SOLUTIONS OF NONSTANDARD INITIAL VALUE PROBLEMS FOR A FIRST ORDER ORDINARY DIFFERENTIAL EQUATION*

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ABSTRACT

Differential equations of the form $y' = f(t,y,y')$, where f is not necessarily linear in its arguments, represent certain physical phenomena and have been known to mathematicians for quite a long time. But a fairly general existence theory for solutions of the above type of problems does not exist because the (nonstandard) initial value problem $y' = f(t,y,y')$, $y(t_0) = y_0$ does not permit an equivalent integral equation of the conventional form. Hence, our aim here is to present a systematic study of solutions of the NSTD IVPs mentioned above.

First, we establish the equivalence of the NSTD IVP with a functional equation and prove the local existence of a unique solution of the NSTD IVP via the functional equation. Secondly, we prove the continuous dependence of the solutions on initial conditions and parameters. Finally, we prove a global existence result and present an example to illustrate the theory.

Key words: Nonstandard Initial Value Problems, existence, uniqueness, solution, scalar-valued connected continuous dependence, initial conditions, parameters, differentiable functional equation, Gronwall's inequality, contraction mapping theorem, fundamental theorem of integral calculus.

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0. INTRODUCTION

Differential equations of the form $y' = f(t,y,y')$ where f is not necessarily linear in its arguments represent certain physical phenomena and have been known by mathematicians for quite a long time. The well-known Clairut's and Chrystal's equations fall into this category [3]. A few authors, notably E.L. Ince [4], H.T. Davis [3] et.al., have given some methods for finding solutions of equations of the above type. In fact, these methods are best described as follows.

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If there exists (t_0,y_0) such that the equation $y' = f(t,y,y')$ can be solved for y' as a single-valued function of (t,y) in a neighborhood of (t_0,y_0), say $y' = g(t,y)$, then the solution of the initial value problem (IVP) $y' = g(t,y)$, $y(t_0) = y_0$, if it exists, is also a solution of the original equation $y' = f(t,y,y')$ (and satisfies the initial condition $y(t_0) = y_0$). Or, if there exists (t_0,y_0), such that the equation $y' = f(t,y,y')$ can be solved for y' as a multi-valued function of (t,y) in a neighborhood of (t_0,y_0), then a (nonunique) solution of the IVP $y' = f(t,y,y')$, $y(t_0) = y_0$ is given by a certain (not necessarily convergent) infinite series. For other cases, to the authors' knowledge, there does not seem to be any method for proving the existence of a solution of the above problem.

One obvious reason why there does not exist a fairly general existence theory for solutions of equations of the above type is that the nonstandard IVP $y' = f(t,y,y')$, $y(t_0) = y_0$ does not permit an equivalent integral equation representation, whereas the IVP $y' = g(t,y)$, $y(t_0) = y_0$ does so. Hence our aim here is to present a systematic study of solutions of the nonstandard IVP mentioned above. First, we shall establish the equivalence of the nonstandard IVP with a functional equation (not the conventional integral equation), and shall prove the local existence of a unique solution of the nonstandard IVP via the functional equation. Secondly, we shall prove the continuous dependence of the solution on initial conditions, and parameters. Also, we shall establish a global existence result. Finally, we shall present an example to illustrate the theory.

Before indicating the section-wise split-up of the work, we shall introduce some notation and make some definitions. Let $\boldsymbol{R}$ denote the real line, and let C^2 denote the two-dimensional complex plane. Let $\boldsymbol{R} \times C^2$ denote the cartesian product space of $\boldsymbol{R}$ and C^2, taken in that order, equipped with the usual product topology. Let D be a connected subset of $\boldsymbol{R} \times C^2$ such that the interior of D, denoted by D^0, is nonempty. For a scalar (real or complex) valued function y defined on an interval I of $\boldsymbol{R}$, let y' denote the derivative of y, if it exists. Let $f(t,y,z)$ be a scalar valued, not necessarily linear, function defined for $(t,y,z) \in D$.

Consider the nonstandard initial value problem (NSTD IVP)

$$y' = f(t,y,y'),\ y(t_0) = y_0 \tag{1}$$

where $(t_0,y_0) \in D^0 \cap$ (ty-plane).

Definition 1: *By a solution y of the NSTD IVP* (1), *we mean a continuously differentiable scalar valued function $y(t)$ defined for $t \in I$, where I is some interval of the real line containing the point 't_0' such that*

i) $y(t_0) = y_0$,

ii) the triplet $(t,y(t),y'(t)) \in D$ for all $t \in I$, and

iii) $y'(t) = f(t,y(t),y'(t))$ holds good for all $t \in I$.

Also, consider the (not necessarily linear) functional equation

$$z(.) = f(.,y_0 + \int_{t_0}^{\cdot} z(s)ds, z(.)). \tag{2}$$

Definition 2: *By a solution z of equation* (2), *we mean a continuous scalar valued function z(t) defined for* $t \in I$, *where I is some interval of the real line containing the point 't_0' such that*

ii') the triplet $(t, y_0 + \int_{t_0}^{t} z(s)ds, z(t)) \in D$ *for all* $t \in D$, *and*

iii') $z(t) = f(t, y_0 + \int_{t_0}^{t} z(s)ds, z(t))$ *holds good for all* $t \in I$.

In section 1, we shall show that the NSTD IVP (1) and equation (2) are equivalent in the sense that IVP (1) has a (unique) solution if and only if equation (2) has a (unique) solution. Also, under usual hypotheses on f, we shall prove, using the contraction mapping theorem, that equation (2) has a unique local solution which, by the equivalence property, implies that IVP (1) has a unique local solution.

In section 2, we shall prove the continuous dependence of the solution of IVP (1) on initial conditions (t_0, y_0), and on functions $f(t,y,y')$. Also, under continuity, uniform boundedness and Lipschitz conditions on f (for f defined on $\boldsymbol{R} \times C^2$), we shall establish a global existence result for the solution of IVP (1).

In section 3, by way of illustration, we shall prove the local existence of a unique solution of a concrete NSTD IVP.

1. EQUIVALENCE OF THE NONSTANDARD IVP (1) AND FUNCTIONAL EQUATION (2), AND THE LOCAL EXISTENCE AND UNIQUENESS THEOREM

The following lemmas establish the equivalence of the NSTD IVP (1) and functional equation (2).

Lemma 1: *The nonstandard IVP (1) has a solution if and only if equation (2) has a solution.*

Proof: Suppose that IVP (1) has a solution y. Then by definition 1, there exists an interval I containing the point 't_0' such that $y(t)$ is continuously differentiable (scalar-valued function) for $t \in I$, $y(t_0) = y_0$, $(t, y(t), y'(t)) \in D$ for all $t \in I$, and such that $y'(t) = f(t, y(t), y'(t))$ holds good for all $t \in I$.

Define $z(t) = y'(t)$, for $t \in I$. Then, clearly, $z(t)$ is a continuous scalar-valued function defined for $t \in I, t_0 \in I$, $(t, y_0.+ \int_{t_0}^{t} z(s)ds\ , z(t)) = (t, y(t), y'(t)) \in D$ for all $t \in I$, and $z(t) = f\,(t, y_0.+ \int_{t_0}^{t} z(s)ds\ , z(t))$ holds good for all $t \in I$. Therefore, by definition 2, $z(t)$ is a solution of equation (2).

Conversely, suppose that equation (2) has a solution z. Then, by definition 2, there exists an interval I containing 't_0' such that $z(t)$ is a continuous scalar-valued function defined for $t \in I$, $(t, y_0.+ \int_{t_0}^{t} z(s)ds\ , z(t)) \in D$ and that $z(t) = f\,(t, y_0.+ \int_{t_0}^{t} z(s)ds\ , z(t))$ holds good for all $t \in$ I.

Define $y(t) = y_0.+ \int_{t_0}^{t} z(s)ds$, for $t \in I$. By the fundamental theorem of integral calculus [1], $y(t)$ is continuously differentiable for $t \in I, t_0 \in I$, and $y'(t) = z(t)$ for all $t \in I$. Therefore, we have that $y(t_0) = y_0$, $.(t, y(t), y'(t)) = (t, y_0.+ \int_{t_0}^{t} z(s)ds\ , z(t)) \in D$ for all $t \in I$, and $y'(t) = f(t,y(t),y'(t))$ holds good for all $t \in I$. Hence, by definition 1, y is a solution of IVP (1). This completes the proof.

Lemma 2: *The nonstandard IVP (1) has a unique solution if and only if equation (2) has a unique solution.*

Proof: Equivalently, we shall show that IVP (1) has more than one solution if and only if equation (2) has more than one solution.

Suppose that IVP (1) has two distinct solutions, say $y_1 \neq y_2$ existing on a common interval I containing 't_0'. Define $z_i(t) = y'_i$, $t \in I$, i=1,2. We claim $z_1 \not\equiv z_2$. For, if possible, let $z_1(t) = z_2(t)$ for all $t \in I$. Then $y'_1(t) - y'_2(t) = 0$ for all $t \in I$, which upon integration implies that $y_1(t) - y_2(t) = k$, a constant, for all $t \in I$, and taking $t = t_0$, we get that $y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0 = k$. Hence $y_1(t) = y_2(t)$ for all $t \in I$, which is a contradiction. Therefore, $z_1 \not\equiv z_2$ and by lemma 1 we get that z_1 and z_2 are two distinct solutions of equation (2).

Conversely, suppose that $z_1 \not\equiv z_2$ are two distinct solutions of equation (2) existing on a common interval containing 't_0'. Define

$$y_i(t) = y_0 + \int_{t}^{t_0} z_i(s)ds,\ i = 1,2.$$

We claim that $y_1 \not\equiv y_2$. For, if possible, let $y_1(t) = y_2(t)$ for all $t \in I$. Then by the fundamental theorem of integral calculus we get that $z_1(t) = z_2(t)$ for all $t \in I$, which is a contradiction. Therefore $y_1 \not\equiv y_2$ and by lemma 1 we get that y_1 and y_2 are two distinct solutions of IVP (1). This completes the proof.

The following corollary follows immediately from lemmas 1 and 2.

Corollary 1: *The nonstandard IVP (1) has a unique solution if and only if equation (2) has a unique solution.*

Next, we shall prove a local existence and uniqueness theorem for the solution of IVP (1).

Let D be a connected subset of $R \times C^2$ defined by

$$D = \{(t,y,z) \in R \times C^2 \mid |t - t_0| \leq a, |y - y_0| \leq b, |z| \leq c\}$$

where a, b and c are positive constants. We note that D^0 is nonempty and $(t_0,y_0) \in D^0 \cap$ (ty-plane). Let f be a scalar valued function defined on D satisfying the following conditions:

i) f is continuous with respect to $(t,y,z) \in D$,

ii) $|f(t,y,z)| \leq c$ for all $(t,y,z) \in D$, and

iii) $|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq k_1|y_1-y_2| + k_2|z_1-z_2|$ for all (t,y_1,z_1), $(t,y_2,z_2) \in D$,

where $k_1 > 0$ and $0 \leq k_2 < 1$ are constants.

Theorem 1 (Local existence and uniqueness theorem): *Under conditions (i)-(iii), the nonstandard IVP (1) has a unique solution existing on the interval* $[t_0 - \alpha, t_0 + \alpha]$, *where* α *is a real number such that*

$$0 < \alpha < \min\left(\frac{1 - k_2}{k_1}, b/c, a\right). \tag{3}$$

Proof: By corollary 1, it is enough to prove the existence of a unique solution z of equation (2) on the interval $[t_0 - \alpha, t_0 + \alpha]$, which we shall accomplish by making use of contraction mapping theorem in a suitable function space.

To this end, let $I = [t_0 - \alpha, t_0 + \alpha]$ where α is a real number satisfying relation (3). Consider the Banach space $C(I)$ of all continuous scalar valued functions defined on I, equipped with the supremum norm given by $\|y\| = \sup_{t \in I} |y(t)|$, $y \in C(I)$. Let $M = \{z \in C(I) \mid \|z\| \leq c\}$. Clearly, M is a nonempty closed subset of $C(I)$. For $z \in M$, we have $|z(t)| \leq c$ for all $t \in I$, and letting

$$y(t) = y_0 + \int_{t_0}^{t} z(s)ds, \; t \in I,$$

we get that $|y(t)-y_0| \leq |\int_{t_0}^{t} |z(s)|ds|$ (by (3)).

Thus for every $t \in I$, the triplet $(t,y(t),z(t)) \in D$. Now define a map $F: M \to C(I)$ by

$$(Fz)(t) = f(t,y_0 + \int_{t_0}^{t} z(s)ds, \, z(t)), \; t \in I.$$

From conditions (i) and (ii), it easily follows that F indeed maps M into itself. To verify that F is a contraction on M, let z_1 and $z_2 \in M$, and consider

$$\begin{aligned}\| Fz_1 - Fz_2\| &= \sup_{t \in I} f(t,y_0 + \int_{t_0}^{t} z_1(s)ds,\, z_1(t)) - f(t,y_0 + \int_{t_0}^{t} z_2(s)ds,\, z_2(t)) \\ &\le \sup_{t \in I} \left\{ (k_1 \left|(\int_{t0}^{t} |z_1(s) - z_2(s)|ds\right| + k_2|z_1(t) - z_2(t)| \right\} \text{ (by } (iii)) \\ &\le \mathrm{k1} \sup_{t \in I} \left|(\int_{t0}^{t} \|z_1 - z_2\|ds\right| + k_2\|z_1 - z_2\| \\ &\le (k_1\alpha + k_2)\|z_1 - z_2\|\end{aligned}$$

By relation (3), $0 \le k_1\alpha + k_2 < 1$, and hence F is a contraction map on M.

Consequently, by contraction mapping theorem [5], F has a unique fixed point (in M). That is, there exists a unique $z \in M$ such that

$$z(t) = (Fz)(t) = f(t, y_0 + \int_{t_0}^{t} z(s)ds,\, z(t)) \text{ for all } t \in I.$$

This completes the proof.

Corollary 2: *Under the hypotheses of theorem 1, IVP (1) has a unique solution existing on the interval* $(t_0 - \beta, t_0 + \beta)$, *where* $\beta = \min(\frac{1-k_2}{k_1}, b/c, a)$.

Proof follows immediately from theorem 1, since the unique solution of IVP (1) exists on the interval $[t_0 - \alpha, t_0 + \alpha]$ for every α such that $0 < \alpha < \beta$.

2. CONTINUOUS DEPENDENCE OF THE SOLUTION OF IVP (1) ON INITIAL CONDITIONS, PARAMETERS, AND THE GLOBAL EXISTENCE THEOREM

Theorem 2 (Continuous dependence of the solution on initial conditions): *Let the hypotheses of the theorem be true. Let* $t'_0 \in (t_0 - \beta, t_0 + \beta)$, *where* $\beta = \min(\frac{1-k_2}{k_1}, b/c, a,$ *and let* $|y_0 - y'_0| < b$. *Let* $\tilde{D} = \{(t,y,z) \in \mathbf{R} \times \mathbb{C}^2 \mid |t-t'_0| \le a,\ |y-y'_0| \le b,\ |z| \le c\}$. *Suppose that* f *is also defined on* $\tilde{D}$ *and satisfies conditions* (i) -(iii) *on* $\tilde{D}$. *Let* y_1 *be the unique solution of the IVP* $y' = f(t,y,y')$, $y(t_0) = y_0$ *existing on the interval* $(t0 - \beta, t0 + \beta)$. *Let* y_2 *be the unique solution of the IVP* $y' = f(t,y,y')$, $y(t'_0) = y'_0$ *existing on the interval* $(t'_0 - \beta, t'_0 + \beta)$. *Let* $r_1 = \max(t_0 - \beta, t'_0 - \beta)$ *and* $r_2 = \min(t_0 + \beta, t'_0 + \beta)$. *Clearly* t_0 *and* $t'_0 \in (r_1, r_2)$, *and both* y_1 *and* y_2 *exist on the interval* (r_1, r_2). *Suppose that* $|t_0 - t'_0| < \delta$ *and* $|y_0 - y'_0| < \delta$ *where* $0 \le \delta < (r_2 - r_1)$, *then*

$$|y_1(t) - y_2(t)| \le \left(1 + c + \frac{k_1(r_2 - r_1)(1 + c)}{1 - k_2} e^{(k_1(r_2 - r_1)/(1 - k_2))}\right)\delta$$

for all $t \in (r_1, r_2)$.

Proof: Let $\phi_1(t) = y'_1(t)$ and $\phi_2(t) = y'_2(t)$. Then

$$\phi_1(t) = f(t, y_0 + \int_{t_0}^{t} \phi_1(s)\, ds, \phi_1(t)), \text{ and } \phi_2(t) = f(t, y'_0 + \int_{t'_0}^{t} \phi_2(s)\, ds, \phi_2(t))$$

for all $t \in (r_1, r_2)$.

Therefore, for $t \in (r_1, r_2)$, we have

$$|\phi_1(t) - \phi_2(t)| = |f(t, y_0 + \int_{t_0}^{t} \phi_1(s)\, ds, \phi_1(t)) - f(t, y'_0 + \int_{t'_0}^{t} \phi_2(s)\, ds, \phi_2(t))|$$

$$\le k_1 |y_0 - y'_0 + \int_{t_0}^{t} \phi_1(s)\, ds - \int_{t'_0}^{t} \phi_2(s) ds| + k_2 |\phi_1(t) - \phi_2(t)| \text{ (by } (iii))$$

$$(4) \qquad \le k_1 |y_0 - y'_0| + k_1 \left|\int_{t_0}^{t'_0} \phi_1(s)\, ds\right| + k_1 \left|\int_{t'_0}^{t} (\phi_1(s) - \phi_2(s))\, ds\right| + k_2 |\phi_1(t) - \phi_2(t)|$$

$$\le k_1\delta + k_1 \left|\int_{t_0}^{t'_0} |\phi_1(s)|\, ds\right| + k_1 \left|\int_{t'_0}^{t} |\phi_1(s) - \phi_2(s)|\, ds\right| + k_2 |\phi_1(t) - \phi_2(t)|$$

$$\le k_1\delta + k_1 c\delta + k_1 \left|\int_{t'_0}^{t} |\phi_1(s) - \phi_2(s)|\, ds\right| + k_2 |\phi_1(t) - \phi_2(t)|$$

which implies that

$$|\phi_1(t) - \phi_2(t)| \le \frac{k_1(1 + c)\delta}{1 - k_2} + \frac{k_1}{1 - k_2} \left|\left(\int_{t'_0}^{t} |\phi_1(s) - \phi_2(s)| ds\right.\right|.$$

Now, by Gronwall's inequality [2], we get from the above that

$$|\phi_1(t) - \phi_2(t)| \le \left(\frac{(1 + c)k_1\delta}{1 - k_2}\right) e^{k_1/(1 - k_2)|t - t'_0|}$$

$$\le \left(\frac{(1 + c)k_1\delta}{1 - k_2}\right) e^{k_1(r_1 - r_2)/(1 - k_2)}$$

for all $t \in (r_1, r_2)$.
Consequently, for $t \in (r_1, r_2)$, we get that

$$\begin{aligned} | y_1(t) - y_2(t) | &= | y_0 + \int_{t_0}^{t} \phi_1(s)\, ds - y'_0 - \int_{t'_0}^{t} \phi_2(s)\, ds | \\ &\le | y_0 - y'_0 | + | \int_{t_0}^{t'_0} | \phi_1(s) |\, ds | + | \int_{t'_0}^{t} | \phi_1(s) - \phi_2(s) |\, ds | \\ &\le \delta + c\delta + \left(\frac{(1+c)k_1\delta}{1-k_2}\right)(r_2 - r_1)e^{k_1(r_2 - r_1)/(1-k_2)} \end{aligned}$$

This completes the proof.
Corollary 3: *Under the hypotheses of theorem 2, if $t, t' \in (r_1, r_2)$ and $| t - t'| < \delta$, then*

$$|y_1(t) - y_2(t')| \le \left(1 + 2c + \left(\frac{k_1(1+c)(r_2 - r_1)}{1-k_2}\right)e^{k_1(r_2 - r_1)/(1-k_2)}\right)\delta.$$

Proof: For $t, t' \in (r_1, r_2)$ and $| t - t'| < \delta$, we have

$$\begin{aligned} | y_1(t) - y_2(t') | &\le | y_1(t) - y_2(t) | + | y_2(t) - y_2(t') | \\ &\le | y_1(t) - y_2(t) | + | y'_0 + \int_{t'_0}^{t} \phi_2(s)\, ds - y'_0 - \int_{t'_0}^{t'} \phi_2(s)\, ds | \\ &\le | y_1(t) - y_2(t) | + | \int_{t}^{t'} |\phi_2(s)|\, ds | \\ &\le | y_1(t) - y_2(t) | + c\delta \\ &\le \left(1 + c + \left(\frac{k_1(1+c)(r_2 - r_1)}{1-k_2}\right)e^{k_1(r_2 - r_1)/(1-k_2)}\right)\delta + c\delta \end{aligned}$$

(by theorem 2).

This completes the proof.
Theorem 3 (Continuous dependence of the solution on parameters): *Let the hypotheses of theorem 1 be true. Let y_1 be the unique solution of the IVP $y' = f(t,y,y')$, $y(t_0) = y_0$ existing on the interval $(t_0 - \beta, t_0 + \beta)$, where $\beta = \min(\frac{1-k_2}{k_1}, b/c, a)$. Let $|t_0 - t'_0| < \beta$ and $|y_0 - y'_0| < \beta$.*

Let $\tilde{D} = \{(t,y,z) \in R \times \mathbb{C}^2 \mid |t - t'_0| \le a_1,\ |y - y'_0| \le b_1,\ |z| \le c_1\}$ *where* a_1, b_1 *are positive constants and* $0 < c_1 \le c$. *We note that* $D \cap \tilde{D} \ne \phi$. *Let* $g(t,y,z)$ *be a scalar valued function defined for* $(t,y,z) \in \tilde{D}$ *satisfying the following conditions:*

i') $g(t,y,z)$ *is continuous with respect to* $(t,y,z) \in \tilde{D}$,

ii') $|g(t,y,z)| \le c_1$ *for all* $(t,y,z) \in \tilde{D}$,

iii') $|g(t,y_1,z_1) - g(t,y_2,z_2)| \le k_3|y_1 - y_2| + k_4|z_1 - z_2|$, *for all* $(t,y_1,z_1), (t,y_2,z_2) \in \tilde{D}$ *where* $k_3 > 0$ *and* $0 \le k_4 < 1$ *are constants, and*

iv) $|f(t,y,z) - g(t,y,z)| \le \theta$ for all $(t,y,z) \in D \cap \tilde{D}$ *where* $\theta \ge 0$ *is a constant.*

Let y_2 *be the unique solution of the IVP* $y' = g(t,y,y')$, $y(t_0') = y_0'$ *existing on the interval* $(t_0' - \tilde{\beta}, t_0' + \tilde{\beta})$ *where* $\tilde{\beta} = \min\left(\frac{1-k_4}{k_3}, b_1/c_1, a_1\right)$. *Let* $\eta = \min(\beta, \tilde{\beta})$. *Futher, assume that* $|t_0 - t'_0| < \eta$. *Let* $r_1 = \max(t_0 - \eta, t'_0 - \eta)$ *and* $r_2 = \min(t_0 + \eta, t'_0 + \eta)$. *Clearly,* t_0 *and* $t'_0 \in (r_1, r_2)$, *and both* y_1 *and* y_2 *exist on the interval* (r_1, r_2). *Now suppose that* $|t_0 - t'_0| < \delta$ *and* $|y_0 - y'_0| < \delta$ *where*

(5) $$0 \le \delta \le \min(\eta, b - c(r_2 - r_1)),$$

then

$$|y_1(t) - y_2(t)| \le (1 + c_1)\delta + \left(\frac{(1+c)k_1\delta + \theta}{1 - k_2}\right)(r_2 - r_1)\, e^{\left(\frac{k_1}{1-k_2}(r_2 - r_1)\right)}$$

for all $t \in (r_1, r_2)$.

Proof: Let $\phi_1 = y'_1$ and $\phi_2 = y'_2$. Then $\phi_1(t) = f(t, y_0 + \int_{t_0}^{t} \phi_1(s)\,ds, \phi_1(t))$ and

$\phi_2(t) = g(t, y_0' + \int_{t_0}^{t} \phi_2(s)\,ds,\ \phi_2(t))$ for all $t \in (r_1, r_2)$. Now for (r_1, r_2), we have that $|t - t_0| \le a$, $|\phi_2(t)| \le c_1 \le c$, and

$$|y'_0 + \int_{t_0}^{t} \phi_2(s)\,ds, - y_0| \le |y_0 - y'_0| + |\int_{t_0}^{t} |\phi_2(s)|\,ds| \le \delta + c_1(r_2 - r_1) \le \delta + c(r_2 - r_1) \le b$$

(by (5)).

Hence $(t, y'_0 + \int_{t_0}^{t} \phi_2(s)\,ds,\ \phi_2(t)) \in D$.

Therefore,

$$|\phi_1(t) - \phi_2(t)| = |f(t, y_0 + \int_{t_0}^{t} \phi_1(s)\,ds, \phi_1(t)) - g(t, y'_0 + \int_{t_0}^{t} \phi_2(s)ds,\ \phi_2(t))|$$

$$\leq |f(t,y_0 + \int_{t_0}^{t}\phi_1(s)ds,\ \phi_1(t)) - f(t,y'_0 + \int_{t_0}^{t}\phi_2(s)ds,\ \phi_2(t))|$$

$$+ |f(t,y'_0 + \int_{t_0}^{t}\phi_2(s)ds,\ \phi_2(t)) - g(t,y'_0 + \int_{t_0}^{t}\phi_2(s)ds,\ \phi_2(t))|$$

$$\leq (1+c)k_1\delta + k_1|\int_{t_0}^{t}|\phi_1(s) - \phi_2(s)|ds| + k_2|\phi_1(t) - \phi_2(t)| + \theta,$$

(by (4) and (*iv*))

which implies that

$$|\phi_1(t) - \phi_2(t)| \leq \left(\frac{(1+c)k_1\delta + \theta}{1-k_2}\right) + \frac{k_1}{1-k_2}|\int_{t'_0}^{t}|\phi_1(s) - \phi_2(s)|ds|.$$

Hence by Gronwall's inequality we get that

$$|\phi_1(t) - \phi_2(t)| \leq \left(\frac{(1+c)k_1\delta + \theta}{1-k_2}\right) e^{\left(\frac{k_1}{1-k_2}(r_2 - r_1)\right)}, \quad \text{for all } t \in (r_1, r_2).$$

Consequently, for $t \in (r_1, r_2)$, we obtain that

$$|y_1(t) - y_2(t)| = |y_0 - y'_0 + \int_{t_0}^{t}\phi_1(s)ds - \int_{t'_0}^{t}\phi_2(s)ds|$$

$$\leq |y_0 - y'_0| + |\int_{t'_0}^{t_0}|\phi_2(s)|ds| + |\int_{t_0}^{t}|\phi_1(s) - \phi_2(s)|ds|$$

$$\leq \delta + c_1\delta + \left(\frac{(1+c)k_1\delta + \theta}{1-k_2}\right)(r_2 - r_1)\, e^{\left(\frac{k_1}{1-k_2}(r_2 - r_1)\right)}.$$

This completes the proof.

Theorem 4 (Global existence theorem) *Let f be a scalar valued function defined on the whole of $R \times C^2$ such that f satisfies conditions (i) -(iii) on $R \times C^2$. Then for every initial data $(t_0, y_0) \in R \times C$, the IVP(1) has a unique solution existing on the entire real line* $(-\infty,+\infty)$.

Proof: For the initial data $(t_0, y_0) \in R \times C$, consider the set

$$D_1 = \{ (t,y,z) \in R \times C^2 \mid |t - t_0| \leq a, |y - y_0| \leq b, |z| \leq c \}$$

where a,b are some positive constants, and c is the bound for f given by (*ii*). From the hypotheses of the theorem, it clearly follows that f satisfies conditions (*i*) - (*iii*) on D_1. Hence, by theorem 1, there exists a unique solution y_1 of the IVP(1) on the interval

$[t_0 - \alpha, t_0 + \alpha]$, for every α such that $0 < \alpha < \min(\frac{1-k_2}{k_1}, b/c, a)$. Now fix α and consider the initial data $(t_0 + \alpha, y_1(t_0 + \alpha))$. Let

$$D_2 = \{(t,y,z) \in R \times C^2 \mid |t - t_0 - \alpha| \le a, |y - y_1(t_0 + \alpha)| \le b, |z| \le c\}.$$

Again f satisfies conditions (*i*) - (*iii*) on D_2, and hence by theorem 1, there exists a unique solution y_2 of the IVP

$$y' = f(t,y,y'), \; y(t_0 + \alpha) = y_1(t_0 + \alpha),$$

on the interval $[t_0, t_0 + 2\alpha]$. Also, by the uniqueness of the solution, we have that $y_1(t) = y_2(t)$, for $t \in [t_0, t_0 + \alpha]$. Now, define

$$y(t) = \begin{cases} y_1(t), \text{ for } t \in [t_0 - \alpha, t_0 + \alpha], \\ y_2(t), \text{ for } t \in [t_0 + \alpha, t_0 + 2\alpha] \end{cases}.$$

Clearly, $y(t)$ is a unique solution of the IVP(1) existing on the interval $[t_0 - \alpha, t_0 + 2\alpha]$.

Next take the new initial data $(t_0 + 2\alpha, y_2(t_0 + 2\alpha))$ and consider the set

$$D_3 = \{(t,y,z) \in R \times C^2 \mid |t - t_0 - 2\alpha| \le a, |y - y_2(t_0 + 2\alpha)| \le b, |z| \le c\}$$

As before, f satisfies conditions (*i*) - (*iii*) on D_3, and hence there exists a unique solution y_3 of IVP

$$y' = f(t,y,y'), \; y(t_0 + 2\alpha) = y_2(t_0 + 2\alpha),$$

on the interval $[t_0 + \alpha, t_0 + 3\alpha]$. Also, we have that $y_2(t) = y_3(t)$, for $t \in [t_0 + \alpha, t_0 + 3\alpha]$. Now, define

$$y(t) = \begin{cases} y_1(t), \text{ for } t \in [t_0 - \alpha, t_0 + \alpha] \\ y_2(t), \text{ for } t \in [t_0 + \alpha, t_0 + 2\alpha] \\ y_3(t), \text{ for } t \in [t_0 + 2\alpha, t_0 + 3\alpha] \end{cases}.$$

Thus the IVP(1) has a unique solution y existing on the interval $[t_0 - \alpha, t_0 + 3\alpha]$. Proceeding in this way indefinitely, we see that the solution y of the IVP(1) can be extended uniquely to the entire interval $[t_0 - \alpha, +\infty)$. Similarly it can be shown that the solution y can be extended uniquely to the interval $(-\infty, t_0 - \alpha]$. This proves that the IVP(1) has a unique solution existing on the entire real line $(-\infty, +\infty)$. Hence the proof is complete.

3. ILLUSTRATIVE EXAMPLE

Consider the IVP

(6) $$y' = \alpha e^t + \beta y + \gamma y_2 \cos|y'|, \; y(0) = 1$$

where α, β, and γ are (real or complex) constants. The problem that is of interest to us comes when $\gamma \neq 0$.

Below, we shall prove that IVP(6) has unique solution existing on (-1, +1) for certain values of α, β, and γ.

Theorem 5: *Suppose that the constants α, β, and $\gamma \neq 0$ satisfy the inequalities*

$$|\alpha|\, e + 2|\beta| + 4|\gamma| \leq 1, \tag{7}$$

$$\textit{and} \quad |\beta| + 8|\gamma| \leq 1. \tag{8}$$

Then the IVP(6) has a unique solution existing on the interval (-1, 1).

Proof: Let

$$D = \{(t,y,z) \in \boldsymbol{R} \times C^2 \mid |t| \leq 1, |y - 1| \leq 1, |z| \leq 1\}. \tag{9}$$

Clearly, D is a connected subset of $\boldsymbol{R} \times C^2$, and $(t_0, y_0) = (0,1) \in D^0 \cap (ty\text{-plane})$. Here, we have $f(t,y,z) = \alpha e^t + \beta y + \gamma y_2 \cos|z|$ and is continuous for $(t,y,z) \in D$. Also, for $(t,y,z) \in D$, we have that

$$\begin{aligned} |f(t,y,z)| &\leq |\alpha||e^t| + |\beta||y| + |\gamma||y_2||\cos|z|| \\ &\leq |\alpha||e| + 2|\beta| + 4|\gamma| && \text{(by (9))} \\ &\leq 1 && \text{(by (7))} \end{aligned}$$

Now, for (t,y_1, z_1) and $(t,y_2,z_2) \in D$, consider

$$\begin{aligned} |f(t,y_1, z_1) - f(t,y_2,z_2)| &= |\beta y_1 - \beta y_2 + \gamma y_1^2 \cos|z_1| - \gamma y_2^2 \cos|z_2|| \\ &\leq |\beta||y_1 - y_2| + |\gamma||y_1^2 \cos|z_1| - y_2^2 \cos|z_2|| \\ &\leq |\beta||y_1 - y_2| + |\gamma|(|y_1^2 - y_2^2||\cos|z_1|| + |y_2^2||\cos|z_1| - \cos|z_2||) \\ &\leq |\beta||y_1 - y_2| + |\gamma|((|y_1| + |y_2|)|y_1 - y_2| \\ &\qquad + |y_2|^2 2\sin\left(\frac{|z_1| + |z_2|}{2}\right)\sin\left(\frac{|z_1| - |z_2|}{2}\right)) \\ &\leq |\beta||y_1 - y_2| + 4|\gamma||y_1 - y_2|^2 + 4|\gamma||z_1 - z_2|^2 \qquad \text{(by (9))} \\ &= k_1|y_1 - y_2| + k_2|z_1 - z_2| \end{aligned}$$

where $k_1 = |\beta| + 4|\gamma|$ and $k_2 = 4|\gamma|$. By inequality (8), we have that $k_1 > 0$ and $0 < k_2 < 1$ (since $\gamma \neq 0$). Thus f satisfies conditions (*i*) -(*iii*) on D, and hence by theorem 1, the IVP(6) has a unique solution existing on the interval $(-\delta,\delta)$ where $\delta = \min\left(\frac{1 - 4|\gamma|}{|\beta| + 4|\gamma|}, 1\right)$. But by inequality (8), we have $\delta = 1$. This completes the proof of the theorem.

Remark 1: Theorem 1 guarantees only the existence of unique solutions of nonstandard IVPs and does not provide methods of finding these solutions explicitly. Nevertheless, in a subsequent paper we shall present numercial methods for finding the (approximate) solutions of the nonstandard IVPs.

Remark 2: The theory developed here also holds good for nonstandard IVPs for

(m-dimensional) vector differential equations and in this case we simply replace the (modulus)$|.|$ by the m-dimensional Euclidean norm $|.|_m$.

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