Research Article

Lower Bound for the Class Number of $\mathbb{Q}\left(\sqrt{n^2 + 4}\right)$

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In this paper, we give an explicit lower bound for the class number of real quadratic field $\mathbb{Q}(\sqrt{d})$, where $d = n^2 + 4$ is a square-free integer, using $\omega(n)$ which is the number of odd prime divisors of $n$.

1. Introduction

Let $d$ be a positive square-free integer and let $h(d)$ and $C_k$ denote the class number and the class group of a real quadratic field $k = \mathbb{Q}(\sqrt{d})$, respectively.

The class number problem of quadratic fields is one of the most intriguing unsolved problems in Algebraic Number Theory and has for a long time inspired the study of lower bounds of $h(d)$.

Many fruitful research studies have been conducted in this direction. Hasse [1] and Yokoi [2, 3] studied lower bounds for class numbers of certain real quadratic fields. Mollin [4, 5] generalized their results for certain real quadratic and biquadratic fields.

In this work, we give a lower bound for $h(n^2 + 4)$, and also we find a necessary and sufficient condition for $k = \mathbb{Q}(\sqrt{n^2 + 4})$ to have class number $\omega(n) + 1$.

2. Notation and Preliminaries

Let $k$ be a real quadratic field and $\zeta_k(s)$ be its Dedekind zeta function. Siegel [6] developed a method of computing $\zeta_k(1 - 2n)$, where $n$ is a positive integer. By specializing Siegel’s formula for a real quadratic field, we obtain the following result.

**Theorem 1** (Zagier [7]). Let $k$ be a real quadratic field with discriminant $D$. Then

$$\zeta_k(-1) = \frac{1}{60} \sum_{|r| = |D|^{1/2}} \sigma_1\left(\frac{D - r^2}{4}\right),$$

where $\sigma_1(r)$ denotes the sum of divisors of $r$.

However, there is another method, according to Lang, of computing special values of $\zeta_k(s)$ if $k$ is a real quadratic field.

Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant $D$ and $H$ an ideal class of $k$. Let $I$ be any integral ideal belonging to $H^{-1}$ with an integral basis $\{r_1, r_2\}$. We put

$$\delta(I) = r_1 r_2' - r'_1 r_2,$$

where $r'_1$ and $r'_2$ are the conjugates of $r_1$ and $r_2$ respectively.

Let $\epsilon$ be the fundamental unit of $k$. Then, $\{\epsilon r_1, \epsilon r_2\}$ is also integral basis of $I$, and thus we can find a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries satisfying

$$\begin{pmatrix} r_1' \\ r_2' \end{pmatrix} = M \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$
Now, we can state Lang’s formula.

**Theorem 2** (Lang [8]). By keeping the abovementioned notation, we have

\[
\zeta_k(-1, H) = \frac{\text{sgn}(I) r_r r'_r}{360N(I)c^3} \left( (a + d)^3 - 6(a + d)N(\epsilon) \right)
\]

\[
- 240c^3 (\text{sgn} c) S^3(a, c) + 180ac^3 (\text{sgn} c) S^2(a, c)
\]

\[
- 240c^3 (\text{sgn} c) S^3(d, c) + 180 dc^2 (\text{sgn} c) S^2(d, c)
\]

(4)

where \( N(I) \) denotes the norm of an ideal \( I \), \( N(\epsilon) \) is the norm of \( \epsilon \), and \( S'(-, -) \) denotes the generalized Dedekind sum as defined in [9].

To use Lang’s formula, we need to determine the values of \( a, b, c, d, \) and generalized Dedekind sum.

**Lemma 1** (see ref [10]). The entries of \( M \) are given by

\[
a = \text{tr} \left( \frac{r_r r'_r}{\delta(I)} \right),
\]

\[
b = \text{tr} \left( \frac{r_r r'_r}{\delta(I)} \right),
\]

\[
c = \text{tr} \left( \frac{r_r r'_r}{\delta(I)} \right),
\]

\[
d = \text{tr} \left( \frac{r_r r'_r}{\delta(I)} \right).
\]

Moreover, \( \det M = N(\epsilon) \) and \( bc \neq 0 \).

Kim [11] obtained the following expressions for generalized Dedekind sum. These expressions are also needed to compute the values of zeta functions for ideal classes of the respective real quadratic fields.

**Lemma 2** (Kim [11]). Let \( m \) be a positive integer. Then, we have

(i) \( S^3(\pm 1, m) = \pm ((-m^4 + 5m^2 - 4)/(120m^3)) \)

(ii) \( S^2(\pm 1, m) = ((m^4 + 10m^2 - 6)/(180m^3)) \)

3. Main Results

Let \( n \) be a positive integer and let \( d = n^2 + 4 \) is a square-free integer. Clearly, \( d \equiv 5 \) (mod 8) and \( n \) is odd. In this case, the fundamental unit of \( k \) is \( \epsilon = (n + \sqrt{d})/2 \) and \( N(\epsilon) = -1 \). If \( p \mid n \), then \( p \) splits in \( k = \mathbb{Q}(\sqrt{d}) \) as

\[
\langle p \rangle = \left( \frac{p + 2 + \sqrt{d}}{2}, \frac{p + 2 - \sqrt{d}}{2} \right).
\]

By [12], Theorem 2.4, we know that

\[
\zeta_k(-1, A) = \frac{n^3 + 11n}{360},
\]

(7)

where \( A \) will always denote the principal ideal class in \( k \).

In this section, we will prove our main results. As a start, we record the following proposition.

**Proposition 1.** Let \( k = \mathbb{Q}(\sqrt{d}) \), where \( d = n^2 + 4 \) is a square-free integer. Let \( p \) be an odd prime divisor of \( n \), and let \( C \) be the ideal class containing \( \langle p, ((p + 2 + \sqrt{d})/2) \rangle \) or \( \langle p, ((p + 2 - \sqrt{d})/2) \rangle \). Then

\[
\zeta_k(-1, C) = \frac{n^3 + n(p^4 + 10p^2)}{360p^2}.
\]

(8)

**Proof.** Let us assume \( I = \langle p, ((p + 2 + \sqrt{d})/2) \rangle \in C^{-1} \). Then, \( \{r_1 = ((p + 2 + \sqrt{d})/2), r_2 = p \} \) is an integral basis for \( I \), and thus \( \delta(I) = p\sqrt{d} \). By Lemma 1, we get

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} = \begin{pmatrix}
  n + p + 1 & d - (p + 2)^2 \\
  4p & n - p - 1
\end{pmatrix}.
\]

(9)

Since \( p \mid n \), \( ((n \pm p)/2) \pm 1 \equiv 1 \pmod{p} \). Hence, by Lemma 2, we obtain

\[
240c^3 (\text{sgn} c) S^3(a, c) = 240 \times p^3 S^3(1, p)
\]

\[
= 2(-p^4 + 5p^2 - 4),
\]

\[
240c^3 (\text{sgn} c) S^3(d, c) = 240 \times p^3 S^3(-1, p)
\]

\[
= -2(-p^4 + 5p^2 - 4),
\]

\[
180ac^3 (\text{sgn} c) S^2(a, c) = 180 \left( \frac{n + p}{2} + 1 \right)^3 S^3(1, p)
\]

\[
= \frac{(p^4 + 10p^2 - 6)(n + p + 2)}{2},
\]

\[
180 dc^3 (\text{sgn} c) S^2(d, c) = 180 \left( \frac{n - p}{2} - 1 \right)^3 S^3(-1, p)
\]

\[
= \frac{(p^4 + 10p^2 - 6)(n - p - 2)}{2}.
\]

(10)

By Theorem 2, we get

\[
\zeta_k(-1, C) = \frac{n^3 + n(p^4 + 10p^2)}{360p^2}.
\]

(11)

\[
\square
\]

We use Proposition 1 in order to prove the following theorem, which gives a lower bound for \( h(n^2 + 4) \).

**Theorem 3.** Let \( n \) be a positive integer, \( \omega(n) \geq 3 \) and let \( d = n^2 + 4 \) is a square-free integer. Then
By keeping the abovementioned notation, we obtain:

\[
\zeta_k(-1,C_i) = \frac{n^3 + n(p_i^4 + 10p_i^2)}{360p_i^2}
\]

(13)

By computing \( \zeta_k(-1, C_i) = \zeta_k(-1, A) \), we arrive at \( n = p_i \). This contradicts the fact that \( \omega(n) \geq 3 \). Similarly, \( \zeta_k(-1, C_i) = \zeta_k(-1, C_j) \) for \( i \neq j \), and then \( n = p_i p_j \). This again contradicts the fact that \( \omega(n) \geq 3 \).

We notice \( \{C_1, \ldots, C_{\omega(n)}\} \) are distinct nonprincipal ideal class in \( k \). This completes the proof. \( \square \)

Now, we give a necessary and sufficient condition for \( k = \mathbb{Q}(\sqrt{n^2 + 4}) \) to have class number \( \omega(n) + 1 \).

**Theorem 4.** By keeping the abovementioned notation, we have \( h(d) = \omega(n) + 1 \) if and only if

\[
\sum_{m=0}^{(n-1)/2} \sigma_1(1 + mn - m^2) = \frac{n^3 + 11n}{12} + \sum_{i=1}^{\omega(n)} n_i^2 + n(p_i^4 + 10p_i^2) \cdot \frac{12p_i^2}{360}.
\]

(14)

**Proof.** Now, by Theorem 1,

\[
\zeta_k(-1) = \frac{1}{60} \sum_{|t| < \sqrt{n^2 + 4}} \sigma_1 \left( \frac{n^2 + 4 - t^2}{4} \right),
\]

\[
= \frac{1}{60} \sum_{t \equiv n \mod 4 \text{ and } t \text{ is odd}} \sigma_1 \left( \frac{n^2 - t^2}{4} \right),
\]

\[
= \frac{1}{30} \sum_{t \equiv n \mod 4 \text{ and } t \text{ is odd}} \sigma_1 \left( \frac{(n-t)(n+t)}{4} \right).
\]

(15)

If we replace \( t \) by \( n-r \), where \( r \) is even, then

\[
\zeta_k(-1) = \frac{1}{30} \sum_{0 \leq r \leq n-1 \text{ and } r \text{ is even}} \sigma_1 \left( 1 + \frac{r(2n-r)}{4} \right).
\]

(16)

Now, we replace \( r \) by \( 2m \), and then we have

\[
\zeta_k(-1) = \frac{1}{30} \sum_{m=0}^{(n-1)/2} \sigma_1 \left( 1 + \frac{mn-m^2}{4} \right).
\]

(17)

Necessary: let \( h(d) = \omega(n) + 1 \). Then, the class group of \( k \) is \( C_k = \{A, C_1, \ldots, C_{\omega(n)}\} \). Now, by definition, we have

\[
\zeta_k(-1) = \sum_{j \in C_k} \zeta_k(-1, J),
\]

\[
\zeta_k(-1) = \zeta_k(-1, A) + \sum_{j \in C_k} \zeta_k(-1, C_i).
\]

(18)

This implies

\[
\zeta_k(-1) = \frac{n^3 + 11n}{12} + \sum_{i=1}^{\omega(n)} n_i^3 + n(p_i^4 + 10p_i^2) \cdot \frac{12p_i^2}{360}.
\]

(19)

Finally, by (17), we obtain

\[
\sum_{m=0}^{(n-1)/2} \sigma_1(1 + mn - m^2) = \frac{n^3 + 11n}{12} + \sum_{i=1}^{\omega(n)} n_i^3 + n(p_i^4 + 10p_i^2) \cdot \frac{12p_i^2}{360}.
\]

(20)

Sufficiency: let

\[
\sum_{m=0}^{(n-1)/2} \sigma_1(1 + mn - m^2) = \frac{n^3 + 11n}{12} + \sum_{i=1}^{\omega(n)} n_i^3 + n(p_i^4 + 10p_i^2) \cdot \frac{12p_i^2}{360}.
\]

(21)

Hence, by (17), we find

\[
\zeta_k(-1) = \frac{n^3 + 11n}{360} + \sum_{i=1}^{\omega(n)} n_i^3 + n(p_i^4 + 10p_i^2) \cdot \frac{12p_i^2}{360}.
\]

(22)

By Theorem 3, we get \( h(d) \geq \omega(n) + 1 \).

Suppose \( h(d) > \omega(n) + 1 \). Then, there exist at least \( \omega(n) + 2 \) ideal classes in \( k \).

Since for any ideal class \( E \), \( \zeta_k(-1, E) > 0 \); thus,

\[
\zeta_k(-1) > \zeta_k(-1, A) + \sum_{j \in C_k} \zeta_k(-1, C_i).
\]

(23)

It is a contradiction. \( \square \)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


