CYCLE TIMES IN A STARLIKE NETWORK WITH STATE-DEPENDENT ROUTING

HANS DADUNA

*Institut für Mathematische Stochastik
Universität Hamburg
Bundesstrasse 55
2000 Hamburg 13, Federal Republic of Germany*

ABSTRACT

For a star like network (central server system) with state dependent branching we compute the cycle time distribution. The Laplace-Stieltjes transform of this distribution is of product form. This allows to define a recursive algorithm for evaluation of cycle time moments of any order.

Key words: cycle times, state-dependent routing, central server system, product-form network, computational algorithms

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INTRODUCTION

The problem of determining routing strategies in networks of queues arises in the design of computing systems, communication networks, repair systems, etc. The purpose of introducing flexible state-dependent routing strategies is to optimally utilize network resources and to minimize network delay and response times.

The models for networks which are analytically tractable are almost all so-called "Product-form networks" (see e.g.: Baskett et al. [1975], Kelly [1979]). Unfortunately it turned out that introducing state-dependent routing into product-form networks usually destroy the product form of the steady state probabilities. For closed queueing networks Towsley [1980] has found a class of network topologies and a versatile class of state-dependent routing probabilities for these networks which lead to a product-form equilibrium distribution. An important member of that class of networks is the so-called central server system (starlike system) (see fig. 1). This starlike network is used to model a central processing unit with a set of peripheral devices or a set of distributed computer systems the communication of which is governed by a central control processor.

The aim of this note is threefold: For the starlike network

- I) we present the class of state dependent routing probabilities defined by Towsley and extend it,
- II) we compute the distribution of a customer's cycle time in the network, and
- III) we derive computational algorithms for the cycle time and its moments.

DESCRIPTION OF THE MODEL

Fig. 1

There are N customers cycling in the system. Service times at node v are $\exp(\mu_v)$ -distributed, $v = 1, \dots, U$, the service discipline is first-come-first-served. All service times are independent. The state space of the network is

$$S_N = \{ (n_1, \dots, n_U) : n_v \geq 0, v = 1, \dots, U, n_1 + \dots + n_U = N \}$$

recording the queue length at the nodes.

The branching probabilities from node 1 (central processor) to the peripheral units are assumed to be of the form

$$(1) \quad b_{1,v}(n_1, \dots, n_U) = h_v(n_v) h(n_2 + \dots + n_U), \quad v = 2, \dots, U,$$

where $h_v(\cdot)$ and $h(\cdot)$ are functions on $\mathbb{N}$. The sequence of branching decisions depends only on the actual state of the system.

Towsley [1980] proved

Theorem 1: Let $X = (X_t : t \geq 0)$ be the Markov process describing the time evolution on state space S_N of the starlike system of Fig. 1. Then the unique equilibrium distribution of X has product-form and is given by

$$(2) \quad p^N(n_1, \dots, n_U) = G_N^{-1} \left\{ \prod_{v=2}^U \left(\prod_{i=0}^{n_v-1} h_v(i) \right) \right\} \left\{ \prod_{i=0}^{N-n_1-1} h(i) \right\} \left\{ \prod_{v=1}^U \mu_v^{-n_v} \right\}, \quad (n_1, \dots, n_U) \in S_N,$$

where G_N is the norming constant (empty products are defined to be 1).

Although the functional form (1) is a severe restriction for the branching probabilities $b_{1,v}(\cdot)$, we shall show, that the class of feasible routing strategies obtained from it is a versatile one. The structure of the functions in (1) is very simple. Towsley [1980] partially stated

Theorem 2: Assuming that all peripheral processors are visited with positive probability by some customers, the functions h and h_v must be of the form

$$(3) \quad h_v(i) = (C i + d_v)_+,$$

and

$$(4) \quad h(i) = \left(i C + \sum_{v=2}^U d_v \right)^{-1},$$

$$v = 2, \dots, U; \quad i = 0, \dots, N-1,$$

where $(x)_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$, is the positive part of x , C is any real constant, and $d_2, \dots, d_U$ are positive constants.

Proof: See Daduna [1985], Theorem 1.

Remarks:

- a) If $h_v(n) = 0$ for some v and n , then X is not irreducible on S_N , but there exists a unique subset $S_N' \subseteq S_N$ such that X is irreducible on S_N' . Henceforth we shall assume that X is defined on S_N' , if necessary.
- b) In applications a severe problem is that of buffer-overflow which yields blocking of parts of the network. Handling blocking is very complicated in

BCMP networks e.g. The model presented here admits blocking in a natural way:

Assume that processors $u_1, u_2, \dots, u_S \in \{2, \dots, U\}$ should be blocked if there are

$s_{u_1}, \dots, s_{u_B}$ customers waiting (including the one just in service).

Then for a given $C < 0$ choose $-d_v = C s_v, v = u_1, \dots, u_B$ and $-d_v > (N - 1)C, v \in \{2, \dots, U\} - \{u_1, \dots, u_B\}$.

CYCLE TIME DISTRIBUTION

Krzesinski [1982] developed a Mean-Value-Algorithm for Towsley's network which in the usual way provides system orientated performance measures: throughput and average queue length, and the customer orientated performance measure: average sojourn time.

For the simple starlike network we are able to determine a customer's cycle time distribution. This is stated in

Theorem 3: (Daduna [1985]) Suppose in the starlike system with state-dependent routing a customer finds at his arrival at node 1 n_v other customers at node $v, v = 1, \dots, U, (n_1 + \dots + n_U = N-1)$, with probability $p^{N-1}(n_1, \dots, n_U)$ (defined similar to (2)), then the distribution of his cycle time, i.e. the time until his subsequent arrival at node 1 has Laplace-Stieltjes transform (LST)

$$(5) \quad g^N(s) = \sum_{\substack{(n_1, \dots, n_U) \\ n_1 + \dots + n_U = N-1}} p^{N-1}(n_1, \dots, n_U) \left(\frac{\mu_1}{\mu_1 + s} \right)^{n_1 + 1} \\ \times \sum_{v=2}^U b_{1,v}(n_1, \dots, n_U) \left(\frac{\mu_v}{\mu_v + s} \right)^{n_v + 1}, \quad s \geq 0.$$

Theorem 3 is in fact a simple corollary of the stronger result of

Theorem 4: Suppose that in the starlike system of Fig. 1 at time $t = 0$ there is at least one customer - henceforth called the "tagged customer C" - at node 1 placed at the tail of the queue, who sees n_v other customers at node $v, v = 1, \dots, U$, with probability $p^{N-1}(n_1, \dots, n_U), n_1 + \dots + n_U = N - 1$.

Let σ be the time of C's first departure from node 1 and let τ be the length of C's first sojourn time in the rest of the network, i.e. in $\{2, 3, \dots, U\}$.

The joint distribution of (σ, τ) has LST

$$(6) \quad g^N(s, t) = \sum_{\substack{(n_1, \dots, n_U) \\ n_1 + \dots + n_U = N-1}} p^{N-1}(n_1, \dots, n_U) \left(\frac{\mu_1}{\mu_1 + s} \right)^{n_1 + 1} \\ \times \sum_{v=2}^U b_{1,v}(n_1, \dots, n_U) \left(\frac{\mu_v}{\mu_v + t} \right)^{n_v + 1}, \quad s, t \geq 0.$$

Remark: The distribution determined by $g^N(s, t)$ is a limiting distribution as well as a stationary distribution:

If $(\sigma_1, \sigma_2, \dots)$ is the sequence of C's sojourn times at node 1 and if $(\tau_1, \tau_2, \dots)$ is the sequence of C's sojourn times in the rest of the network, then both sequences are ergodic and we have weak convergence $L(\sigma_n, \tau_n) \rightarrow L(\sigma, \tau)$.

If the system is started as in theorem 4 then the sequence $((\sigma_n, \tau_n) : n = 1, 2, \dots)$ is stationary although X is not stationary. For a discussion see Kelly/Pollett [1983] or Melamed [1982].

Proof of Theorem 4: Let P_{N-1} be the measure governing the system's evolution if the initial distributions of X is prescribed by $p^{N-1}(\cdot)$ and the condition that C is placed at the end of the tail of the queue of node 1; let $E_{N-1}(\cdot)$ denote expectations with respect to P_{N-1} . It follows:

$$g^N(s, t) = E_{N-1} e^{-s\sigma - t\tau} = \\ = \sum_{\substack{(n_1, \dots, n_U) \\ n_1 + \dots + n_U = N-1}} \int_{[0, \infty)} e^{-sx} E_{N-1}(e^{-t\tau} \mid \sigma = x, X_{\sigma^-} = (n_1 + 1, n_2, \dots, n_U)) \\ dP_{N-1}(\sigma \leq x \mid X_{\sigma^-} = (n_1 + 1, n_2, \dots, n_U)) P_{N-1}(X_{\sigma^-} = (n_1 + 1, n_2, \dots, n_U)).$$

The strong Markov property of a suitable underlying Markov process yields

$$E_{N-1}(e^{-t\tau} \mid \sigma = x, X_{\sigma^-} = (n_1 + 1, n_2, \dots, n_U)) \\ = E_{N-1}(e^{-t\tau} \mid X_{\sigma^-} = (n_1 + 1, n_2, \dots, n_U))$$

$$= \sum_{v=2}^U b_{1,v}(n_1+1, n_2, \dots, n_U) \left(\frac{\mu_v}{\mu_v+t} \right)^{n_v+1}, t \geq 0.$$

It is routine (but tedious) to prove

$$\begin{aligned} P_{N-1}(X_{\sigma^-} = (n_1+1, n_2, \dots, n_U)) \\ = p^{N-1}(n_1, n_2, \dots, n_U), \quad n_1 + n_2 + \dots + n_U = N-1. \end{aligned}$$

This yields

$$\begin{aligned} (7) \quad E_{N-1} e^{-s\sigma - t\tau} &= \\ &= \sum_{\substack{(n_1, \dots, n_U) \\ n_1 + \dots + n_U = N-1}} p^{N-1}(n_1, \dots, n_U) E_{N-1}(e^{-s\sigma} | X_{\sigma^-} = (n_1+1, \dots, n_U)) \\ &\quad \times \sum_{v=2}^U b_{1,v}(n_1+1, \dots, n_U) \left(\frac{\mu_v}{\mu_v+t} \right)^{n_v+1} \end{aligned}$$

To compute $E_{N-1}(e^{-s\sigma} | X_{\sigma^-} = (n_1+1, \dots, n_U))$ it suffices to determine the measure

$$P_{N-1}(\sigma \leq x, X_{\sigma^-} = (n_1+1, \dots, n_U)), \quad n_1 + \dots + n_U = N-1, \quad x \geq 0,$$

i. e. the joint distribution of C's sojourn time at node 1 and the queue length processes just before C's departure from node 1.

To determine that measure we introduce a Markov process $Y = (Y_t : t \geq 0)$ which describes C's progress through the network from time $t = 0$ until he enters one of the nodes $2, 3, \dots, U$.

Y lives on

$$\bar{S}_N = \{ ((\tilde{n}_1, C, n_1), n_2, \dots, n_U) : \tilde{n}_1 + n_1 + n_2 + \dots + n_U = N-1 \} \cup \{n^*\}.$$

If $Y_t = ((\tilde{n}_1, C, n_1), n_2, \dots, n_U)$ then at time t C is still present at node 1 having $\tilde{n}_1$ other customers behind and n_1 customers before him at node 1, while n_v customers are at node v , $v = 2, 3, \dots, U$. n^* is an absorbing state: Y jumps into n^* if C departs from node 1.

So $P_{N-1}(\sigma \leq x, X_{\sigma^-} = (n_1+1, n_2, \dots, n_U))$ is the probability that absorption of Y takes place before time $x \geq 0$ and the state of Y just before absorption is $((n_1, C, 0), n_2, \dots, n_U)$. For fixed $(n_1, \dots, n_U)$ $P_{N-1}(\sigma \leq x, X_{\sigma^-} = (n_1+1, n_2, \dots, n_U))$ has a density given by

$$P_{N-1} (Y_x = ((n_1, C, 0), \dots, n_U)) \mu_1 dx.$$

Therefore,

$$(8) \quad E_{N-1} (e^{-s\sigma} | X_{\sigma^-} = (n_1 + 1, \dots, n_U)) =$$

$$\begin{aligned} &= \int_{[0, \infty)} e^{-sx} \mu_1 P_{N-1} \left(Y_x = \left((n_1, C, 0), \dots, n_U \right) \right) dx = \\ &= \mu_1 \hat{p} \left((n_1, C, 0), \dots, n_U; s \right), s \geq 0, \end{aligned}$$

where

$$\begin{aligned} &\int_{[0, \infty)} e^{-sx} P_{N-1} \left(Y_x = \left((\tilde{n}_1, C, n_1), \dots, n_U \right) \right) dx \\ &= \hat{p} \left((\tilde{n}_1, C, n_1), \dots, n_U; s \right), \tilde{n}_1 + n_1 + \dots + n_U = N - 1, s \geq 0, \end{aligned}$$

is the Laplace transform (LT) of $P_{N-1} (Y_{(.)} = ((\tilde{n}_1, C, n_1), \dots, n_U))$.

Abbreviating

$$\begin{aligned} p \left(x; (\tilde{n}_1, C, n_1), \dots, n_U \right) &:= P_{N-1} \left(Y_x = \left((\tilde{n}_1, C, n_1), \dots, n_U \right) \right), \\ &\text{for } \tilde{n}_1 + n_1 + \dots + n_U = N - 1, \end{aligned}$$

we have

$$(9') \quad p \left(0; (\tilde{n}_1, C, n_1), n_2, \dots, n_U \right) = \delta_{o\tilde{n}_1} p^{N-1} (n_1, n_2, \dots, n_U),$$

and for $x > 0$

$$\begin{aligned} (9'') \quad &\frac{d}{dx} p \left(x; (\tilde{n}_1, C, n_1), \dots, n_U \right) \\ &= - \sum_{v=1}^U \left(1 - \delta_{on_v} \right) \mu_v p \left(x; (\tilde{n}_1, C, n_U), \dots, n_U \right) \\ &+ \left(1 - \delta_{o\tilde{n}_1} \right) \sum_{v=2}^U p \left(x; (\tilde{n}_1 - 1, C, n_1), \dots, n_v + 1, \dots, n_U \right) \mu_v \\ &+ \sum_{w=2}^U \left(1 - \delta_{on_w} \right) p \left(x; (\tilde{n}_1, C, n_1 + 1), \dots, n_w - 1, \dots, n_U \right) \\ &\times \mu_1 b_{1,w} \left(\tilde{n}_1 + 2 + n_1, \dots, n_U \right), \end{aligned}$$

$$\tilde{n}_1 + n_1 + \dots + n_U = N - 1.$$

Applying Laplace transformation we obtain from (9', 9''):

$$\begin{aligned}
 (10) \quad & \hat{p} \left((\tilde{n}_1, C, n_1), \dots, n_U; s \right) \left(\sum_{v=1}^U (1 - \delta_{on_v}) \mu_v + s \right) \\
 &= (1 - \delta_{on_1}) \sum_{v=2}^U \hat{p} \left((\tilde{n}_1 - 1, C, n_1), \dots, n_v + 1, \dots, n_U; s \right) \mu_v \\
 &\quad + \sum_{w=2}^U (1 - \delta_{on_w}) \hat{p} \left((\tilde{n}_1, C, n_1 + 1), \dots, n_w - 1, \dots, n_U; s \right) \\
 &\quad \times \mu_1 b_{1,w} (\tilde{n}_1 + 2 + n_1, \dots, n_U) \\
 &\quad + \delta_{on_1} p^{N-1} (n_1, \dots, n_U), s \geq 0,
 \end{aligned}$$

$$\tilde{n}_1 + n_1 + \dots + n_U = N - 1.$$

For fixed $s \geq 0$ the linear system (10) has a solution

$$\begin{aligned}
 (11) \quad & \hat{p} \left((\tilde{n}_1, C, n_1), \dots, n_U; s \right) \\
 &= \frac{1}{\mu_1} p^{N-1} (\tilde{n}_1 + n_1, \dots, n_U) \left(\frac{\mu_1}{\mu_1 + s} \right)^{\tilde{n}_1 + 1}, \\
 &\tilde{n}_1 + n_1 + \dots + n_U = N - 1,
 \end{aligned}$$

which is verified by substitution.

Inserting (11) into (8) and then (8) into (7) yields our result.

Remark: Formulas (5) and (6) show a product form representation for the cycle time LST (as well as for the LST of the joint sojourn times of a customer. Having the simple proofs of theorem 4 (and 3 in Daduna [1983]) and recalling the results of Daduna [1982] and Kelly/Polllett [1983] our present theorem seems necessarily to be true because the steady state distribution is of product form and C can not be overtaken during one cycle.

But the latter observations can be made in a starlike system with state-independent routing and state-dependent service times at node 1 without having a product-form cycle time distribution.

COMPUTATIONAL ALGORITHM

The LST (5) $g^N(s)$ of the cycle time distribution can easily be inverted to obtain the exact steady-state probabilities

$$P_{N-1}(\sigma + \tau \leq x), x \geq 0.$$

On the other hand it often suffices to compute moments of this distribution which can be done by evaluating at $0+$ the derivatives of $g^N(\cdot)$.

Reiser [1981] observed that in cyclic exponential queues it is possible to evaluate moments of cycle times of any order by a procedure which is recursive in the number of customers cycling, the number of queues and the order of moments.

We derive a similar procedure for the starlike system. Therefore we investigate the following system which is derived from the starlike system by elimination of the central processor

Fig. 2

where $b_v(n_2, n_3, \dots, n_U) := b_{1,v}(1, n_2, n_3, \dots, n_U)$, $n_2 + \dots + n_U = N - 1$, and $(n_2, n_3, \dots, n_U)$ is the vector of queue lengths the branching customer observes just before his arrival - he himself is not counted. (For details of the model description see section I.)

We have the following data of that network:

- (i) The steady state probabilities of the joint queue length process are

$$(12) \quad q_U^N(n_2, \dots, n_U) =$$

$$= F_{N,U}^{-1} \left\{ \prod_{v=2}^U \left(\prod_{i=0}^{n_v-1} h_v(i) \right) \right\} \left\{ \prod_{i=0}^{N-1} h(i) \right\} \left\{ \prod_{v=2}^U \mu_v^{-n_v} \right\}, n_2 + \dots + n_U = N,$$

where the functions $h_v(\cdot)$, $h(\cdot)$ are given in theorem 2, and $F_{N,U}$ is the norming constant.

- (ii) The steady-state cycle time of a customer, i.e. the time between two successive passages through the branching point of the network, has LST

$$(13) \quad f_U^N(s) = \sum_{\substack{(n_2, \dots, n_U) \\ n_2 + \dots + n_U = N-1}} q_U^{N-1}(n_2, \dots, n_U) \sum_{v=2}^U b_{1,v} \left(1, n_2, \dots, n_U\right) \left(\frac{\mu_v}{\mu_v + s} \right)^{n_v+1}, \quad s \geq 0$$

(For an interpretation of formula (13) see the remark after theorem 4.)

The product form of $q_U^N(\cdot)$ and $f_U^N(\cdot)$ leads directly to

Lemma 5: If in the system of Fig. 2 nodes $V+1, V+2, \dots, U$ are eliminated ($V \geq 2$) and N customers are cycling, let

$$q_V^N(n_2, \dots, n_V), \quad n_2 + \dots + n_V = N,$$

denote the steady-state distribution of the joint queue length process and

$$f_V^N(s), \quad s \geq 0$$

denote the LST of the steady-state cycle time distribution. Then

$$(14) \quad f_V^N(s) = \sum_{k=0}^{N-1} \frac{F_{N-1-k,U-1}}{F_{N-1,U}} \prod_{i=0}^{k-1} (C_i + d_v) \mu_v^{-k} \frac{\prod_{i=0}^{N-1-k} \left(C_i + \sum_{v=2}^{V-1} d_v \right)}{\prod_{i=0}^{N-1} \left(C_i + \sum_{v=2}^V d_v \right)} f_{V-1}^{N-k}(s) \\ + \frac{C_k + d_v}{C_{N-1-k} + \sum_{v=2}^{V-1} d_v} \left(\frac{\mu_v}{\mu_v + s} \right)^{k+1}, \quad s \geq 0.$$

Proof: Simple rearrangement of sums.

In the same way as lemma 5 we can prove

Theorem 6: The LST of the steady-state cycle time distribution can be derived recursively using the relation

$$(15) \quad (\mu_1 + s) G_{N-1} g^N(s) = G_{N-2} g^{N-1}(s) + \mu_1 G_{N-1,U} f_U^N(s), \quad N \geq 2, U \geq 2,$$

and the boundary conditions

$$f_2^N(s) = \left(\frac{\mu_2}{\mu_2 + s} \right)^N, \quad N \geq 1,$$

$$g^1(s) = \frac{\mu_1}{\mu_1 + s} \sum_{v=2}^U \frac{d_v}{\sum_{v=2}^U d_v} \frac{\mu_v}{\mu_v + s}.$$

Remarks:

- a) Theorem 6 is an analogon for sojourn time distributions (in closed queueing networks) of Buzen's algorithm which was developed for the computation of the norming constants in closed queueing networks (see Buzen [1973]).
- b) The norming constants $G_N, F_{N,U}, N \geq 1, U \geq 1$ can be computed via an analogon to Buzen's algorithm which can be written easily - see Krzesinski (1982).

In the spirit of Reiser's [1981] paper we obtain from theorem 6

Corollary 7: Let $M^N(r)$ be the r -th moment of the steady-state cycle time when N customers are cycling and

$$K_U^N(r)$$

be the r -th moment of the steady-state cycle time in the system of Fig. 2 with queues 2, ..., U when N customers are cycling. Then

$$\begin{aligned} \mu_1 M^N(r) G_{N-1} + M^N(r-1) G_{N-1} &= \\ &= M^{N-1}(r) G_{N-2} + \mu_1 K_U^N(r) F_{N-1,U}, \quad N \geq 2, \quad U \geq 2, \end{aligned}$$

and

$$K_2^N(r) = \frac{\Gamma(N+r)}{\mu_2^r \Gamma(N)}, \quad N \geq 1,$$

$$M^1(1) = \sum_{v=2}^U \frac{d_v}{\left(\sum_{v=2}^U d_v \right)} \left(\frac{1}{\mu_1} + \frac{1}{\mu_v} \right).$$

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