Research Article

Rational Cohomology Algebra of Mapping Spaces between Complex Grassmannians

Paul Antony Otieno ¹, Jean Baptiste Gatsinzi,² and Vitalis Onyango-Otieno¹

¹Strathmore Institute of Mathematical Sciences, Strathmore University, 59857 Nairobi, Kenya

²Department of Mathematics, Botswana International University of Science and Technology, Private Bag 16, Palapye, Botswana

Correspondence should be addressed to Paul Antony Otieno; paotieno@strathmore.edu

Received 28 November 2019; Accepted 26 December 2019; Published 29 January 2020

Academic Editor: Ricardo Estrada

Copyright © 2020 Paul Antony Otieno et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the complex Grassmannian $\text{Gr}(k, n)$ of k -dimensional subspaces of $\mathbb{C}^n$. There is a natural inclusion $i_{n,r}: \text{Gr}(k, n) \hookrightarrow \text{Gr}(k, n+r)$. Here, we use Sullivan models to compute the rational cohomology algebra of the component of the inclusion $i_{n,r}$ in the space of mappings from $\text{Gr}(k, n)$ to $\text{Gr}(k, n+r)$ for $r \geq 1$ and in particular to show that the cohomology of $\text{map}(\text{Gr}(n, k), \text{Gr}(n, k+r); i_{n,r})$ contains a truncated algebra $\mathbb{Q}[x]/x^{r+n+k^2-nk}$, where $|x| = 2$, for $k \geq 2$ and $n \geq 4$.

1. Introduction

The complex Grassmann $\text{Gr}(k, n)$ is the set of k -planes through the origin in $\mathbb{C}^n$. Moreover,

$$\text{Gr}(k, n) \cong \frac{U(n)}{U(k) \times U(n-k)}, \quad (1)$$

where $U(n)$ is the unitary group ([1], chap. 18). There is a canonical inclusion $i_{n,r}: \text{Gr}(k, n) \hookrightarrow \text{Gr}(k, n+r)$ which is induced by $i_r: \mathbb{C}^n \rightarrow \mathbb{C}^{n+r}$ defined by $i(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$.

The study of the rational homotopy type of function spaces started with Thom in the case where the codomain is an Eilenberg–MacLane space [2]. The first description of a Sullivan model of function spaces is due to Haefliger [3]. Moreover, a model of function spaces between complex projective spaces was given by Moller and Raussen using Postnikov tower [4]. However, there is no explicit and complete description of the homotopy type of the component of the inclusion $\text{Gr}(k, n) \hookrightarrow \text{Gr}(k, n+r)$ in the space of mappings from $\text{Gr}(k, n)$ to $\text{Gr}(k, n+r)$, $r \geq 1$ and $n \geq 4$. We begin our work with a model of the inclusion.

In our previous work, we studied the rational homotopy of $\text{map}(\text{Gr}(2, n), \text{Gr}(2, n+r); i_{n,1})$ where $i_{n,1}: \text{Gr}(2, n) \rightarrow \text{Gr}(2, n+r)$ is the canonical inclusion. We showed

among other things that $\text{map}(\text{Gr}(2, n), \text{Gr}(2, n+1); i_{n,1})$ has the rational homotopy type of a product of odd spheres. Under some assumptions on r , the cohomology algebra of $\text{map}(\text{Gr}(2, n), \text{Gr}(2, n+r); i_{n,r})$ contains either a polynomial algebra or a truncated algebra over a generator of degree 2.

In this paper, we consider the inclusion $\text{Gr}(k, n) \rightarrow \text{Gr}(k, n+r)$, $k \geq 3$ [5].

2. Sullivan Models

Henceforth, we work on the field of rational numbers, $\mathbb{Q}$.

Definition 1. A differential graded algebra (dga) is a graded algebra $A = \bigoplus_{n \geq 0} A^n$ endowed with a derivation d , of degree +1, such that $d^2 = 0$. The pair (A, d) is called a cochain algebra. A graded algebra A is commutative if $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$ for $a, b \in A$ ([6], chap. 3).

Definition 2. A Sullivan algebra is a commutative cochain algebra of the form $(\wedge V, d)$ where $V = \bigoplus_{k \geq 1} V_k$ and $V = \bigcup_{k=0}^{\infty} V(k)$, $V(0) \subset V(1) \subset \dots$ such that $dV(k) \subset \wedge V(k-1)$. A Sullivan model for a commutative cochain algebra (A, d) is a quasi-isomorphism $m: (\wedge V, d) \rightarrow (A, d)$ from a Sullivan algebra $(\wedge V, d)$. A Sullivan algebra (or

model) is said to be minimal if the differential is decomposable, that is, $\text{Im } d \subset \wedge^+ V \cdot \wedge^+ V$.

Moreover, if $H^0(A) = \mathbb{Q}$, then (A, d) has a minimal model which is unique up to isomorphism. If X is a nilpotent space and $A_{\text{PL}}(X)$ the commutative differential graded algebra (cdga) of piecewise linear forms on X , then a Sullivan model of X is a Sullivan model of $A_{\text{PL}}(X)$ ([6], chap.12). A space X is formal if there is a quasi-isomorphism $(\wedge V, d) \rightarrow H^*(X, \mathbb{Q})$. Moreover, complex Grassmann manifolds are formal [7].

The cohomology ring $H^*(\text{Gr}(k, n), \mathbb{Q})$ of $\text{Gr}(k, n)$ has a presentation

$$H^*(\text{Gr}(k, n), \mathbb{Q}) = \frac{\wedge(c_1, c_2, \dots, c_k)}{(h_{n-k+1}, \dots, h_n)}, \quad (2)$$

a quotient of the polynomial ring generated by $c_1, c_2, \dots, c_k$, $|c_i| = 2i$, modulo the ideal generated by the elements $h_j, n - k + 1 \leq j \leq n$. Here, h_j is defined as the $2j^{\text{th}}$ degree term in Taylor's expansion of $(1 + c_1 + c_2 + c_3 + \dots + c_k)^{-1}$, where $(1 + c_1 + c_2 + c_3 + \dots + c_k)$ is the total Chern class and $c_i \in H^{2i}(\text{Gr}(k, n), \mathbb{Q})$ for $1 \leq i \leq k$ [8].

For instance, $H^*(\text{Gr}(2, 4), \mathbb{Q}) = \wedge(x_2, x_4)/(h_3, h_4)$, where $h_3 = 2x_2x_4 - x_2^3$ and $h_4 = x_4^2 - 3x_2^2x_4 + x_2^4$. As $(2x_2x_4 - x_2^3, x_4^2 - 3x_2^2x_4 + x_2^4)$ forms a regular sequence, the Sullivan model of $\text{Gr}(2, 4)$ is hence given by $(\wedge(x_2, x_4, x_5, x_7), d)$ with $dx_2 = dx_4 = 0$, $dx_5 = 2x_2x_4 - x_2^3$ and $dx_7 = x_4^2 - 3x_2^2x_4 + x_2^4$.

3. L_∞ Models of Function Spaces

Definition 3. On a graded vector space L , an L_∞ structure, usually denoted by $(L, \{l_k\}_{k \geq 1})$, is a collection of linear maps, $\{l_k\}_{k \geq 1}$ called brackets, where $l_k: \otimes^k L \rightarrow L$ such that the following conditions are satisfied:

- (1) l_k are graded skew symmetric, that is, for any k permutation σ ,

$$l_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \text{sgn}(\sigma) \epsilon_\sigma l_k(x_1, \dots, x_k), \quad (3)$$

where ϵ_σ is the sign given by the Koszul convention.

- (2) The generalised Jacobi identity holds, that is,

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i, n-i)} \text{sgn}(\sigma) \epsilon_\sigma (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0, \quad (4)$$

where $S(i, n-i)$ denotes the $(i, n-i)$ shuffles which are permutations $\sigma \in S_n$ such that $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n)$.

An L_∞ algebra $(L, \{l_k\}_{k \geq 1})$ is minimal if $l_1 = 0$. L_∞ structures are in one-to-one correspondence with codifferentials on the non unital, free commutative coalgebra $\wedge^+ sL$ in which s denotes suspension, that is $(sL)_k = L_{k-1}$ [9].

Note that if L is a graded vector space of finite type, an L_∞ structure on L induces a commutative differential graded algebra structure $C^\infty(L) = (\wedge(sL)^\sharp, d)$, $d = \sum_{j \geq 1} d_j$, where $d_j: V \rightarrow \wedge^j V$ is defined by $\langle d_j v; sx_1 \wedge \dots \wedge sx_j \rangle = (-1)^\epsilon \langle v; sl_j(x_1, \dots, x_j) \rangle$ and $\epsilon = (-1)^{|v| + \sum_{k=1}^{j-1} (j-k)|x_k|}$ [10]. This is a generalisation of the Quillen cochain functor to L_∞ algebras ([6], Chap.23).

Further, an L_∞ algebra L is an L_∞ model of a simply connected space X , if $C^\infty(L)$ is a Sullivan model of X [10].

Definition 4. Let $\phi: (A, d_A) \rightarrow (B, d_B)$ be a morphism of cochain algebras. A ϕ -derivation θ of degree n is a linear map $A^* \rightarrow A^{*-n}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{|a|} \phi(a)\theta(b)$. We denote by $\text{Der}_n(A, B, \phi)$ the vector space of all ϕ -derivations of degree n . Define a differential of chain complexes $\delta: \text{Der}_n(A, B, \phi) \rightarrow \text{Der}_{n-1}(A, B, \phi)$, by $\delta\theta = d_B\theta - (-1)^{|\theta|} \theta d_A$.

If $\phi: (\wedge V, d) \rightarrow (B, d)$ is a morphism of commutative differential graded algebras, then there is an isomorphism of vector spaces:

$$\text{Der}(\wedge V, B, \phi) \xrightarrow{\cong} \text{Hom}(V, B). \quad (5)$$

Moreover, if $\{v_1, \dots, v_n\}$ is a basis for V , we will denote by (v_i, b_i) the ϕ -derivation θ such that

$$\begin{cases} \theta(v_i) = b_i, & b_i \in B, \\ \theta(v_j) = 0, & i \neq j. \end{cases} \quad (6)$$

We now follow [10] for the definition below.

Consider positive derivations defined by

$$\widetilde{\text{Der}}_i(\wedge V, B, \phi) = \begin{cases} \text{Der}_i(\wedge V, B, \phi), & \text{if } i \geq 2, \\ Z(\text{Der}_i(\wedge V, B, \phi)), & \text{if } i = 1. \end{cases} \quad (7)$$

If $(A, d) = (\wedge V, d)$ is a Sullivan algebra, then given $j \geq 2$ and $\varphi_1, \dots, \varphi_j \in \widetilde{\text{Der}}(\wedge V, B, \phi)$, of degrees $p_1, \dots, p_j$, we define their brackets of length j , $[\varphi_1, \dots, \varphi_j] = \lambda_j(\varphi_1, \dots, \varphi_j) \in \text{Der}(\wedge V, B, \phi)$ by

$$[\varphi_1, \dots, \varphi_j](v) = (-1)^{p_1 + \dots + p_{j-1}} \sum_{i_1, \dots, i_j} \epsilon \phi \cdot (v_1 \cdots \widehat{v}_{i_1} \cdots \widehat{v}_{i_j} \cdots v_k) \varphi(v_{i_1}) \cdots \varphi_j(v_{i_j}), \quad (8)$$

where $dv = \sum v_1 \cdots v_k$ and ϵ is the sign given by the Koszul convention. These operations may be desuspended to define a set of linear maps $\{l_j\}_{j \geq 1}$ each of degree $j-2$ on $s^{-1}\widetilde{\text{Der}}(\wedge V, B, \phi)$ as follows:

For $j = 1$,

$$l_1: s^{-1}\widetilde{\text{Der}}(\wedge V, B, \phi) \rightarrow s^{-1}\widetilde{\text{Der}}(\wedge V, B, \phi), \quad l_1(s^{-1}\theta) = -s^{-1}\delta\theta. \quad (9)$$

For $j = 2$, define

$$l_j: s^{-1}\widetilde{\text{Der}}(\wedge V, B, \phi) \otimes \dots \otimes s^{-1}\widetilde{\text{Der}}(\wedge V, B, \phi) \rightarrow s^{-1}\widetilde{\text{Der}}(\wedge V, B, \phi), \quad (10)$$

by $l_j(s^{-1}\varphi_1, \dots, s^{-1}\varphi_j) = (-1)^{\alpha} s^{-1}\lambda_j(\varphi_1, \dots, \varphi_j)$ and $\alpha = \sum_{n=1}^{j-1} (j-n) + |\varphi_n|$.

Let $f: X \rightarrow Y$ be a continuous function between 1-connected spaces and $\text{map}(X, Y; f)$ the component of f in the space of continuous maps from X to Y . Let $\phi: (\wedge V, d) \rightarrow B$ be a cdga model of f . Lemma 3.3 of [10] proves that $(s^{-1}\widetilde{\text{Der}}(\wedge V, B, \phi), \{l_j\}_{j \geq 1})$ is an L_∞ model of

$$\text{map}(X, Y; f). \quad (11)$$

4. Inclusion of $\text{Gr}(3, n)$ in $\text{Gr}(3, n+r)$

Theorem 1. Let $i_{n,r}: \text{Gr}(3, n) \rightarrow \text{Gr}(3, n+r)$ be the canonical inclusion. If $r > 2n-4$, then the rational cohomology algebra of $\text{map}(\text{Gr}(3, n), \text{Gr}(3, n+r); i_{n,r})$ contains a truncated algebra $\mathbb{Q}[x]/(x^{r-2n+9})$, where $|x| = 2$.

Proof. The cohomology algebras of $\text{Gr}(3, n)$ and $\text{Gr}(3, n+r)$ are given by $H^*(\text{Gr}(3, n), \mathbb{Q}) = \wedge(y_2, y_4, y_6)/(h_{n-2}, h_{n-1}, h_n)$ and $H^*(\text{Gr}(3, n+r), \mathbb{Q}) = \wedge(x_2, x_4, x_6)/(h_{n+r-2}, h_{n+r-1}, h_{n+r})$, respectively. Recall that $\text{Gr}(3, n)$ is a smooth manifold of dimension $6n-18$; hence, $H^m(\text{Gr}(3, n), \mathbb{Q}) = 0$ for $m \geq 6n-18$.

The minimal model of $\text{Gr}(3, n+r)$ is

$$(\wedge V, d) = (\wedge(x_2, x_4, x_6, x_{2n+2r-5}, x_{2n+2r-3}, x_{2n+2r-1}), d), \quad (12)$$

with $d(x_2) = d(x_4) = d(x_6) = 0$, $d(x_{2n+2r-5}) = h_{n+r-2}$, $d(x_{2n+2r-3}) = h_{n+r-1}$, and $d(x_{2n+2r-1}) = h_{n+r}$. A model of the inclusion is given by

$$\phi: (\wedge V, d) \rightarrow B = \frac{\wedge(y_2, y_4, y_6)}{(h_{n-2}, h_{n-1}, h_n)}, \quad (13)$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_6) = y_6$, and $\phi(x_{2n+2r-5}) = \phi(x_{2n+2r-3}) = \phi(x_{2n+2r-1}) = 0$. Let $L = \text{Der}(\wedge V, B, \phi)$. Consider $\gamma_2 = (x_6, y_4) \in L$. Then $q_1 = \lambda_k(\gamma_2, \dots, \gamma_2)(x_{2n+2r-5})$, $q_2 = \lambda_k(\gamma_2, \dots, \gamma_2)(x_{2n+2r-3})$, and $q_3 = \lambda_k(\gamma_2, \dots, \gamma_2)(x_{2n+2r-1})$ are polynomials of degree at least $2n+2r-2k-4$. If $k < r-2n+7$, then q_1 , q_2 , and q_3 will be of degree $> 6n-18$. Therefore, $\lambda_k(\gamma_2, \dots, \gamma_2) = 0$ for $k = 1, \dots, r-2n+7$. In $C^\infty(s^{-1}L)$, let $\gamma_2^* = \rho_2$. Then $[\rho_2^i] \neq 0$ for $i = 1, \dots, r-2n+7$. $\square$

Theorem 2. Consider the inclusion $i_{n,r}: \text{Gr}(3, n) \rightarrow \text{Gr}(3, n+r)$. If $r > 5n-14$, then the cohomology algebra of $\text{map}(\text{Gr}(3, n), \text{Gr}(3, n+r); i_{n,r})$ contains a polynomial algebra over a generator of degree 2.

Proof. Let

$$\begin{aligned} \phi: (\wedge(x_2, x_4, x_6, x_{2n+2r-5}, x_{2n+2r-3}, x_{2n+2r-1}), d) \\ \rightarrow \frac{\wedge(y_2, y_4, y_6)}{(h_{n-2}, h_{n-1}, h_n)}, \end{aligned} \quad (14)$$

be a model of the inclusion. Assume $n \geq 6$, in particular, if $r \geq 5n-14$, then any odd derivation is of degree at least

$2n+2r-5 - (6n-18) = 2r-4n+13 \geq 13$. Hence, $L_{11} = L_9 = L_7 = \dots = L_1 = 0$. As $L_1 = 0$, then the derivation $\gamma_2 = (x_6, y_4)$ is a cycle.

It is sufficient to show that $\lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2) = 0$ for $m \geq 2$.

We consider the case when $n+r$ is even, the other case is dealt with in a similar way. For $m > n+r/2$, $\lambda_m(\gamma_2, \dots, \gamma_2) = 0$. Assume $2 \leq m \leq n+r/2$, and let

$$\begin{aligned} r &= \lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2n+2r-5}), \\ s &= \lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2n+2r-3}), \\ t &= \lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2n+2r-1}). \end{aligned} \quad (15)$$

The polynomials r , s , and t are of total degree at least $2n+2r-2m-4$. Assume now that $r \geq 5n-14$, then $2n+2r-2m-4 > 6n-18$. Therefore, $\lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2) = 0$ for all $m \geq 1$. Hence, $H^*(C^\infty(s^{-1}L))$ contains a polynomial algebra over a generator of degree 2. $\square$

5. The General Case

We can generalise the above results.

Theorem 3. If $r > nk - k^2 - n + 2k - 1$, then the rational cohomology algebra of $\text{map}(\text{Gr}(n, k), \text{Gr}(n, k+r); i_{n,r})$ contains a truncated algebra $\mathbb{Q}[x]/x^{r+n+k^2-nk}$ where $|x| = 2$, for $k \geq 2$ and $n \geq 4$.

Proof. Consider $i_{n,r}: \text{Gr}(k, n) \rightarrow \text{Gr}(k, n+r)$. Let

$$(\wedge V, d) = (\wedge(x_2, x_4, x_6, \dots, x_{2k}, x_{2(n+r-k)+1}, \dots, x_{2(n+r)-1}), d) \quad (16)$$

be the minimal Sullivan model of $\text{Gr}(k, n+r)$ with $d(x_2) = d(x_4) = d(x_6) = \dots = d(x_{2k}) = 0$, $dx_{2(n+r-k)+1} = h_{n+r-k+1}, \dots, dx_{2(n+r)-1} = h_{n+r}$. Moreover,

$$B = H^*(\text{Gr}(k, n), \mathbb{Q}) = \frac{\wedge(y_2, y_4, y_6, \dots, y_{2k})}{(h_{n-k+1}, \dots, h_k)}. \quad (17)$$

A Sullivan model of $i_{n,r}$ is given by $\phi: (\wedge V, d) \rightarrow B$, where $\phi(x_2) = y_2, \phi(x_4) = y_4, \phi(x_6) = y_6, \dots, \phi(x_{2k}) = y_{2k}$, and $\phi(x_{2n+2r-2k+1}) = \dots = \phi(x_{2n+2r-1}) = 0$.

Note that the lowest odd degree derivation is of degree $2k-1$ as $k \geq 2$; then, $L_1 = 0$. Define $\gamma_2 = (x_{2k}, y_{2k-2})$. Then γ_2 is a cycle. Moreover,

$$\begin{aligned} b_1 &= \lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2(n+r-k)+1}), \\ &\vdots \\ b_k &= \lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2(n+r)-1}), \end{aligned} \quad (18)$$

are polynomials of degree $\geq 2n+2r-2k+2-2m$. As $\text{Gr}(k, n)$ is of dimension $2k(n-k)$, if $m < r+n+k^2+1-nk-k$, then $b_1, b_2, \dots, b_k$ are of degree $> 2k(n-k)$.

Therefore, $\lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2) = 0$ for $m = 2, \dots, (r+n+k^2+1-nk-k)$. In $C^\infty(s^{-1}L)$, let $\gamma_2^* = \varphi_2$. Then $[\varphi_2^i] \neq 0$ for $i = 1, \dots, (r+n+k^2+1-nk-k)$. $\square$

Theorem 4. If $r > (n - k)(k - 1) + 2$, then the cohomology algebra of map $(\text{Gr}(k, n), \text{Gr}(k, n + r); i_{n,r})$ contains a polynomial algebra over a generator of degree 2.

Proof. Let

$$\begin{aligned} \phi: \wedge V &= \left(\wedge (x_2, x_4, x_6, \dots, x_{2k}, x_{2(n+r-k)+1}, \dots, x_{2(n+r)-1}), d \right) \\ &\longrightarrow B = \frac{\wedge (y_2, y_4, y_6, \dots, y_{2k})}{(h_{n-k+1}, \dots, h_k)}, \end{aligned} \quad (19)$$

be a model of the inclusion. By hypothesis, $r > (n - k)(k - 1) + 2$ and $n \geq 4$. Then any odd derivation is of degree at least $2n + 2r - 2k - 2nk + 2k^2 + 1$. Hence, $L_1 = L_3 = \dots = L_f = 0$ where $f = 2n + 2r - 2k - 2nk + 2k^2 - 1$. As defined in the proof of Theorem 3, consider $\gamma_2 = (x_{2k}, y_{2k-2})$. It suffices to show that $\lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2) = 0$ for $m \geq 1$. Consider $n + r$ even, the other case is dealt with in a similar way.

For $m > n + r/2$, $\lambda_m(\gamma_2, \dots, \gamma_2) = 0$. Assume $2 \leq m \leq n + r/2$,

$$\begin{aligned} c_1 &= \lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2n+2r-2k+1}), \\ &\vdots \\ c_k &= \lambda_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2n+2r-1}), \end{aligned} \quad (20)$$

are polynomials of total degree $\geq 2n + 2r - 2k + 2 - 2m$. If $r > (n - k)(k - 1) + 2$, then $2n + 2r - 2k + 2 - 2m > -k + 2 - k^2 - 1 + nk$. Hence, $c_1 = \dots = c_k = 0$.

This implies that $-k + 2 - k^2 - 1 + nk > 2k(n - k)$. Therefore, $\lambda_m(\gamma_2, \dots, \gamma_2) = 0$ for all $m \geq 0$. This shows that $H^*(C^\infty(L))$ contains a polynomial algebra over a generator of degree 2. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] P. W. Michor, "Topics in differential geometry," in *Graduate Studies in Mathematics*, vol. 93, American Mathematical Society, Providence, RI, USA, 2008.
- [2] R. Thom, "L'homologie des espaces fonctionnels," *Colloque de Topologie Algébrique*, pp. 29–39, Masson & Cie, Paris, France, 1957.
- [3] A. Haefliger, "Rational homotopy of the space of sections of a nilpotent bundle," *Transactions of the American Mathematical Society*, vol. 273, no. 2, Article ID 667163, 609 pages, 1982.
- [4] J. M. Moller and M. Raussen, "Rational homotopy of spaces of maps into spheres and complex projective spaces," *Transactions of the American Mathematical Society*, vol. 292, no. 2, Article ID 808750, pp. 721–732, 1985.
- [5] J. B. Gatsinzi, P. A. Otieno, and V. Onyango-Otieno, "Rational homotopy of mapping spaces between complex Grassmannians," *Quaestiones Mathematicae*, pp. 1–12, 2019.
- [6] Y. Félix, S. Halperin, and J.-C. Thomas, "Rational homotopy theory," in *Graduate Texts in Mathematics*, vol. 205, Springer-Verlag, New York, NY, USA, 2001.
- [7] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, "Real homotopy theory of Kähler manifolds," *Inventiones Mathematicae*, vol. 29, no. 3, Article ID 0382702, pp. 245–274, 1975.
- [8] M. Hoffman, "Endomorphisms of the cohomology of complex Grassmannians," *Transactions of the American Mathematical Society*, vol. 281, no. 2, Article ID 722772, 745 pages, 1984.
- [9] T. Lada and M. Markl, "Strongly homotopy lie algebras," *Communications in Algebra*, vol. 23, no. 6, Article ID 1327129, pp. 2147–2161, 1995.
- [10] U. Buijs, Y. Félix, and A. Murillo, " L_∞ rational homotopy of mapping spaces," *Revista Matemática Complutense*, vol. 26, no. 2, Article ID 3068613, pp. 573–588, 2013.