MARKOV CHAINS WITH QUASITOEPLITZ TRANSITION MATRIX: FIRST ZERO HITTING

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ABSTRACT

This paper continues the investigation of Markov Chains with a quasitoeplitz transition matrix. Generating functions of first zero hitting probabilities and mean times are found by the solution of special Riemann boundary value problems on the unit circle. Duality is discussed.

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1. INTRODUCTION.

Here we continue the investigation started in [1] of Markov chains with a quasitoeplitz transition matrix, elements of which depend on the difference of indices except for the first column and the first several rows. The primary focus of this paper is the problem of first zero hitting (FZH) in terms of generating functions (GF) of probabilities and mean times.

The practical importance of this problem is in its obvious connection with the problems of first emptying and response time arising in applications. There is also another reason for investigation: duality of this problem and state probabilities evaluation.

The FZH problem was only studied in that simplest particular case

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when the transition matrix was in fact toeplitz (except, of course, in the first column). In this case FZH is in fact a problem of the first entering nonpositive area for some random walk on the integers, so relevant GF's were found either in a purely probabilistic manner (see, for example, [2]), or by the solution of a Riemann boundary value problem (for example, Dukhovny [3] for a random walk that depends on a parameter).

In this paper we study a quasitoeplitz case arising in applications with feedback, so there is no structure of a random walk at some early states. We transform the FZH problem to nonuniform Riemann boundary value problems for the GF's of probabilities and mean times. As the sums of these do not converge, we deal in fact with boundary problems in a wide sense, that is in special Banach spaces (see [4]).

We also prove here that FZH mean times do not exceed some linear function of a starting state's number.

Having solved the boundary value problems, we obtain some finite systems of linear equations and prove that they have only one solution.

In conclusion we discuss a duality between transient and steady state probabilities from one side, and FZH probabilities and mean times' increments, respectively, from the other, imposed by the method of boundary value problems.

2. DEFINITIONS.

We assume here that we have a Markov chain $\{\xi_k\}$ with a quasitoeplitz transition matrix $A = (a_{ij})$ with "natural" properties (see [1], pp. 72-73). We retain the notation of [1] for $A_i(z) = \sum_{j=0}^{\infty} a_{ij}z^j$, $H(z) = \sum_{-\infty}^{\infty} h_jz^j$, level n such that $\forall i \geq n$ $a_{ij} = h_{j-i}$, $j > 0$; $a_{i0} = \sum_{-\infty}^i h_j$; operators T^+ and T^- that cut out parts with nonnegative and negative exponents of Laurent power series, D^i that gets the i th coefficient of the series, space L_1 of Laurent power series with absolutely summable coefficients, unit circle Γ : $|z| = 1$, its interior Γ^+ and exterior Γ^- .

Let us introduce q_i^k , $i > 0$, $k > 0$, the probability of reaching zero for the first time in k steps starting from i , and $\mu_i = \sum_{k=1}^{\infty} kq_i^k$, the mean number of steps in the FZH route starting from i , which is known to converge if the chain is ergodic. We also denote $q_i^0 = -1$, $i > 0$.

Let $x \in (0, 1)$, $|z| > 1$, and denote

$$\begin{aligned} q_i(x) &= \sum_{k=0}^{\infty} q_i^k x^k, \\ \mathbf{Q}(\mathbf{x}) &= (0, q_1(x), q_2(x), \dots)^T, \\ Q(x, z) &= \sum_{i=1}^{\infty} q_i(x) z^{-i}, \\ \mu(z) &= \sum_{i=1}^{\infty} \mu_i z^{-i}. \end{aligned}$$

We introduce the linear Banach space L_{∞} of formal Laurent power series with bounded coefficients with a norm

$$\| \sum_{i=-\infty}^{\infty} a_i z^i \| = \sup_i |a_i|$$

and a formal product of series (convolution)

$$a(z)b(z) = \sum_{i=-\infty}^{\infty} z^i \sum_{j=-\infty}^{\infty} a_{i-j} b_j$$

provided the absolute convergence of sums

$$\sum_{j=-\infty}^{\infty} a_{i-j} b_j, \quad \forall i.$$

Note that if $a(z) \in L_1$ and $b(z) \in L_{\infty}$ then their convolution obviously belongs to L_{∞} .

Whatever i is, a total FZH probability $\sum_{k=1}^{\infty} q_i^k$ does not exceed 1, so

$$(1) \quad -1 = q_i(0) \leq q_i(x) \leq q_i(1) \leq 0 \quad x \in [0, 1]$$

hence $Q(x, z) \in L_{\infty}$.

Denote $e(z) = \sum_{i=1}^{\infty} z^{-i} \in L_{\infty}$.

Lemma 1. Let $a^+(z) = \sum_{i=0}^{\infty} a_i z^i \in L_1$. Then $T^- a^+(z) e(z) = a^+(1) e(z)$.

Proof. According to the definitions of T^- and the convolution

$$T^- a^+(z) e(z) = \sum_{i=-\infty}^{-1} z^i \sum_{j=-\infty}^{\infty} a_{i-j} D^j e(z) = \sum_{i=-\infty}^{-1} z^i \sum_{j=0}^{\infty} a_j = a^+(1) e(z).$$

3. FIRST ZERO HITTING PROBABILITIES.

Every FZH route from $i > 0$ is either a direct step to zero or consists of one step to some $j > 0$ and a FZH route from j , so

$$(2) \quad q_i^{k+1} = \sum_{j=1}^{\infty} a_{ij} q_j^k \quad k \geq 0, i > 0.$$

Multiplying both sides by x^{k+1} and summing over k from 0 to ∞ we obtain

$$(3) \quad q_i(x) = x \sum_{j=1}^{\infty} a_{ij} q_j(x) + (x-1) \quad i > 0$$

or, in vector form,

$$(4) \quad \mathbf{Q}(\mathbf{x}) = xA^1\mathbf{Q}(\mathbf{x}) + (x-1)\mathbf{1},$$

where $\mathbf{1} = (0, 1, 1, \dots)$, and A^1 differs from A only by elements of the first row and column, which are all zeros except $a_{00}^1 = 1$.

Proceeding from (4) we now find the GF $Q(x, z)$.

Theorem 1. The GF $Q(x, z)$, $x \in (0, 1)$, $|z| > 1$, of the first zero hitting probabilities of the Markov chain $\{\xi_k\}$ with a quasitoeplitz transition matrix with $n > 0$ is defined by the following formulas:

$$(5) \quad Q(x, z) = z^{1-n} R^-(x, z) \\ [(x-1)R^+(x, 1)e(z) + \sum_{i=1}^{n-1} q_i(x)T^+ z^{n-1-i} R^-(x, z)^{-1}]$$

$$(6) \quad R^{\pm}(x, z) = \exp(-T^{\pm} \ln[1 - xH(z)])$$

and the unknowns $q_i(x)$, $i = 1, 2, \dots, n-1$ are the only solution of the system of linear equations

$$(7) \quad q_i(x) - x \sum_{j=1}^{n-1} q_j(x) D^{n-1} A_i(z) R^-(x, z) T^+ z^{n-1-j} R^-(x, z) \\ = (x-1)[1 + xR^+(x, 1)D^{n-1}A_i(z)R^-(x, z)e(z)] \quad i = 1, 2, \dots, n-1$$

($n = 0$ and $n = 1$ obviously generate the same FZH problems).

Proof. According to the definitions made

$$(8) \quad \sum_{j=1}^{\infty} a_{ij} q_j(x) = \begin{cases} D^0 A_i(z) Q(x, z) & i < n \\ D^{-i} H(z) Q(x, z) & i \geq n. \end{cases}$$

Note that the convolutions $A_i(z)Q(x, z)$ and $H(z)Q(x, z)$ exist and belong to L_{∞} as $A_i(z)$ and $H(z)$ by definition belong to L_1 . Therefore, the subset of equalities (3) for $i \geq n$ yields

$$(9) \quad Q_2(x, z) = xT^{-}H(z)[Q_2(x, z) + z^{n-1}Q_1(x, z)] + (x-1)e(z)$$

where

$$\begin{aligned} Q_1(x, z) &= \sum_{i=1}^{n-1} q_i(x) z^{-1} \\ Q_2(x, z) &= \sum_{i=n}^{\infty} q_i(x) z^{n-1-i}. \end{aligned}$$

We transform (9) to the formal equality of power series

$$(10) \quad \begin{aligned} Q_2(x, z) \sum_{i=1}^{n-1} [1 - xH(z)] \\ = F^+(x, z) + (x-1)e(z) - [1 - xH(z)]z^{n-1}Q_1(x, z) \end{aligned}$$

where

$$F^+(x, z) = -xT^+H(z)Q_2(x, z) + T^+[1 - xH(z)]z^{n-1}Q_1(x, z).$$

In (10) the power series $Q_2(x, z)$ and $F^+(x, z)$ correspond to functions analytic in Γ^- and Γ^+ , respectively; but they need not converge on Γ , so the equality (10) is the formal equality of the power series, not their sums. Therefore (10) must be treated as a Riemann boundary value problem in a wide sense.

Let us multiply (10) by $R^+(x, z)$ and apply T^- to both sides. Due to (6) on Γ : $|z| = 1$, $R^+(x, z)R^-(x, z) = [1 - xH(z)]^{-1}$, so with the help of Lemma 1 we obtain:

$$Q_2(x, z)R^-(x, z)^{-1} = (x-1)R^+(x, 1)e(z) - \sum_{i=1}^{n-1} q_i(x)T^-z^{n-1-i}R^-(x, z)^{-1},$$

which yields (5).

To obtain (7) we transform some first equalities of (3) as follows:

$$q_i(x) = xD^0 A_i(z)Q(x, z) + (x - 1) \quad i = 1, 2, \dots, n - 1$$

and substitute (5) for $Q(x, z)$.

Had (7) another solution, then (5) would generate another $\hat{Q}(x, z)$ satisfying (7) and (9). As $R^-(x, z)^{-1}$ belongs to L_1 , $\hat{Q}(x, z)$ defined by (5) belongs to L_∞ . Therefore, the vector $\hat{Q}(x)$ of coefficients of its Laurent expansion, that satisfies (4) because of (7) and (9), is a column with bounded elements. As matrix A is stochastic, the norm of the operator xA^1 in the Banach space of columns with bounded elements does not exceed $|x| < 1$, so (4) has the only solution in this space. Hence $\hat{Q}(x)$ and $Q(x)$ must be equal. This completes the proof of Theorem 1.

Let q_0^k be the probability of the first return to zero in k steps and $q_0(x) = \sum_{k=1}^{\infty} x^k q_0^k - 1$.

Corollary. The function $q_0(x)$ is expressed by

$$(11) \quad q_0(x) = (x - 1)[1 + xR^+(x, 1)D^{n-1}A_0(z)R^-(x, z)e(z)] \\ + x \sum_{j=1}^{n-1} q_j(x)D^{n-1}A_0(z)R^-(x, z)T^+z^{n-1-j}R^-(x, z)^{-1}.$$

Indeed, every route of the first return to zero in $k > 1$ steps consists of one step to some $j > 0$ and then a FZH route from j , so

$$(12) \quad q_0^{k+1} = \sum_{j=1}^{\infty} a_{0j}q_j^k \quad k > 0$$

$$(13) \quad q_0^1 = a_{00} = - \sum_{j=1}^{\infty} a_{0j} + 1 = \sum_{j=1}^{\infty} a_{0j}q_j^0 + 1.$$

Multiplying both sides of (12) by x^{k+1} , summing over k from 1 to ∞ and taking into account (13), we obtain

$$q_0(x) = x \sum_{j=1}^{\infty} a_{0j}q_j(x) + (x - 1) = xD^0 A_0(z)Q(x, z) + (x - 1)$$

which leads to (11) due to (5).

4. FZH MEAN TIMES: ORDER OF GROWTH.

In this section we estimate the order of growth for FZH mean times depending on the number of the starting state. This estimate makes it possible to find $\mu(z)$ by the same method used for $Q(x, z)$, ensuring the existence of all necessary convolutions.

Let s_m be the set of states $0, 1, \dots, m$; $d^m(i, j)$, $i > m$, $j \leq m$, be the first-descent-to- s_m route from i to j , and $D^m(i, j)$ be the set of all such routes. We introduce functions $N(d)$, the number of steps in the route d , and $PR(d)$, the probability of this route. Obviously, if d is a composition of d_1 and d_2 then

$$(14) \quad PR(d) = PR(d_1)PR(d_2)$$

$$(15) \quad N(d) = N(d_1) + N(d_2).$$

Denote

$$(16) \quad \begin{aligned} \chi_{ij}^m &= \sum_{d \in D^m(i, j)} PR(d) \\ \lambda_{ij}^m &= \sum_{d \in D^m(i, j)} PR(d)N(d) \end{aligned}$$

$$(17) \quad \lambda_i^m = \sum_{j=0}^m \lambda_{ij}^m$$

so that (17) defines the mean length of the first descent from i to s_m .

Lemma 2.

$$(18) \quad \mu_i = \lambda_i^m + \sum_{j=1}^m \chi_{ij}^m \mu_j \quad i > m.$$

Proof. Every FZH route from i is $d^0(i, 0)$ and obviously is either $d^m(i, 0)$ or a composition of $d^m(i, j)$ and $d^0(j, 0)$, $j = 1, 2, \dots, m$. Now (18) readily follows from (16) and (17).

Let us apply (18) to the chain $\{\xi_k\}$ with $n = 1$ and denote here λ_i instead of μ_i . For $m = 1$ we have here

$$(19) \quad \lambda_{i+1} = \lambda_{i+1}^1 + \lambda_1 \chi_{i+1,1}^1$$

Lemma 3. The FZH mean times for the ergodic Markov chain $\{\xi_k\}$ with a quasitoeplitz transition matrix with $n = 1$ satisfy the inequality

$$(20) \quad \lambda_i < \lambda_{i+1} < \lambda_i + \lambda_1 \quad i > 0.$$

Proof. Note that for this chain

$$\lambda_i^m = \lambda_{i-m}^0 = \lambda_{i-m} \quad i > m > 0$$

because all the transitions above zero depend on the difference of states. Hence $\lambda_{i+1}^1 = \lambda_i$, so (19) leads to (20).

Now we are back to an arbitrary $n > 0$.

Lemma 4. The FZH mean times for the ergodic Markov chain $\{\xi_k\}$ with quasitoeplitz transition matrix do not exceed some linear function of starting state's numbers.

Proof. As the transition matrix is quasitoeplitz, all the transitions above $n - 1$ depend on a difference of states, so

$$\lambda_i^{n-1} = \lambda_{i-n+1} \quad i \geq n.$$

Hence when $m = n - 1$ (18) reduces to

$$(21) \quad \mu_i = \lambda_{i-n+1} + \sum_{j=1}^{n-1} \chi_{ij}^{n-1} \mu_j.$$

Note that the sum of coefficients at μ_j on the right side is less than the total probability of first descent to s_{n-1} , i.e., less than 1; because of (20) $\lambda_{i-n+1} < (i - n + 1)\lambda_1$. Hence

$$(22) \quad \mu_i < (i - n + 1)\lambda_1 + \sup_{j=1,2,\dots,n-1} \{\mu_j\}$$

which proves the statement of Lemma 4.

Let us introduce

$$\begin{aligned} \nu_i &= \begin{cases} \mu_i & i \leq n \\ \mu_i - \mu_{i-1} & i > n \end{cases} \\ \nu(z) &= \sum_{i=n}^{\infty} \nu_i z^{n-1-i}. \end{aligned}$$

Corollary. $\nu(z) \in L_{\infty}$.

Indeed, according to (20) and (21), for all $i > n$

$$|\nu_i| = |\mu_i - \mu_{i-1}| < \lambda_1 + 2\sup_{j=1,2,\dots,n-1} \mu_j.$$

5. FZH MEAN TIMES: THE GENERATING FUNCTION.

Theorem 2. The GF $\mu(z)$, $|z| > 1$, of the FZH mean times for the ergodic Markov chain $\{\xi_k\}$ with a quasitoeplitz transition matrix with "natural properties" (see [1], pp. 72–73) is defined by the formulas

$$(23) \quad \mu(z) = z^{2-n} e(z) R^-(z) [R^+(1) e(z) + \sum_{i=1}^{n-1} \mu_i T^+ z^{n-i-2} (z-1) R^-(z)^{-1}]$$

$$(24) \quad R^\pm(z) = \exp \left(-T^\pm \ln z e(z) [1 - H(z)] \right)$$

where the unknown μ_i , $i = 1, 2, \dots, n-1$, $n > 0$, are the only solution of the system of linear equations

$$(25) \quad \begin{aligned} \mu_i - \sum_{j=1}^{n-1} \mu_j D^{n-2} A_i(z) R^-(z) e(z) T^+ z^{n-j-2} (z-1) R^-(z)^{-1} \\ = 1 + R^+(1) D^{n-2} A_i(z) e^2(z) R^-(z) \quad i = 1, 2, \dots, n-1, n > 0. \end{aligned}$$

Proof. Every FZH or first-return-to-zero route is either one step to zero or consists of one step to some $j > 0$ and then a FZH route from j . Hence for all i

$$(26) \quad \mu_i = \sum_{j=1}^{\infty} a_{ij} (\mu_j + 1) + a_{i0} = \sum_{j=1}^{\infty} a_{ij} \mu_j + 1.$$

For $i > n-1$ we can rewrite (26) as

$$(27) \quad \mu_i = D^0 z^i H(z) \mu(z) + 1.$$

According to the definition of $\{\nu_i\}$ we have

$$(28) \quad \mu(z) = \sum_{i=1}^{n-1} \mu_i z^{-i} + z^{2-n} \nu(z) e(z).$$

Therefore, multiplying both sides of (27) by z^{n-1-i} and summing over i from n to ∞ we obtain

$$z \nu(z) e(z) = T^- z^{n-1} \mu(z) H(z) + e(z).$$

Denoting $F^+(z) = T^+ z^{n-1} \mu(z)[1 - H(z)]$, due to (28) we have

$$(29) \quad \nu(z)z[1 - H(z)]e(z) = F^+(z) + e(z) - \sum_{i=1}^{n-1} \mu_i z^{n-1-i}[1 - H(z)].$$

Note that this is the equality of series, not of their sums, which converge in different areas. $\nu(z)$ is analytic in Γ^- and $F^+(z)$ is analytic in Γ^+ , though both belong to L_∞ , $\nu(z)$ due to the corollary of Lemma 4, and $F^+(z)$ because of Lemma 4 and the absolute convergence of $\sum_{j=-\infty}^{\infty} |j|h_j$, which is one of the required "natural properties." Therefore (29) is a Riemann boundary value problem in L_∞ .

As was shown in [1], the criterion of ergodicity implies that the functions $R^\pm(z)$ defined by (24) belong to L_1 , so when we multiply both sides of (29) by $R^+(z)$ all the convolutions exist and belong to L_∞ . Then we apply T^- to both sides and with the help of Lemma 1 obtain

$$\nu(z)R^-(z)^{-1} = R^+(1)e(z) - T^- \sum_{i=1}^{n-1} \mu_i z^{n-i-2}(z-1)R^-(z)^{-1}$$

which, with (28) yields (23).

The first n equalities of (26) may be written as

$$\mu_i = D^0 A_i(z)\mu(z) + 1 \quad i = 0, 1, \dots, n-1$$

from which, with the use of (23), we obtain (25).

Suppose the system (25) has a solution other than μ_i , $i = 1, 2, \dots, n-1$, say $\hat{\mu}_i$. Then (23) would generate another sequence $\hat{\mu}_i$, $i = 0, 1, \dots$, such that the vector $\Delta M = \{\hat{\mu}_i - \mu_i\}^T$ would be bounded (as $R^-(z)^{-1}$ and $R^-(z)$ belong to L_1).

Obviously the sequence $\{\hat{\mu}_i - \mu_i\}$ satisfies the equations

$$(30) \quad (\hat{\mu}_i - \mu_i) = \sum_{j=1}^{\infty} a_{ij}(\hat{\mu}_j - \mu_j) \quad i = 0, 1, \dots$$

Let us multiply both sides of (30) by p_i , which is the steady state probability of state i , and sum over i from 0 to ∞ . With ΔM bounded all the sums present converge absolutely, so we can change the order of summation and make use of the equilibrium equations

$$p_j = \sum_{i=0}^{\infty} p_i a_{ij} \quad j = 0, 1, \dots$$

to obtain

$$\sum_{i=0}^{\infty} p_i(\hat{\mu}_j - \mu_j) = \sum_{i=0}^{\infty} p_i \sum_{j=1}^{\infty} a_{ij}(\hat{\mu}_j - \mu_j) = \sum_{i=0}^{\infty} p_i(\hat{\mu}_i - \mu_i) - p_0(\hat{\mu}_0 - \mu_0).$$

Due to the conditions of descent (see [1]) $p_0 > 0$, so $\hat{\mu}_0 = \mu_0$. Thus we can rewrite (30) as $\Delta M = A \Delta M$. According to Chung [5], this means that ΔM must be collinear with $(1, 1, \dots)^T$, so with $\hat{\mu}_0 - \mu_0 = 0$, ΔM must be zero. This concludes the proof of Theorem 2.

Corollary. The mean time of the first return to zero is

$$(31) \quad \mu_0 = 1 + R^+(1)D^{n-2}A_0(z)e^2(z)R^-(z) + \sum_{i=1}^{n-1} \mu_i D^{n-2}A_0(z)R^-(z)e(z)T^+ z^{n-i-2}(z-1)R^-(z)^{-1}.$$

This follows, with the use of (23), from (26) with $i = 0$ written as

$$\mu_0 = D^0 A_0(z)\mu(z) + 1.$$

6. DUALITY.

In conclusion we discuss the relationship of duality between transient and steady state probabilities from one side, and FZH probabilities and mean times increments, respectively, from the other. This relationship is imposed in a natural way by the method of Riemann boundary value problems.

Following [4] we consider two Riemann boundary value problems

$$\begin{aligned} a_1^+(z)K_1(z) &= a_1^-(z) + b_1(z), & z \in \Gamma \\ a_2^+(z)K_2(z) &= a_2^-(z) + b_2(z), & z \in \Gamma \end{aligned}$$

dual if and only if

- (a) Banach spaces where solutions must be found for these problems are dual;
- (b) $K_1(z)K_2(z) = 1$.

When the dual problems have unique solutions, their duality imposes the duality of these solutions. We can easily see that with this in mind the GF's of the transient state probabilities and FZH probabilities are dual. Indeed, they belong to the dual Banach spaces L_1 and L_∞ , and the coefficients of their respective boundary value problems (see (10) in the present paper and (15) in [1]) are $[1 - xH(z)]$ and $[1 - xH(z)]^{-1}$, respectively.

The same duality exists between the steady state probabilities and the sequence of increments of FZH mean times. Their GF's belong, as was proved above, to L_1 and L_∞ , respectively. The coefficients of their respective boundary value problems are $[1 - H(z)]ze(z)$ and $(1 - z^{-1})[1 - H(z)]^{-1}$.

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