)\cosh(2y) - y\cos(2x)\sinh(2y) + \frac{1}{8}(\cosh(4y) - \cos(4x))\\
&\leq 1 - x\sin(2x)\cosh\left(2\sqrt{1-x^2}\right) - \sqrt{1-x^2}\cos(2x)\sinh\left(2\sqrt{1-x^2}\right) + \frac{1}{8}\left(\cosh\left(4\sqrt{1-x^2}\right) - \cos(4x)\right),\\
&\qquad \text{for } -1\leq x\leq 1.
\end{aligned} \tag{29}$$

Now, let us define $g(x): = 1 - x\sin(2x)\cosh(2\sqrt{1-x^2}) - \sqrt{1-x^2}\cos(2x)\sinh(2\sqrt{1-x^2}) + (1/8)(\cosh(4\sqrt{1-x^2}) - \cos(4x))$. A direct computation shows that on the interval $[-1,1]$, $g'(x) = 0$ if and only if $x = 0$. This can also be seen informally by graphing either $g(x)$ or $g'(x)$. Therefore, g takes on its maximum value at $x = 0$ and we have

$$\begin{aligned}|z-\sin(z)\cos(z)|^2 &\leq g(x)\leq g(0) = 1-\sinh(2)\\
&+\frac{1}{8}(\cosh(4)-1),\quad \text{for all} < 1 - 1\leq x\leq 1.\end{aligned} \tag{30}$$

Thus, the maximum of $|F_t(z)|$ is less than 1 for all $z\in\overline{D_0}$, so certainly $|F_t(z)|<1$ for all $z\in D_0$. Furthermore, $F(0)=0$, thus by Schwarz's lemma, for all $z\in D_0$, $|F_t(z)|\leq|z|$. Note that the only point for which $|F_t(z)| = |z-\sin(z)\cos(z)| = |z|$ is when $z = z_0$, so excluding z_0 from the domain will yield a strict inequality. That is, for all $z\in\mathbb{C}\cap D_0\backslash\{z_0\}$, we have $|F_t(z)|<|z|$, i.e., $|F_t(z)-z_0|<|z-z_0|$.

We now extend this to the general case where $z_n = n\pi\,\forall n\in\mathbb{Z}$. Consider $D_n = \{z: |z-z_n|<1\}$ and let $w = z-z_n$ for $z\in\overline{D_n}$. Then, we have $|w|\leq 1$, $F_t(0)=0$ and $|F_t(w)| = |w-\sin(w)\cos(w)|<1$ by the same reasoning as above, and thus by Schwarz's lemma,

$$\begin{aligned}\left|F_t(w)\right| &\leq |w|\left|z-z_n-\sin(z-z_n)\right|\leq\left|z-z_n\right|\\
&\Longleftrightarrow\left|z-\sin(z)\cos(z)-z_n\right|\leq\left|z-z_n\right|\text{for all } z\in D_n.\end{aligned} \tag{31}$$

Hence,

$$\left|F_t(z)-z_n\right| < \left|z-z_n\right|\forall z\in\mathbb{C}\cap D_n\backslash\{z_n\}. \tag{32}$$

Based on Figure 4, further exploration could entail an attempt to bound monotonically converging seed values in $A_F^*(0)$ within the ellipse $(x/\pi/2)^2 + (y/\beta)^2 < 1$, where $\beta: = \{|x|: \sinh(2x) = 4x\}$, or equivalently, $|z+c|+|z-c|<\pi$ where

$$c = \sqrt{\left(\frac{\pi}{2}\right)^2 - \beta^2}. \tag{33}$$

□

3. Conclusion

The fractal image created from iterating the Newton map of $t(z) = \tan(z)$ is symmetric about both the x- and y–axis as well as with respect to each attracting fixed point. In general, that which can be said about the dynamics surrounding $z_0 = 0$ can be said about the dynamics about $z_n = n\pi$. Indeed, as was shown, monotonically convergent seed values are bounded inside each primary basin of attraction by a circle of radius at least one centered at z_n. These bounds can be extended along the real axes to $-\pi/2 + z_n$ and $\pi/2 + z_n$, and along the line $z = z_n + iy$ to $|\sinh(2b)| < |4b|$ for $b\in\mathbb{R}$. This leads one to conjecture that the unit circle is a minimal bound which could be extended to the ellipse: $|z-c| + |z-c| < \pi$ where $c = \sqrt{(\pi/2)^2 - \beta^2}$. Furthermore, one could possibly show that each connected component of $A_F(n\pi)$ is bounded by a scaled version of this ellipse.

Indeed, as we have highlighted by the exploration of the Newton maps of two (seemingly) simple functions ($\sin(z)$ and $\tan(z)$), complex dynamical systems have numerous interesting properties available for study.

Data Availability

No data were produced during this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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