A SHARP UPPER BOUND FOR DISCRETE INEQUALITIES OF GRONWALL-BELLMAN TYPE

E. Thandapani

*Department of Mathematics
Madras University P.G. Centre
Salem-636 011, Tamil Nadu, India*

ABSTRACT

A sharp upper bound is given for solutions of a discrete inequality of the Gronwall-Bellman type. The bound, which is the exact solution of the corresponding discrete equation, is obtained by reducing the equation to a system of difference equations.

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Key Words: Difference inequality, exact solution, difference equations.

1. INTRODUCTION.

It is well known that the discrete analog of Gronwall-Bellman lemma and its generalization, Bihari's lemma established by Jones [4] and Sugiyama [6] (also see [1], [2], [3], [5]), play an important role in the theory of difference equations and numerical analysis. They have been used, for example, to obtain upper bounds of solutions and in [7] it is shown that sharp upper bounds can be obtained by reducing the integral inequalities to a system of differential inequalities. The purpose of this note is to demonstrate how similar estimates can be derived for solutions of the discrete inequality

$$(1) \quad u(x) \leq b_0(x) + b_1(x) \sum_{j=0}^n \sum_{x_1=0}^{x-1} a_0(x_1) \sum_{x_2=0}^{x_1-1} a_1(x_2) \dots \sum_{x_{j+1}=0}^{x_j-1} a_j(x_{j+1}) u(x_{j+1})$$

where $x_0 = x$. When $n = 1$, this inequality was considered by Pachpatte [5] for which he gave an upper bound for u which is not sharp. Further,

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this type of inequality was considered by Agarwal and Thandapani [1] and their upper bound is also not sharp.

2. MAIN RESULT.

Here we shall obtain a sharp upper bound for solutions of (1). The procedure is to find the exact solution of the corresponding discrete equation by reducing it to a system of difference equations.

Let N denote the set $\{0, 1, \dots\}$ and define the operator

$$(2) \quad T_{k-1}u(x) = \sum_{j=k-1}^n \sum_{x_k=0}^{x-1} a_{k-1}(x_k) \dots \sum_{x_{j+1}=0}^{x_j-1} a_j(x_{j+1})u(x_{j+1})$$

with $x_{k-1} = x$ for each $k = 1, 2, \dots, n+1$. It is easily seen that

$$(3) \quad \Delta T_{k-1}u(x) = a_{k-1}(x)[u(x) + T_k u(x)] \quad k = 1, 2, \dots, n+1$$

where Δ is the operator defined by $\Delta z(x) = z(x+1) - z(x)$. Further, let A denote the $n+1$ by $n+1$ matrix

$$(4) \quad A(x) = \begin{bmatrix} 1 + b_1 a_0 & a_0 & 0 & \dots & 0 & 0 \\ b_1 a_1 & 1 & a_1 & \dots & 0 & 0 \\ \vdots & & & & & \\ b_1 a_{n-1} & 0 & 0 & \dots & 1 & a_{n-1} \\ b_1 a_n & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

and B the column vector

$$(5) \quad B = [a_0 b_0, a_1 b_0, \dots, a_n b_0]^T$$

Theorem: Let u , b_0 , b_1 and a_j be real-valued, non-negative numbers defined on N , and $a_j \neq 0$, $j = 0, 1, \dots, n$. If u satisfies (1), then for $x \in N$

$$(6) \quad u(x) \leq b_0(x) + b_1(x)v_1(x)$$

where v_1 is the first component of the $n+1$ -vector function

$$(7) \quad V(x) = \sum_{s=0}^{x-1} Y(x)Y^{-1}(s)B(s)$$

and $Y(x)$ is the fundamental matrix satisfying

$$(8) \quad Y(x+1) = A(x)Y(x).$$

Proof: With the operator (2) define

$$(9) \quad v_k(x) = T_{k-1}u(x)$$

and note that $v_k(0) = 0$, $k = 1, 2, \dots, n+1$. By (3) we have

$$(10) \quad \begin{aligned} \Delta v_k(x) &= a_{k-1}(x)[u(x) + v_{k+1}(x)] \\ &\leq a_{k-1}(x)[b_0(x) + b_1(x)v_1(x) + v_{k+1}(x)] \quad k = 1, 2, \dots, n+1 \end{aligned}$$

since $u(x) \leq b_0(x) + b_1(x)T_0u(x) = b_0(x) + b_1(x)v_1(x)$ and $v_{n+2} \equiv 0$. Thus treating $v_1, v_2, \dots, v_{n+1}$ as components of an $n+1$ -vector function V , the system (10) can be rewritten as

$$(11) \quad V(x+1) \leq A(x)V(x) + B(x), \quad V(0) = 0$$

where A and B are defined by (4) and (5), respectively. $V(0) = 0$ denotes the zero column vector. As is well known, the solution of the initial value problem

$$V(x+1) = A(x)V(x) + B(x), \quad V(0) = 0$$

is given by (7).

It is clear that if equality holds in (1), then the same is true in the above derivation, so that $w = b_0 + b_1v_1$ is the exact solution of the corresponding difference equation.

As a simple example, consider (1) when $n = 2$, $b_0 = b_1 = a_0 = a_1 = 1$ and $a_2 = 2$. The system (11) yields

$$\begin{bmatrix} v_1(x+1) \\ v_2(x+1) \\ v_3(x+1) \end{bmatrix} \leq \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

with $v_1(0) = v_2(0) = v_3(0) = 0$ for which a fundamental matrix $Y(x)$ can be readily obtained. We find

$$v_1(x) = \frac{2}{7}3^x + \frac{5}{7}\cos\frac{\pi}{3}x + \frac{11}{21}\sqrt{3}\sin\frac{\pi}{3}x - 1.$$

Hence we have $u \leq v_1(x) + 1 = w(x)$. This bound is sharp since $w(x)$ is the solution of the corresponding difference equation which is equivalent to the problem

$$w(x+3) - 4w(x+2) + 4w(x+1) - 3w(x) = 0$$

$$w(0) = 1, \quad w(1) = 2, \quad w(2) = 3.$$

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