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Research Letter

Analytical Structuring of Periodic and Regular Cascading Solutions in Self-Pulsing Lasers

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A newly proposed strong harmonic-expansion method is applied to the laser-Lorenz equations to analytically construct a few typical solutions, including the first few expansions of the well-known period-doubling cascade that characterizes the system in its self-pulsing regime of operation. These solutions are shown to evolve in accordance with the driving frequency of the permanent solution that we recently reported to illustrate the system. The procedure amounts to analytically construct the signal Fourier transform by applying an iterative algorithm that reconstitutes the first few terms of its development.

1. Introduction

The standard laser equations describe semiclassical atom-field interactions inside a unidirectional ring cavity. In the case of a homogeneously broadened system, these equations are transformed, after adequate approximations, into a simple set of three nonlinearly coupled differential equations, the so-called laser-Lorenz equations [1–7]. Despite their amazing simplicity, these equations deliver a remarkably broad range of solutions in their unstable regime of operation, ranging from stable period one to typical hierarchical cascades ending in erratic time traces.

The pulsing solutions of these equations must be found by numerical integration [5]. That was the generally accepted line of thoughts which prevailed among the scientific community involved in the laser dynamic issue ever since Haken proved the analogy between the single-mode laser equations and the Lorenz model of fluid turbulence [3]. Immediately after the recognition of such an analogy, an unprecedented rush into computers ensued (see [2, 5], and references therein for more details). Despite the huge literature that was devoted to the system for the past three decades, the interest towards the subject does not seem to dwindle away over the years [8–11].

In connection with analytical aspects of the laser-Lorenz dynamics, we have recently revisited the system with the application of a strong harmonic expansion method [10], which revealed to be quite genuine to handle some features of the system unstable behavior. The procedure, carried out to third-order in field amplitude, has led us to extract, for the first time in the dynamic literature, a second eigenfrequency, which has been shown to be an intrinsic property of the unstable permanent solution of the system. This new natural frequency, added to the well-known transient frequency, which is rooted in standard first-order small signal analyses, gives the system all the properties of a two-frequency scheme, from which most of the physical interpretation of its dynamics have been pulled out [11].

This paper aims at demonstrating that the analytical information extracted from such third-order expansion analysis naturally yields a simple method to construct periodic solutions. Typical as well as peculiar examples, including the well-known period-doubling that structures following an increase of the excitation parameter are directly built.

2. Reminder of the Laser-Lorenz Equations

In their simplest normalized form, the well-known single-mode homogeneously broadened laser equations write [2, 3, 10, 11]

$$\begin{aligned}\dot{E}(t) &= -\kappa\{E(t)+2CP(t)\},\\ \dot{P}(t) &= -P(t)+E(t)D(t),\\ \dot{D}(t) &= -\wp\{D(t)+1+E(t)P(t)\},\end{aligned} \tag{1}$$

where $E(t)$, $P(t)$, $D(t)$ represent, respectively, the laser field-amplitude, the polarization, and the population inversion of the amplifying medium, κ and $\wp$ are, respectively, the cavity decay-rate and the population relaxation rate, both scaled to the polarization relaxation rate, while $2C$ is an excitation parameter associated with the external pumping mechanism that is responsible for population inversion. No attempt is made, here, to recall the full properties of these equations, since these equations were described in many books and volume chapters, in addition to the so many journal contributions.

3. Typical Hierarchies of Numerical Solutions

Regular and irregular pulsing solutions are known to characterize (1) for low $\wp$ values. Let us center our attention on a period-doubling sequence that structures itself following an increase of the excitation parameter $2C$, obtained with cavity and material parameters $\kappa = 3$ and $\wp = 0.1$.

Figure 1 represents the evolution of the laser-field variable along with the corresponding frequency spectra, obtained with a standard fast Fourier transform algorithm. Up to $2C = 18.4$, the solution is a symmetric period one solution (see Figure 1(a)) and the frequency spectrum consists of components at Δ, 3Δ, 5Δ, and so forth (see Figure 1(d)). Δ is the fundamental angular frequency of the signal-time trace. Increasing the excitation parameter transforms the symmetric solution into an asymmetric signal, an example obtained with $2C = 27.7$ is represented in Figure 1(b). Note that the corresponding frequency spectrum (see Figure 1(e)) exhibits additional components at 2Δ, 4Δ, and 6Δ. This constitutes the first signature of a period-doubling sequence which is observed in the output intensity. For $2C = 29.5$, the asymmetric signal shows period-two oscillations (see Figure 1(c)), while the frequency spectrum shows additional components at 1.5Δ, 2.5Δ, 3.5Δ, and so forth (see Figure 1(f)). The corresponding intensity is a period-four signal (as may be checked from Figure 2(c)).

Let us point out that, for all the solutions represented in Figure 1, the field amplitude undergoes perpetual switching around a zero-mean value. This behavior constitutes the basis of the strong harmonic expansion method, which is outlined in the following section.

Representations of the population-inversion variable show an evolution according to the following scheme. Up to $2C = 18.4$, the time trace is a regular period one oscillation, with sharp components in the frequency spectrum at 2Δ, 4Δ, 6Δ, and so forth. An increase of $2C$ transforms the period-one solution into a period-two signal, the corresponding frequency spectrum shows additional components at Δ, 3Δ, 5Δ, and so forth. For $2C = 29.5$, a period-four oscillation characterizes the solution while the frequency spectrum consists of the previous components and emerging small peaks at 0.5Δ, 1.5Δ, 2.5Δ, and so forth.

Before constructing the corresponding analytical structures, let us turn into basic analytical considerations. In order to characterize the essential features of the pulsing solutions, we will first give a brief outline of the essential steps of an adapted strong harmonic-expansion method that yields analytical expressions for the angular frequency of the pulsing state as well as for the first few harmonic components of the corresponding analytical solutions.

4. Strong Harmonic-Expansion Analysis of the Long-Term Solutions

Adapted to the long-term solutions of Figure 1 are the following Fourier expansions:

$$E(t) = \sum_n E_n \cos(n\Delta t), \tag{2a}$$

$$P(t) = \sum_n P_{ipn}\cos(n\Delta t) + P_{opn}\sin(n\Delta t), \tag{2b}$$

$$D(t) = D_{dc} + \sum_n d_{ipn}\cos(n\Delta t) + d_{opn}\sin(n\Delta t), \tag{2c}$$

where the indexes ip and op, respectively, stand for in phase and out of phase.

When the above expansions are inserted into (1), one finds a hierarchical set of algebraic relations. These show that the field and polarization expansions (2a) and (2b) both contain odd terms while the population inversion development (2c) contains even terms, in conformity with the nonlinear coupling of (1) and with the spectra shown in Figure 1. The first-, third-, and fifth-order terms for the laser-field expansion (2a) are derived after lengthy but straightforward calculations and take the form

$$E_1 = 2\sqrt{\frac{(1+\kappa)(\wp^2+4\Delta^2)(1+\Delta^2)}{\wp[\kappa(1+\wp-\Delta^2)+\wp(1-\Delta^2)-4\Delta^2]}}, \tag{3a}$$

$$E_3 = -2C\Gamma_d T_{3d}\left[\wp^2(1-3\Delta^2)-8\wp\Delta^2\right]\frac{E_1^3}{4}, \tag{3b}$$

$$\begin{aligned}E_5 = -2C\Gamma_d\Gamma_4\Gamma_5\{&f(\Delta)E_1^5+g(\Delta)E_1^2E_3+h(\Delta)E_1^4E_3\\ &+q(\Delta)E_1E_3^2+s(\Delta)E_1^3E_3^2\},\end{aligned} \tag{3c}$$

with

$$\Gamma_d = \frac{1}{1+\Delta^2+E_1^2/2}, \tag{3d}$$

$$T_{3d} = \frac{1}{(1+9\Delta^2)(\wp^2+4\Delta^2)}. \tag{3e}$$

The above procedure, limited to third-order, also gives a closed-form expression for the long-term pulsation in terms of the fixed decay rates κ and $\wp$, and the adjustable excitation parameter $2C$, which writes (for details, see [10]):

$$\Delta = \sqrt{\frac{(2C-1)\kappa\wp(2+\wp)-3(\kappa+1)\wp^2}{8(\kappa+1)-\wp(2\kappa+\wp+4)}}. \tag{4}$$

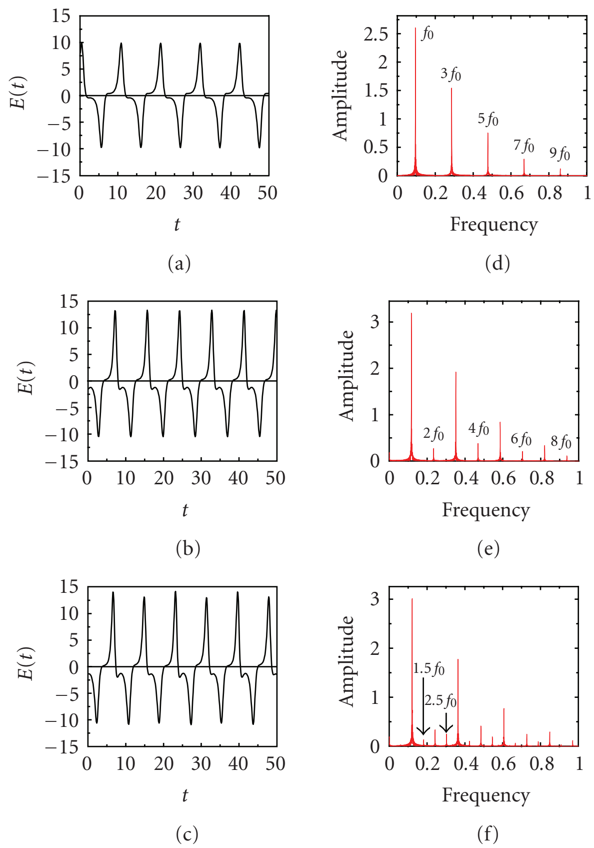

Figure 1: Typical hierarchy of pulsing solutions obtained with increasing excitation parameter and corresponding frequency spectra. (a) Symmetric solution with period one, obtained with $2C = 18.4$, (b) asymmetric period one solution, obtained with $2C = 27.9$, (c) asymmetric solution with period two, obtained with $2C = 29.5$, (d) frequency spectrum of the period one solution, (e) frequency spectrum of the asymmetric solution, showing the emergence of small even harmonics, and (f) frequency spectrum of the asymmetric period two solution, where, in addition to the even harmonics of (e), additional peaks appear at intermediate values between the odd and even components (subharmonic components).

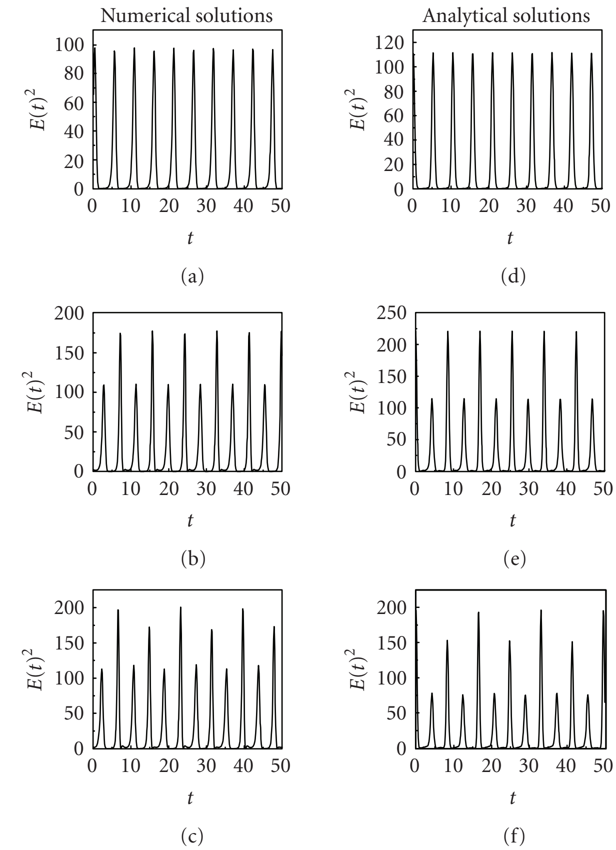

Figure 2: Comparison between the field-intensities, obtained (a), (b), (c) numerically and (d), (e), (f) analytically. The period-doubling sequence appears clearly in both series of traces.

Let us point out, at this stage, that this last expression carries a fundamental significance in pulse structuring of the *permanent pulsing state*, whereas the usual *small-signal analyses only give the transient oscillations of the relaxing solutions.*

5. Analytical structuring of the pulsing solutions

5.1. First example: period-one solution

Let us focus on the period-one example of Figure 1(a) and see how the analytical solution structures itself with increasing order in laser-field expansion. The long-term operating frequency is estimated from (4). Indeed if one limits expansion (2a) to first order only, the solution is a mere sinusoidal trace, while the inclusion of the third-order field component as evaluated from (3a) and (3b), respectively, gives plain indications on how the solution structures itself. For example, for the regular solution obtained just above the instability threshold at $2C = 10$, $\kappa = 3$, and $\wp = 0.1$, the values of these components evaluate as $E_1 = 5.2$ and $E_3 = 1.73$, while the corresponding long- term frequency is $\Delta \approx 0.42$. Thus, to third order, the analytical field expansion writes

$$E(t) = 5.2\cos(0.42t) + 1.73\cos(3 \times 0.42t). \tag{5}$$

In order to obtain a better fit, the calculations are extended towards fifth order in field amplitude. For the same parameter values, (3c) yields $E_5 = 0.8$. To fifth order, the analytical field expansion thus writes

$$\begin{aligned} E(t) = {} & 5.2\cos(0.42t) + 1.73\cos(3 \times 0.42t) \\ & + 0.8\cos(5 \times 0.42t). \end{aligned} \tag{6}$$

The very lengthy and time-consuming calculations (with an increased rate of error occurrence with increasing order-term evaluation) required to obtain the fifth-order terms are of no encouragement to attempt higher-order-terms determination any further. Instead, a much more convenient way to find the amplitudes of higher-order terms consists in a direct evaluation of the components peak height from the frequency spectra of the corresponding numerical solutions. Such a procedure straightforwardly yields the analytical solution up to the desired order. The highest order term is dictated by the importance of the corresponding frequency component emerging from the frequency spectrum. The examples of Figure 1 suggest an expansion up to the 11th term. Extending the expansion to ninth order still shows some differences with the exact numerical solution. The adapted expansion requires taking into account all the terms up to the eleventh order. This demonstrates the importance of each order in pulse structuring.

The final analytical expansion of the period-one oscillations thus writes

$$E(t) = \sum_{n=0}^{n=5} E_{2n+1}\cos((2n+1)\Delta t). \tag{7}$$

The odd-order expansion (7) remains valid as long as the solution consists of symmetric period one oscillations, that is, from $2C_{2\text{th}} \approx 10$ (instability threshold) up to $2C = 18.4$. Indeed, the field-amplitude components depend on the excitation parameter and must be evaluated separately for each excitation level. The values of these components obtained with $2C = 18.4$ are directly extracted from the corresponding Fourier components of Figure 1(d).

Now, let us focus on the asymmetric and the period-two signals of Figures 1(b) and 1(c), respectively. The following subsection demonstrates that even for these seemingly more complicated examples; the above procedure appears to be fairly-well adaptable to find the corresponding analytical Fourier expansions.

5.2. Construction of the period-doubling cascade

The asymmetric solution of Figure 1(b) stems from the appearance of additional and intermediate even components in the corresponding frequency spectrum (see Figure 1(e)). Thus, obviously an asymmetric regular solution must contain odd as well as even frequency components that stem from stronger nonlinear interactions between the laser-field variable, which evolves with an odd expansion structure, with the population inversion whose Fourier development carries an even form. The field expansion thus writes

$$E(t) = \sum_{n=0}^{N} E_{2n+1}\cos((2n+1)\Delta t) + \sum_{n=1}^{N} E_{2n}\cos(2n\Delta t). \tag{8}$$

The amplitudes are evaluated from the fast Fourier transform of Figure 1(e). When these amplitudes are inserted into (8), one finds, with the use of (4), the complete analytical representation of Figure 1(b).

Finally, the period-two solution of Figure 1(c) is constructed by inferring terms that contain frequency components of the form 1.5Δ, 2.5Δ, 3.5Δ, and so forth. These terms emerge from the corresponding frequency spectrum of Figure 1(f). The period-two solution of Figure 1(c) is described by the following field-expansion:

$$\begin{aligned} E(t) = {} & \sum_{n=0}^{N} E_{2n+1}\cos((2n+1)\Delta t) + \sum_{n=1}^{N} E_{2n}\cos(2n\Delta t) \\ & + \sum_{n=1}^{N} E_{n}\cos\left(\frac{2n+1}{2}\Delta t\right). \end{aligned} \tag{9}$$

The amplitudes are obtained from the fast Fourier transform of Figure 1(f), and the obtained solution carries the same structure as Figure 1(c).

As a final illustration of the period-doubling cascade, we represent the field-intensity signals corresponding to the hierarchy of Figure 1 along with their analytical counterparts for comparison. These are shown in Figure 2. The first few period-doubling sequences show a perfect match between the numerical and the analytical solutions. These representations demonstrate that the numerical period-doubling sequence peculiar to (1) can be constructed analytically with an adapted strong-harmonic expansion method. Indeed, the method is applicable inside the whole control-parameter space that exhibits periodic solutions. Some other interesting examples, obtained with the more complex infinite-dimensional system, are given in [12].

6. Conclusion

We have extended a recently introduced strong harmonic-expansion technique to the laser-Lorenz equations to find an analytical description of a few typical solutions that characterize the self-pulsing regime of the single mode homogeneously broadened laser operating in a bad-cavity configuration. The method presented here is fairly general to be applicable to other differential sets of equations that qualitatively possess the same self-pulsing properties. In particular, the well-known integro-differential "Maxwell-Bloch" equations adapted to describe the unstable state of a single-mode inhomogeneously broadened laser have been, as well, handled with the same approach [12]. However, the involved algebra is much more complicated, and the analytical calculations were limited to third order in field amplitude.

Even in the much simpler cases, presented in this paper, the very lengthy and awkward algebra involved in the determination of high-order terms has not allowed us to evaluate higher than the fifth-order components. Despite these limitations, we have proposed a fairly simple method to determine the complete solutions by inferring the field-amplitudes directly from the field-amplitude spectra obtained with a fast Fourier transform of the temporal signals. The method has been applied to describe, analytically, the first few solutions of a period-doubling sequence, peculiar to the Lorenz equations, which takes place following an increase of the excitation parameter. Even though most of the analysis focused on fixed control parameters (cavity-decay rate and population inversion relaxation rate), the method contains enough generality to be extended to the entire parameter space that exhibits regular pulsing solutions. The examples chosen in Section 5 show strong enough evidence of the efficacy of the strong harmonic expansion technique. *It is worth to mention again and stress on the fact* that the main clue behind the construction of analytical self-pulsing solutions is rooted on the analytical expression of the driving angular frequency, which in turn is rooted in the first- and third-order terms of the harmonic expansions. Once, the driving frequency is evaluated, finding the complete solutions merely amounts to evaluate the laser-field components of the corresponding Fourier expansions.

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